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ORIGINAL ARTICLE

Utility maximization in an illiquid market in continuous time

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Abstract A utility maximization problem in an illiquid market is studied. The financial market is assumed to have temporary price impact with finite resilience. After the formulation of this problem as a Markovian stochastic optimal control problem a dynamic programming approach is used for its analysis. In particular, the dynamic programming principle is proved and the value function is shown to be the unique discontinuous viscosity solution. This characterization is utilized to obtain numerical results for the optimal strategy and the loss due to illiquidity.

Keywords Liquidity risk · Price impact · Weak dynamic programming · Hamilton–Jacobi–Bellman equation · Viscosity solution · Comparison theorem

1 Introduction

This paper studies the classical Merton problem of utility maximization in a financial market with price impact. We chose to study the problem of optimal investment in finite horizon. In the frictionless market, this problem is to maximize the expected utility from terminal wealth

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$$E\Big[U\Big(y+\int_0^T z_u dS_u\Big)\Big]$$

over all admissible trading strategies z with a given utility function U (Merton 1969, 1971).

The random variable $y+\int_0^T z_u dS_u$ is the gains process and in an illiquid market it has to be modelled by taking this friction into consideration. As in our previous paper in discrete time (Soner and Vukelja 2013), we adopt the approach developed in Roch and Soner (2013). This model is an extension of the one introduced by Çetin et al. (2004) and it allows for resilience. Indeed, we assume that a purchase of z number of shares moves the ask price a_t to $a_t+2m_t z$ for some impact process m_t. This stochastic process m_t is related to the depth of the limit order book and a brief discussion is given in the next section. Hence, when buying z number of shares the total amount paid is $a_t z+m_t z^2$, where the quadratic part is due to illiquidity. In addition to the immediate quadratic cost, the impacted price does not relax back to the original price with infinite speed as it is the case in Çetin et al. (2004). We refer to the survey of Gökay et al. (2011) and the references therein for more information.

We continue by describing the state dynamics. In the classical Merton problem, the only state variables needed are the wealth process and stock price. In fact, one may even not use the stock if its dynamics is homogenous. In our model, to describe the evolution of the state one also needs the wealth like process Y_t and the stock price S_t. However, we also require to trace the portfolio position z_t and a proxy l_t for the impact made on the price. In the next section, we give the precise definitions.

For these state processes, we use the dynamics derived in Roch and Soner (2013). The starting point of their derivation is a constant dept limit order book. Let an adapted process $1/2m_t$ to be this constant dept. Then, a direct calculation shows that a purchase or sale of Δz amount of stock causes a trading cost of $m_t(\Delta z)^2$ and price impact of $2m_t(\Delta z)$. Another key ingredient of this model is an exponential relaxation of the price impact. This is modelled through a resilience parameter (or a process) of κ_u. These observations led (Roch and Soner 2013) to postulate the following dynamics (a brief discussion is also given in the next section),

$$\begin{aligned} dS_u &= \mu S_u du + \sigma S_u dW_u \\ dl_u &= -\kappa_u l_u du + 2m_u dz_u \\ dY_u &= z_u(dS_u - \kappa_u l_u du) - z_u^2 dm_u, \end{aligned}$$

where the process z is assumed to be of finite variation and is the control.

Since the market impact is random, it is not a priori clear that the problem is free from manipulation. Following Roch and Soner (2013), in Lemma 2.1 we show that under reasonable conditions on the coefficients (e.g., for sufficiently large resilience) the liquidity risk is non-negative in the mean.

In this paper, we specialize to the case when S_t has the classical Black–Scholes dynamics, κ is constant and $m_t = MS_t$ for some constant $M>0$. Then, a simplification is possible by introducing

$$\eta_t := l_t - 2MS_t z_t.$$

This allows us to describe the problem using only three state variables S_t, η_t and Y_t. This reduction in the number of state variables makes the numerical computations possible. Moreover, after this reduction one can also allow for larger class of controls by allowing the process z to be the control without any regularity, such as bounded variation, assumptions.

We also introduce the optimal stochastic control problem in Sect. 2. Although a no-arbitrage result is proved, it is not a priori clear that the value function is finite. We achieve this by first constructing a smooth supersolution to the corresponding dynamic programming equation. Then, a classical verification argument shows that this supersolution is an upper bound for the value function. This is the content of Corollary 2.1. We subsequently prove in Theorem 2.1 the weak dynamic programming principle using a covering argument. This in turn shows that the value function is a viscosity solution of the associated dynamic programming equation. These results are proved in Theorems 2.2 and 2.3.

It is well documented in the viscosity literature that the comparison result is key to use the dynamic programming equation in the analysis of the control problem. In particular, it would essentially imply that any monotone scheme, such as the one we employ, converges due to the seminal paper of Barles and Souganidis (1991) and the limiting function is the value function of the control problem. Hence, it is very desirable to have an appropriate comparison result for the corresponding dynamic programming equation. However, the classical comparison results always imply the continuity of the value function. Since we do not know whether that is true for the control problem under investigation, the standard statement and proof had to be changed. Indeed, in Theorem 2.4, we prove a new comparison result that allows for discontinuous solutions. We believe that this is a novel feature of our result and might be useful in other contexts in which the solution is not continuous. These more relaxed comparison results, also allow us to prove the convergence of the numerical schemes in a relaxed sense. Namely, the convergence is pointwise at the points of continuity of the value function $v(t, x)$. At other points the possible limiting values of the numerical scheme are contained in the interval $[v_*(t, x), v^*(t, x)]$. These results are similar to the convergence results for conservation law equations in which the solutions are possibly discontinuous and at the point of discontinuity, pointwise convergence can not be expected.

This paper is organized as follows. The stochastic optimization problem is defined in Sect. 2. In the same section the dynamic programming equation is derived and the smooth supersolution is constructed. There we also prove the weak dynamic programming principle and the comparison theorem. In Sect. 3 we give some numerical results for the optimal trading strategy and the value function. "Appendix" provides the proof of the classical supersolution.

2 Expected utility from terminal wealth

In this section, we state the dynamics derived in Roch and Soner (2013) and for their derivation, we refer the reader to that paper or to our previous article (Soner and Vukelja 2013). In particular, as in Roch and Soner (2013) we assume that there is no bid-ask spread. Then, there are four state variables (S_t, l_t, Y_t, z_t). S_t is the un-impacted stock price and l_t is the price impact which can be both positive or negative and it decays exponentially to zero when there is no more trading. The variable z_t is the number of shares in the portfolio and the wealth like process Y_t is the post liquidation value of the portfolio. It essential that Y_t is not the marked to market value.

We continue with the technical description. Let $\Omega = \mathcal{C}([0,T],\mathbb{R})$ be the canonical space. Then, we denote by $(\Omega, \mathcal{F}, P)$ a complete probability space, where P is the Wiener measure and $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ is the augmentation of the filtration generated by the Brownian motion $(W_t)_{t\geq 0}$. The financial market consists of a risky and a risk-free asset. The risk-free asset is taken to be a numeraire and for simplicity we assume that the spot rate of interest is zero.

Given a state space $\tilde{\mathcal{O}} := \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}$, where $\mathbb{R}_+ = (0,\infty)$, we denote for a fixed time horizon $T \in (0,\infty)$ the time-augmented state space by $[0,T] \times \tilde{\mathcal{O}}$. For each $(t,\tilde{x}) \in [0,T] \times \tilde{\mathcal{O}}$ and $t \in [0,T]$, and a real valued, adapted, bounded variation process α of $[t,T]$, we consider an $\mathbb{F}$-adapted dynamical system $\tilde{X}_t(\omega) := (S_t(\omega), l_t(\omega), Y_t(\omega), z_t(\omega))$ and describe the continuous time dynamics of $\tilde{X}_t(\omega)$ through the following stochastic differential equations, for $u \in (t,T]$,

$$
\begin{aligned}
dS_u &= \mu S_u du + \sigma S_u dW_u && (1)\\
dl_u &= -\kappa l_u du + 2m_u d\alpha_u && (2)\\
dY_u &= z_u(dS_u - \kappa l_u du) - z_u^2 dm_u && (3)\\
dz_u &= d\alpha_u,
\end{aligned}
$$

where $\kappa, \mu, \sigma > 0$ and m_u is a continuous $\mathbb{F}$-adapted process and $\tilde{x} := (s,l,y,z) \in \tilde{\mathcal{O}}$ is the initial data at time t. The process α is the control and we now assume that it is simply adapted with some technical conditions specified below. The parameters μ and σ are classical, κ is the resilience and the process m_u is related to the illiquidity of the stock. We will specialize to the case $m_u = MS_u$ with some constant M in then next subsection. This specification will allow us to reduce the dimension of the problem to three.

The difference in the liquidation value obtained in a frictionless market $y + \int_0^t z_u dS_u$ and our current liquidity risk setup Y_t is given by

$$
L_t = \int_0^t \kappa z_u l_u du + \int_0^t z_u^2 dm_u.
$$

The process L_t is the liquidity cost associated to the depth $\frac{1}{2m_u}$ and the resilience κ of the limit order book (LOB). This leads to

$$
Y_t = y + \int_0^t z_u dS_u - L_t.
$$

The following result provides the structure of the liquidity cost.

Lemma 2.1 *Let $\eta_t = l_t - 2m_t z_t$ and assume $\eta_0 = 0$. If m_t is a non-negative constant, then $L_t \geq 0$ for all $t \geq 0$ P-a.s. In general, if $\Theta_t := \frac{\Lambda_t^2}{m_t}$ with $\Lambda_t = e^{-\kappa t}$ is a supermartingale, then $E[L_t] \geq 0$ for all $t \geq 0$. Furthermore, L_t has the representation*

$$L_t = \frac{\eta_t^2}{4m_t} - \frac{\eta_0^2}{4m_0} - \frac{1}{4}\int_0^t l_u^2 \Lambda_u^{-2} d\Theta_u.$$

The proof can be found in Roch and Soner (2013) Theorem 4.1.

Remark 2.1 If κ is a non-negative constant and $m_t = MS_t$ with $M > 0$, then Θ_t is a supermartingale under P (resp. under the equivalent martingale measure Q) if only if

$$\kappa > \frac{\sigma^2 - \mu}{2} \quad \left(\text{resp. } \kappa > \frac{\sigma^2}{2}\right).$$

Proposition 2.1 *Let Q be the equivalent martingale measure, $\alpha \geq 0$ be a constant. Suppose that $\eta_0 = 0$, $y_0 = 0$. If Θ_t is a Q-supermartingale of class (DL), $Y_T \geq 0$ and $Y_s \geq -\alpha$ Q-a.s. for all $s \in [0, T]$, then $Y_s \geq 0$ P-a.s. for all $s \in [0, T]$.*

Proof Since Θ_t is a Q-supermartingale, there exists by the Doob-Meyer decomposition theorem a Q-martingale M_t and a decreasing predictable process A_t with $A_0 = 0$ such that $\Theta_t = M_t + A_t$. Furthermore, by Lemma 2.1,

$$Y_t = \int_0^t z_u dS_u - \frac{\eta_t^2}{4m_t} + \frac{1}{4}\int_0^t l_u^2 \Lambda_u^{-2} d\Theta_u.$$

Since A_t is decreasing we arrive at

$$Y_t + \frac{\eta_t^2}{4m_t} - \frac{1}{4}\int_0^t l_u^2 \Lambda_u^{-2} dA_u = y + \int_0^t z_u dS_u + \frac{1}{4}\int_0^t l_u^2 \Lambda_u^{-2} dM_u \geq -\alpha. \tag{4}$$

The right-hand side of the Eq. (4) is a Q-local martingale, thus the left-hand side is a Q-local martingale too. But since it is bounded from below it is a Q-supermartingale. Therefore, also Y_t is a Q-supermartingale and

$$0 \leq E[Y_T | \mathcal{F}_s] \leq Y_s, \quad Q\text{-a.s. } \forall\, s \in [0, T].$$

Since Q is equivalent to P, we deduce that $Y_s \geq 0$ P-a.s. for all $s \in [0, T]$. □

Remark 2.2 By the above result (assuming that κ is sufficiently large), Y_t is a Q-super-martingale, which implies $E_Q[Y_T] \leq Y_0$, if $\eta_0 = 0$. Then, this supermartingale property of Y_t implies that there cannot be a process Y so that $Q(Y_T \geq Y_0) = 1$ and $Q(Y_T > Y_0) > 0$. Since Q and P are equivalent measures, this implies that there is no arbitrage.

If, however, $\eta_0 \neq 0$, it is possible to create "local arbitrage", in the sense that the resulting value function is larger than the Merton value function. This is illustrated in Sect. 3 by some numerical computations. However, it is not possible to scale this "local-arbitrage" by increasing the portfolio position z. Indeed, larger values of z would imply larger l values, negating the advantage one has due to non-zero η_0. This is consistent with the liquidity premium derived by Çetin et al. (2010) and Gökay and Soner (2012).

2.1 The value function

In what follows, we always assume that

$$m_t = MS_t,$$

for a constant $M > 0$ and $\kappa > \frac{\sigma^2-\mu}{2}$. The assumption on κ implies that Θ_t is a supermartingale, see Remark 2.1. Following the results of Soner and Vukelja (2013) we introduce a new adapted state variable $\eta_t = l_t - 2MS_t z_t$. By (2) we arrive at

$$d\eta_t = (-\kappa\eta_t - 2M(\kappa+\mu)z_t S_t)\,dt - 2M\sigma z_t S_t dW_t, \tag{5}$$

whereas (3) becomes

$$dY_t = \Big(-\kappa\eta_t z_t - 2M\kappa z_t^2 S_t + \mu z_t S_t(1-Mz_t)\Big)dt + \sigma z_t S_t(1-Mz_t)dW_t. \tag{6}$$

This reduces the state space to three state variables S_t, η_t and Y_t which evolve in time through (1), (5) and (6). In this setup the control variable is z_t and not α_t. The control variable z_t is valued in $\mathbb{R}$. We define $\mathcal{O} := \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+$, where $\mathbb{R}_+ = (0,\infty)$ and set $\mathcal{O}_T := [0,T) \times \mathcal{O}$. Then, we will denote the time-augmented state space by $[0,T] \times \mathcal{O}$. We use the notation $X_t(\omega) := (S_t(\omega), \eta_t(\omega), Y_t(\omega))$ and write

$$dX_t = b(S_t, \eta_t, z_t)dt + \sigma(S_t, z_t)dW_t, \tag{7}$$

where

$$b(S_t,\eta_t,z_t) = \begin{pmatrix} \mu S_t \\ -\kappa\eta_t - 2M(\kappa+\mu)z_t S_t \\ -\kappa\eta_t z_t - 2M\kappa z_t^2 S_t + \mu z_t S_t(1-Mz_t) \end{pmatrix},$$

$$\sigma(S_t,z_t) = \begin{pmatrix} \sigma S_t \\ -2M\sigma z_t S_t \\ \sigma z_t S_t(1-Mz_t) \end{pmatrix}$$

with initial data $x = (s,\eta,y) \in \mathcal{O}$. Notice that the measurable functions $b : \overline{\mathbb{R}}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ do not depend on Y_t. Next, we introduce the set of admissible strategies. Let

$$C : \mathbb{R}_+ \to \mathbb{R}_+$$

be a Lipschitz continuous function and non-decreasing in y. In the following $\{X_u^{t,x,z}\}_{u\in[t,T]}$ denotes the unique solution of (7) where $X_t^{t,x,z} = x$. Then, the set of admissible strategies $\mathcal{A}^{(C)}(t,x)$ is the collection of all adapted processes z on $[t,T]$ satisfying,

(i) $|z_u| \le C\left(Y_u^{t,x,z}\right)$ and $Y_u^{t,x,z} > 0$ P-a.s. $\forall\, u \in [t,T]$

(ii) $E\left[U\left(Y_T^{t,x,z}\right)^-\right] < \infty$.

The first constraint above is a technical one. The main reason behind it is to obtain a well defined Hamiltonian in the dynamic programming equation. Although bounding the portfolio position is reasonable, one should allow this bound to grow with the growing portfolio value as we have done above.

The above set of stochastic differential equations have a strong solution. Indeed, for bounded S_t, the functions b and σ are Lipschitz continuous. Moreover, S_t is an uncontrolled process which is explicitly known. Hence, by a straightforward localization argument we can construct global strong solutions.

Notice that

$$X_\cdot^{t,x,z} \quad \text{has continuous paths,}$$

and that the set of admissible strategies $\mathcal{A}^{(C)}(t,x)$ is not empty for $(t,x) \in [0,T] \times \mathcal{O}$. Furthermore, if $(t_n, x_n) \to (t,x)$ then

$$\sup_{r\in[0,T]} d\left(X_r^{t_n,x_n,z}, X_r^{t,x,z}\right) \to 0 \quad \text{in probability and } P\text{-a.s.},$$

where $X_r^{t,x,z} = x$ for $r \leq t$ and $d(\cdot,\cdot)$ is the Euclidean metric. Note that we implicitly assume $z \in \mathcal{A}^{(C)}(t_n, x_n)$.

Next, we define the value function. We use the standard CRRA utility function,

$$U(y) = \begin{cases} \frac{y^p}{p}, & p < 1, p \neq 0, \\ \log(y), & p = 0. \end{cases} \tag{8}$$

Then, the value function of interest is

$$v(t,x) = \sup_{z\in\mathcal{A}^{(C)}(t,x)} E\left[U\left(Y_T^{t,x,z}\right)\right] =: \sup_{z\in\mathcal{A}^{(C)}(t,x)} J(t,x,z). \tag{9}$$

Note, that v may depend on the upper bound function $C(\cdot)$. We suppress this dependence in our notation.

2.2 Dynamic programming equation

We introduce now the Hamilton–Jacobi–Bellman equation associated to our optimal stochastic control problem (9). We denote by $\mathcal{S}_3$ the set of symmetric 3×3 matrices and by A' the transpose of matrix A. For all $q \in \mathbb{R}^3$, $N \in \mathcal{S}_3$ the Hamiltonian related to (9) is given by

$$H(x,q,N) = \sup_{|z|\leq C(y)} \left\{ b(s,\eta,z)q + \frac{1}{2}\mathrm{tr}(\sigma(s,z)\sigma'(s,z)N) \right\}.$$

Observe that H depends on $C(\cdot)$ and again we suppress this in our notation. Note that the general theory of viscosity applies for nonlinear partial differential equations on an open domain. The corresponding dynamic programming equation is

$$-\partial_t v(t,x) - H\left(x, D_x v(t,x), D_x^2 v(t,x)\right) = 0, \quad (t,x) \in \mathcal{O}_T \tag{10}$$

with the final condition

$$v(T,x) = U(y)$$

and a state space constraint at $y = 0$ as defined in Soner (1986). Namely, if for a smooth function φ the difference $v - \varphi$ has a maximum at some point (t_0, s_0, η_0, y_0) with $y_0 = 0$, then φ is a sub solution of the equation at this point. Notice that even if v is smooth, since the maximum may be attained at a boundary point, the y-derivatives of v and φ may not agree at this point. Still the sub solution property persists and this is a boundary condition. We refer the reader to Fleming and Soner (2006) for a more detailed discussion.

2.2.1 A supersolution

Let p be the exponent in the utility function U defined in (8). Then, consider the function $\phi : [0,T] \times \mathcal{O} \to \mathbb{R}$,

$$\phi(t,x) = \begin{cases} f(t)\frac{1}{p}\left(\beta y + \frac{\eta^2}{s}\right)^p, & p < 1, p \neq 0, \\ \log\left(\beta y + \frac{\eta^2}{s}\right) + c(T-t), & p = 0, \end{cases} \tag{11}$$

where $f(t) = e^{cp(T-t)}$; c and β are positive constants that will be chosen later. Note that ϕ depends on the power p. In the proof of Theorem 2.4 below, we will write $\phi^{(p)}$ instead of ϕ only in order to indicate the exponent. For now we suppress this in our notation.

Lemma 2.2 *Let $\phi(t,x)$ be as in (11). For all β and c sufficiently large, ϕ is a supersolution of*

$$-\partial_t \phi(t,x) - H\left(x, D_x\phi(t,x), D_x^2\phi(t,x)\right) \geq 0, \quad (t,x) \in \mathcal{O}_T.$$

Proof Let $p < 1$, $p \neq 0$ and set $\tilde{\phi}(s,\eta,y) = \frac{1}{p}\left(\beta y + \frac{\eta^2}{s}\right)^p$. We need to show that

$$I(\phi) := -\partial_t\phi + \inf_{|z| \leq C(y)} \{-\mathcal{L}^z \phi\} \geq 0,$$

where

$$\begin{aligned}\mathcal{L}^z\phi &:= b(s,\eta,z)\nabla\phi + \frac{1}{2}tr\left(\sigma(s,z)\sigma(s,z)'D_x^2\phi\right)\\ &= \mu s\phi_s + \frac{1}{2}\sigma^2 s^2\phi_{ss} - (\kappa\eta + 2M(\kappa+\mu)zs)\,\phi_\eta + \Big(-\kappa\eta z - 2M\kappa s z^2\\ &\quad + \mu z s(1-Mz)\Big)\,\phi_y - 2M(\sigma z s)^2(1-Mz)\phi_{\eta y} - 2M(\sigma s)^2 z\phi_{s\eta}\\ &\quad +\ (\sigma s)^2 z(1-Mz)\phi_{sy} + 2(M\sigma s z)^2\phi_{\eta\eta} + \frac{1}{2}(\sigma s z)^2(1-Mz)^2\phi_{yy}.\end{aligned}$$

It is sufficient to show that for a constant α depending on β and p

$$\mathcal{L}^z\tilde{\phi} \le \alpha\left(\beta y + \frac{\eta^2}{s}\right)^p. \tag{12}$$

Indeed, if (12) holds, then

$$I(\phi) = cp\phi + f(t)\inf_{|z|\le C(y)}\{-\mathcal{L}^z\tilde{\phi}\} \ge cp\phi - \alpha p\phi = p\phi(c-\alpha).$$

Then for all $c \ge \alpha$, the function ϕ is a supersolution. The inequality (12) follows from a straightforward but tedious calculation given in "Appendix". The case $p = 0$ is proved analogously. □

Applying the operator $\mathcal{L}^z$ to the above supersolution ϕ we obtain an upper bound for the value function.

Corollary 2.1 *Let β and c be as in Lemma 2.2 and*

$$\phi(t,x) = \begin{cases} e^{cp(T-t)}\frac{1}{p}\left(\beta y + \frac{\eta^2}{s}\right)^p & \text{if } p<1,\ p\neq 0,\\ \log\left(\beta y + \frac{\eta^2}{s}\right) + c(T-t) & \text{if } p = 0.\end{cases}$$

Then, for every $(t,x) \in [0,T]\times\mathcal{O}$, $z \in \mathcal{A}^{(C)}(t,x)$ and stopping time $\tau \in [t,T]$, $\phi\left(\tau, X_\tau^{t,x,z}\right)$ is integrable and

$$E\left[\phi\left(\tau, X_\tau^{t,x,z}\right)\right] \le \phi(t,x). \tag{13}$$

In particular,

$$v(t,x) \le \phi(t,x), \quad \forall\,(t,x)\in[0,T]\times\mathcal{O}.$$

Proof Assume $p < 1$, $p \neq 0$. Let τ be an $\mathbb{F}$-stopping time. Itô's rule applied to $\phi(t, X_t)$ and Lemma 2.2 yield

$$\begin{aligned}\phi(t,x) &= \phi(\tau, X_\tau) - \int_t^\tau \left(\partial_t \phi(u, X_u) + \mathcal{L}^z \phi(u, X_u)\right) du \\ &\quad + \sigma \int_t^\tau \left(2M\phi_\eta z_u S_u - \phi_y S_u z_u (1 - M z_u) - \phi_s S_u\right) dW_u \\ &\geq \phi(\tau, X_\tau) + \sigma \int_t^\tau \left(2M\phi_\eta z_u S_u - \phi_y S_u z_u (1 - M z_u) - \phi_s S_u\right) dW_u.\end{aligned}$$

Let $H_u := \sigma(2M\phi_\eta z_u S_u - \phi_y S_u z_u(1 - M z_u) - \phi_s S_u)$ and c_n be a sequence converging to infinity. Define a stopping time $\tau_n := \inf\{u \geq t : |H_u| \geq c_n\} \wedge \tau$ such that $\tau_n \nearrow \tau$. Then, the stochastic integral $\int_t^{\tau_n} H_u dW_u$ is a martingale. Let $z \in \mathcal{A}^{(C)}(t,x)$ and set $x^\varepsilon := (s, \eta, y + \varepsilon)$, where $\varepsilon > 0$. Thus, for all $z \in \mathcal{A}^{(C)}(t,x)$ we have $Y_\cdot^{t,x^\varepsilon,z} = Y_\cdot^{t,x,z} + \varepsilon$ and

$$X_\cdot^{t,x^\varepsilon,z} = \begin{pmatrix} S_\cdot^{t,s} \\ \eta_\cdot^{t,s,\eta,z} \\ Y_\cdot^{t,x,z} + \varepsilon \end{pmatrix}.$$

Let $\alpha^{n,\varepsilon} := \frac{1}{p}\left(\beta(Y_{\tau_n}^{t,x,z} + \varepsilon) + \frac{(\eta_{\tau_n}^{t,s,\eta,z})^2}{S_{\tau_n}^{t,s}}\right)^p$, then

(i) $\alpha^{n,\varepsilon} \geq \frac{(\beta\varepsilon)^p}{p}$ P-a.s.
(ii) $\alpha^{n,\varepsilon} \longrightarrow \frac{1}{p}\left(\beta(Y_\tau^{t,x,z} + \varepsilon) + \frac{(\eta_\tau^{t,s,\eta,z})^2}{S_\tau^{t,s}}\right)^p$ P-a.s. as $n \to \infty$.

Thus, by Fatou we arrive at

$$\begin{aligned}\phi(t, x^\varepsilon) &\geq \liminf_{n\to\infty} E\left[e^{cp(\tau_n - t)}\frac{1}{p}\left(\beta\left(Y_{\tau_n}^{t,x,z} + \varepsilon\right) + \frac{\left(\eta_{\tau_n}^{t,s,\eta,z}\right)^2}{S_{\tau_n}^{t,s}}\right)^p\right] \\ &\geq E\left[e^{cp(\tau - t)}\frac{1}{p}\left(\beta\left(Y_\tau^{t,x,z} + \varepsilon\right) + \frac{\left(\eta_\tau^{t,s,\eta,z}\right)^2}{S_\tau^{t,s}}\right)^p\right] \\ &= E\left[\phi\left(\tau, X_\tau^{t,x^\varepsilon,z}\right)\right].\end{aligned}$$

We prove (13) by letting $\varepsilon \to 0$. The final statement follows by taking $\tau = T$ and observing that $U(Y_T^{t,x,z}) \leq \phi(T, X_T^{t,x,z})$. The case $p = 0$ can be proved analogously by using

$$\alpha^{n,\varepsilon} = \log\left(\beta\left(Y_{\tau_n}^{t,x,z} + \varepsilon\right) + \frac{\left(\eta_{\tau_n}^{t,s,\eta,z}\right)^2}{S_{\tau_n}^{t,s}}\right)$$

instead. $\square$

The above supersolution allows us to construct admissible controls $z \in \mathcal{A}^{(C)}(t,x)$. Indeed, consider an adapted process $\pi : [0,\infty) \times \Omega \to [-1,1]$ and the SDE

$$dX_u = b(S_u, \eta_u, \pi_u C(Y_u))du + \sigma(S_u, \pi_u C(Y_u))dW_u, \quad u \in (t,T]. \tag{14}$$

Define for all $N \in \mathbb{N}$

$$\tau^{(N)} = \inf \left\{ u \in [t,T] : \ Y_u^{t,x,\pi} = N, \, P\text{-a.s.} \right\}, \tag{15}$$

where we set $\tau^{(N)} = T$, if $\left\{ u \in [t,T] : \ Y_u^{t,x,\pi} = N \ P\text{-a.s.} \right\} = \emptyset$. We also introduce

$$\theta := \theta^{t,x,\pi} = \inf \left\{ u \in [t,T] : \ Y_u^{t,x,\pi} = 0 \ P\text{-a.s.} \right\}, \tag{16}$$

where again $\theta^{t,x,\pi} = T$, if the above set is empty. It is clear that the SDE (14) has a strong solution on $[0, \tau^{(N)} \wedge \theta]$. Indeed, set

$$C^{(N)}(y) = \begin{cases} C(y), & 0 < y < N, \\ C(N), & y \geq N \end{cases}$$

and

$$b^{(N)}(s,\eta,\pi) := b\left(s, \eta, \pi C^{(N)}(y)\right), \quad \sigma^{(N)}(s,\pi) := \sigma\left(s, \pi C^{(N)}(y)\right).$$

The SDE

$$dX_u = b^{(N)}(S_u, \eta_u, \pi_u)du + \sigma^{(N)}(S_u, \pi_u)dW_u$$

has a strong solution $X^{(N)}_\cdot$ on $[0, \tau^{(N)} \wedge \theta]$, since $b^{(N)}(\cdot), \sigma^{(N)}(\cdot)$ satisfy the Lipschitz condition. It is clear that $X_\cdot = X^{(N)}_\cdot$ on $[0, \tau^{(N)} \wedge \theta]$, where $X_\cdot$ denotes the solution of (14). We argue below that $\tau^{(N)} \wedge \theta$ converges to $T \wedge \theta$ showing that $X_\cdot$ is a strong solution of (14) on $[0, T \wedge \theta]$.

Lemma 2.3 *Let $\tau^{(N)}$ be as in (15) and θ as in (16). Then,*

$$\lim_{N \to \infty} P\left(\tau^{(N)} \wedge \theta = T \wedge \theta\right) = 1.$$

Proof Fix $(t,x) \in [0,T] \times \mathcal{O}$ and let ϕ be the upper bound with the parameter $p' = \frac{1}{2}$. Set $X_u := X_u^{t,x,z}$, where $z_u := \pi_u C(Y_u)$. Then by Corollary 2.1

$$E\left[\phi\left(\tau^{(N)} \wedge \theta, X_{\tau^{(N)} \wedge \theta}\right)\right] \leq \phi(t,x).$$

Furthermore, note that $p' = \frac{1}{2}$ implies $\phi \geq 0$. Then for all $(t, x) \in [0, T] \times \mathcal{O}$ there exists a positive constant $\tilde{c}$ so that

$$\begin{aligned}\tilde{c}\sqrt{N} P\left(\tau^{(N)} < \theta\right) &\leq E\left[\phi\left(\tau^{(N)}, X_{\tau^{(N)}}\right) \chi_{\{\tau^{(N)}<\theta\}}\right] \\ &\leq E\left[\phi(\tau^{(N)}, X_{\tau^{(N)}})\chi_{\{\tau^{(N)}<\theta\}}\right] + E\left[\phi\left(\theta, X_\theta\right) \chi_{\{\theta<\tau^{(N)}\}}\right] \\ &= E\left[\phi\left(\tau^{(N)} \wedge \theta, X_{\tau^{(N)}\wedge\theta}\right)\right] \\ &\leq \phi(t, x).\end{aligned}$$

Observe that,

$$\left\{\tau^{(N)} \geq \theta\right\} = \left\{\tau^{(N)} \geq T \wedge \theta\right\} \subseteq \left\{\tau^{(N)} \wedge \theta \geq T \wedge \theta\right\} = \left\{\tau^{(N)} \wedge \theta = T \wedge \theta\right\},$$

where the last equality follows from $\tau^{(N)} \leq T$ P-a.s.. Thus,

$$\begin{aligned}\lim_{N\to\infty} P\left(\tau^{(N)} \wedge \theta = T \wedge \theta\right) &\geq \lim_{N\to\infty} P\left(\tau^{(N)} \geq T \wedge \theta\right) \\ &= 1 - \lim_{N\to\infty} P\left(\tau^{(N)} < T \wedge \theta\right) = 1.\end{aligned}$$

□

Corollary 2.2 *There exists a strong solution of the SDE*

$$dX_u = b\left(S_u, \eta_u, \pi_u C(Y_u)\right) du + \sigma\left(S_u, \pi_u C(Y_u)\right) dW_u$$

on $[0, T \wedge \theta]$.

Proof The SDE (14) has a strong solution on $[0, \tau^{(N)} \wedge \theta]$ for all $N \in \mathbb{N}$. Thus, it also has a strong solution on $[0, \lim_{N\to\infty} \tau^{(N)} \wedge \theta]$. Since $\tau^{(N)}$ is increasing, $\tau^{(N)} \wedge \theta$ converges to some τ^* P-a.s. By Lemma 2.3 we know that $\tau^{(N)} \wedge \theta$ converges to $T \wedge \theta$ in probability, which implies that $\tau^* = T \wedge \theta$, P-a.s. Thus, the SDE (14) has a strong solution on $[0, T \wedge \theta]$. □

Finally, for any $\epsilon > 0$ small, we set θ^ϵ to be the exit from (ϵ, ∞) instead of $(0, \infty)$ and set

$$z^\epsilon{}_u := \begin{cases} \pi_u C(Y_u), & \text{on } u \in [t, \theta^\epsilon], \\ 0, & \text{on } u \in [\theta^\epsilon, T]. \end{cases}$$

The above control $z^\epsilon \in \mathcal{A}^C(t, x)$.

2.2.2 Weak dynamic programming principle

A key tool for the analysis of stochastic control problems is the dynamic programming principle which holds a.s. for any stopping time $\tau \in [t, T)$

$$\text{“}v(t, x) = \sup_{z \in \mathcal{A}_t} E\left[v\left(\tau, X_\tau^{t,x,z}\right)\right]\text{”}.$$

The proof of the statement above requires that v is measurable. However, when the value function v is known to have some regularity, then the measurability arguments are not needed and the proof can be simplified. For more details on that we refer to Fleming and Soner (2006). As an alternate approach (Bouchard and Touzi 2011) introduced a weak version of the dynamic programming principle (WDPP). This weaker statement avoids the measurable selection argument and can be used when the value function has no a priori regularity. The WDPP for generalized state constraints was studied by Bouchard and Nutz (2012). They first show the WDPP for expectation constraints and apply the results to state constraints. In the following we will prove the WDPP for our optimal stochastic control problem directly. We denote by $\mathcal{T}$ the collection of all $\mathbb{F}$-stopping times with values in $[t, T]$.

Theorem 2.1 *Let $(t, x) \in [0, T] \times \mathcal{O}$ and $\tau \in \mathcal{T}$ be a stopping time.*

(i) *Let $\varphi : [0, T]\times\mathcal{O} \to \mathbb{R}$ be a measurable function such that $v \le \varphi$ and $\inf \varphi > -\infty$ for all $(t, x) \in [0, T] \times \mathcal{O}$. Then,*

$$v(t, x) \le \sup_{z \in \mathcal{A}^{(C)}(t,x)} E\left[\varphi\left(\tau, X_\tau^{t,x,z}\right)\right]. \tag{17}$$

(ii) *Let $\varphi : [0, T] \times \mathcal{O} \to \mathbb{R}$ be a continuous function such that $v \ge \varphi$. Then, $\varphi\left(\tau, X_\tau^{t,x,z}\right)^+$ is integrable for every $z \in \mathcal{A}^{(C)}(t, x)$ and*

$$v(t, x) \ge \sup_{z \in \mathcal{A}^{(C)}(t,x)} E\left[\varphi\left(\tau, X_\tau^{t,x,z}\right)\right]. \tag{18}$$

Proof (i) Fix $(t, x) \in [0, T] \times \mathcal{O}$ and let $\tau \in \mathcal{T}$. We start by defining a map $\mathcal{I} : \Omega \times \Omega \to \Omega$ by,

$$\mathcal{I}(\omega, \omega')_s := (\omega \otimes_\tau \omega')_s = \begin{cases} \omega_s, & s < \tau(\omega), \\ \omega_{\tau(\omega)} + \omega'_s - \omega'_{\tau(\omega)}, & s \ge \tau(\omega). \end{cases}$$

In view of the properties of the Wiener measure, the measure defined on the Borel subsets A of Ω by,

$$\bar{P}(A) := P \times P\left(\left(\omega, \omega'\right) \in \Omega \times \Omega \ : \ \mathcal{I}\left(\omega, \omega'\right) \in A\right)$$

is again a Wiener measure. We fix $\omega \in \Omega$ and define for a given control $z \in$

$\mathcal{A}^{(C)}(t,x)$ a new control

$$z_u^{\omega}\left(\omega'\right) = z_u\left(\omega \otimes_\tau \omega'\right), \quad \forall\, \omega' \in \Omega \text{ and } u \geq t.$$

Then,

$$\begin{aligned} E\left[U\left(Y_T^{\tau(\omega), X^{t,x,z}_{\tau(\omega)}(\omega), z^\omega}\right)\right] &= \int_\Omega U\left(Y_T^{\tau, X^{t,x,z}_\tau, z^\omega}(\omega')\right) P(d\omega') \\ &= \int_\Omega U\left(Y_T^{t,x,z}(\omega \otimes_\tau \omega')\right) P(d\omega'). \end{aligned}$$

In particular

$$\omega \mapsto E\left[U\left(Y_T^{\tau(\omega), X^{t,x,z}_{\tau(\omega)}(\omega), z^\omega}\right)\right]$$

is $\mathcal{F}_\tau$-measurable. Moreover, for any $\mathcal{F}_\tau$-measurable bounded function h,

$$h\left(\omega \otimes_\tau \omega'\right) = h(\omega), \quad \forall\, \omega, \omega' \in \Omega.$$

Hence, for any integrable function g and h as above, by Fubini's theorem and the property of the measure $\bar{P}$ defined above, we obtain,

$$\begin{aligned} &\int_\Omega h(\omega)\left[\int_\Omega g\left(\omega \otimes_\tau \omega'\right) P(d\omega')\right] P(d\omega) \\ &= \int_{\Omega\times\Omega} h\left(\omega \otimes_\tau \omega'\right) g\left(\omega \otimes_\tau \omega'\right) P(d\omega')\, P(d\omega) \\ &= \int_\Omega h(\bar{\omega}) g\left(\bar{\omega}\right) \bar{P}(d\bar{\omega}) \\ &= E[hg]. \end{aligned}$$

Using the above with

$$g(\omega) := E\left[U\left(Y_T^{\tau(\omega), X^{t,x,z}_{\tau(\omega)}(\omega), z^\omega}\right)\right]$$

we arrive at

$$\begin{aligned} E\left[hg\right] &= \int_\Omega h(\omega)\left[\int_\Omega U\left(Y_T^{\tau, X^{t,x,z}_\tau, z^\omega}\left(\omega'\right)\right) P(d\omega')\right] P(d\omega) \\ &= \int_\Omega h(\omega)\left[\int_\Omega U\left(Y_T^{t,x,z}\left(\omega \otimes_\tau \omega'\right)\right) P(\omega')\right] P(d\omega) \\ &= E[hg]. \end{aligned}$$

Hence,

$$E\left[U\left(Y_T^{t,x,z}\right)|\mathcal{F}_\tau\right](\omega) = E\left[U\left(Y_T^{\tau(\omega), X_{\tau(\omega)}^{t,x,z}(\omega), z^\omega}\right)\right] = J\left(\tau(\omega), X_\tau^{t,x,z}(\omega), z^\omega\right).$$

We also note that z^ω belongs to $\mathcal{A}^{(C)}(\tau(\omega), X_{\tau(\omega)}^{t,x,z}(\omega))$. Therefore,

$$\begin{aligned} E\left[U\left(Y_T^{t,x,z}\right)|\mathcal{F}_\tau\right](\omega) &= J\left(\tau(\omega), X_\tau^{t,x,z}(\omega), z^\omega\right) \\ &\le v\left(\tau(\omega), X_\tau^{t,x,z}(\omega)\right) \\ &\le \varphi\left(\tau(\omega), X_\tau^{t,x,z}(\omega)\right), \quad P\text{-a.s.} \end{aligned}$$

Since φ is bounded from below, taking the expectation on both sides reveals

$$v(t,x) = \sup_{z\in\mathcal{A}^{(C)}(t,x)} J(t,x,z) \le \sup_{z\in\mathcal{A}^{(C)}(t,x)} E\left[\varphi\left(\tau, X_\tau^{t,x,z}\right)\right].$$

(ii) By assumption $\varphi \le v$. Notice that Corollary 2.1 implies $\varphi \le \phi$. Furthermore, by Corollary 2.1 $\phi(\tau, X_\tau^{t,x,z})$ is integrable. Thus, $\varphi(\tau, X_\tau^{t,x,z})^+$ is integrable too. Next, we prove the second statement. Fix $\varepsilon > 0$ and choose for all $(t,x) \in [0,T]\times\mathcal{O}$ an admissible strategy $z(t,x) \in \mathcal{A}^{(C)}(t,x)$ such that

$$J(t,x,z(t,x)) \ge v(t,x) - \frac{\varepsilon}{2}.$$

Note that $v \ge \varphi$ implies

$$J(t,x,z(t,x)) \ge \varphi(t,x) - \frac{\varepsilon}{2}. \tag{19}$$

Choose $0 < \delta(t,x) =: \delta_0 \le y$ continuous, such that

$$|\varphi(t,x) - \varphi(t,\bar{x})| < \frac{\varepsilon}{2}, \tag{20}$$

where $\bar{x} = (s, \eta, y + \delta_0)$. For a given point (t', x') and any process

$$\pi \in \mathcal{A} := \{\pi_u : [0,\infty)\times\Omega \to [-1,1],\ \mathbb{F}\text{-adapted}\},$$

by Corollary 2.2 there exists a strong solution X on $[t', T\wedge\theta]$ of

$$dX_u = b(S_u, \eta_u, \pi_u C(Y_u))du + \sigma(S_u, \pi_u C(Y_u))dW_u$$

with initial data $X_{t'} = x'$. Set

$$z_u{}^\pi\left(t', x'; \pi\right) := \pi_u C\left(Y_u^{t',x',\pi}\right). \tag{21}$$

Then, it is clear that $X_\cdot = X_\cdot^{t',x',z^\pi}$. Also for $z \in \mathcal{A}^{(C)}(t,x)$, $\pi_u := z_u / C\left(Y_u^{t,x,z}\right)$ is in the admissible class $\mathcal{A}$. So in the following we will only use the notations

$$\pi_\cdot^z(t,x) := z_\cdot(t,x)/C\left(Y_\cdot^{t,x,z(t,x)}\right),$$

and $z^{\pi^z} = z$.
For a given $\delta_0 > 0$ define a new control $\pi^{\delta_0}(t',x';\pi)$ by

$$\pi_u^{\delta_0}\left(t',x';\pi\right) = \begin{cases} \pi_u, & u < \theta^{\delta_0}\left(t',x';\pi\right), \\ 0, & u \geq \theta^{\delta_0}\left(t',x';\pi\right), \end{cases}$$

where $\theta^{\delta_0}(t',x';\pi) = \inf\left\{u \in [t,T] : Y_u^{t',x',\pi} = \delta_0\right\}$. Also define,

$$z_u^{\delta_0}\left(t',x';\pi\right) := \pi_u^{\delta_0}\left(t',x';\pi\right) C\left(Y_u^{t',x',z^{\pi_u^{\delta_0}(t',x';\pi)}}\right). \tag{22}$$

Note that $z_u^{\delta_0}(t',x';\pi)$ trivially satisfies the upper bound $C(Y_u^{t',x',z_u^{\delta_0}(t',x';\pi)})$. Since by construction, $Y_\cdot^{t',x',z_u^{\delta_0}(t',x';\pi)} \geq \delta_0 > 0$ for all $(t',x') \in [0,T] \times \mathcal{O}$, we conclude that $z_u^{\delta_0}(t',x';\pi)$ is in $\mathcal{A}^{(C)}(t',x')$. We claim that for any $\pi \in \mathcal{A}$,

$$\liminf_{(t',x')\to(t,x)} Y_u^{t',x',z^{\delta_0}(t',x';\pi)} \geq Y_u^{t,x,z^{\delta_0}(t,x;\pi)}, \quad P\text{-a.s.} \tag{23}$$

Indeed, let $z(t',x';\pi)$ be as in (21). Then,

$$\liminf_{\substack{(t',x')\to(t,x)\\ u'\to u}} Y_{u'}^{t',x',z(t',x';\pi)} = Y_u^{t,x,z(t,x;\pi)}, \quad P\text{-a.s.}, \tag{24}$$

$$Y_u^{t',x',z^{\delta_0}(t',x';\pi)} = \begin{cases} Y_u^{t',x',z(t',x';\pi)}, & u < \theta^{\delta_0}(t',x';\pi), \\ \delta_0, & u \geq \theta^{\delta_0}(t',x';\pi), \end{cases}$$

and

$$\liminf_{(t',x')\to(t,x)} \theta^{\delta_0}\left(t',x';\pi\right) \geq \theta^{\delta_0}\left(t,x;\pi\right), \quad P\text{-a.s.} \tag{25}$$

Since $Y_u^{t',x',z^{\delta_0}(t',x';\pi)} \geq \delta_0$, by considering the two cases $u \geq \theta^{\delta_0}(t,x;\pi)$ and its opposite, (24) and (25) imply (23). We now use the lower bound $U(Y_T^{t',x',z^{\delta_0}(t',x';\pi)}) \geq U(\delta_0)$ and Fatou's Lemma to conclude that

$$J\left(t,x,z^{\delta_0}(t,x;\pi)\right) \leq \liminf_{(t',x')\to(t,x)} J\left(t',x',z^{\delta_0}\left(t',x';\pi\right)\right).$$

Thus, for fixed $(t,x) \in [0,T] \times \mathcal{O}$ the map $(t',x') \mapsto J(t',x',z^{\delta_0}(t',x';\pi(t,x))$ is lower semicontinuous and

$$\mathcal{O}_{(t,x)} = \Big\{ \left(t',x'\right) \in [0,T] \times \mathcal{O} : \; |\varphi\left(t',x'\right) - \varphi(t,x)| < \frac{\varepsilon}{2}, \qquad J\left(t',x',z^{\delta_0}\left(t',x';\pi(t,x)\right)\right) > J(t,x,z(t,x)) \Big\} \tag{26}$$

is a family of open sets. Note that it is not clear if $(t,x) \in \mathcal{O}_{(t,x)}$, since $z^{\delta_0}(t',x';\pi(t,x))$ may not be equal to $z(t,x)$. But we will show that $(t,\bar{x})$ with $\bar{x} = (s,\eta,y+\delta_0)$ is in $\mathcal{O}_{(t,x)}$. Indeed,

$$Y_u^{t,\bar{x},z(t,x)} = \delta_0 + Y_u^{t,x,z(t,x)}, \quad \forall\, u \in [t,T].$$

Thus, $Y_u^{t,\bar{x},z(t,x)} > \delta_0$ for all $u \in [t,T]$ P-a.s., so the exit time $\theta^{\delta_0}(t,\bar{x};z(t,x)) = T$. This implies $\pi^{\delta_0}(t,\bar{x};\pi(t,x)) = \pi(t,x)$ and

$$Y_T^{t,\bar{x},z^{\delta_0}(t,\bar{x};\pi(t,x))} > Y_T^{t,x,z(t,x)}.$$

After noting that J is increasing in y, we arrive at

$$J\left(t,\bar{x},z^{\delta_0}(t,\bar{x};\pi(t,x))\right) > J(t,x,z(t,x)).$$

In view of (20) this implies that $\bar{x} := (s,\eta,y+\delta_0) \in \mathcal{O}_{(t,x)}$. Recall that $\delta_0 = \delta(t,x)$ is chosen so that it is continuous and $\delta_0 \leq y$. These facts imply that the family of open sets in (26) forms an open cover of $[0,T] \times \mathcal{O}$. Indeed, for fixed t,s,η the mapping $y \mapsto f(y) := y + \delta_0 = y + \delta(t,x)$ is continuous with

$$\lim_{y \to 0} f(y) = 0 \leq f(y) \leq \lim_{y \to \infty} f(y) = \infty.$$

Therefore, for any given $\bar{y} > 0$ there is y so that $f(y) = \bar{y}$. Then,

$$(t,\bar{x}) = (t,s,\eta,\bar{y}) = (t,s,\eta,f(y)) \in \mathcal{O}_{(t,s,\eta,y)} = \mathcal{O}_{(t,x)}.$$

Thus, there exists a sequence $(t_n,x_n) \in [0,T] \times \mathcal{O}$ such that $\cup_n \mathcal{O}_{(t_n,x_n)} = [0,T] \times \mathcal{O}$. Set

$$C_1 := \mathcal{O}_{(t_1,x_1)}, \quad C_{n+1} := \mathcal{O}_{(t_{n+1},x_{n+1})} \setminus \cup_{k=1}^n \mathcal{O}_{(t_k,x_k)}.$$

Then, for all $(t,x) \in [0,T] \times \mathcal{O}$ there exists an integer $i(t,x) = \min\{n \mid (t,x) \in \mathcal{O}_{(t_n,x_n)}\}$ such that $(t,x) \in C_{i(t,x)}$. Note that $i(t,x)$ is by construction measurable. Fix $(t,x) \in [0,T] \times \mathcal{O}$ and $z \in \mathcal{A}^{(C)}(t,x)$. Let $X_u := X_u^{t,x,z}$ be the corresponding controlled process. Then, $(\tau(\omega), X_{\tau(\omega)}^{t,x,z}(\omega)) \in [0,T] \times \mathcal{O}$. Furthermore, set $i(\omega) := i(\tau(\omega), X_\tau(\omega))$. Since the C_j's are disjoint and countably many, it follows that

$$\left(\tau(\omega), X^{t,x,z}_{\tau(\omega)}(\omega)\right) \in \cup_{j\geq 1} C_j, \quad P\text{-a.s.}$$

We define

$$\pi^{\omega}_u := \pi_u^{\delta(t_{i(\omega)}, x_{i(\omega)})}\left(\tau(\omega), X^{t,x,z}_{\tau}(\omega); \pi(t_{i(\omega)}, x_{i(\omega)})\right).$$

Then, consider the process

$$\tilde{\pi}_u(t,x)(\omega) := \begin{cases} \pi_u(\omega) := z_u(\omega)/C(Y^{t,x,z}_u(\omega)), & u < \tau(\omega), \\ \pi^{\omega}_u, & u \geq \tau(\omega). \end{cases}$$

Finally, set

$$\tilde{z}_u(t,x) := \tilde{\pi}_u(t,x) C\left(Y^{t,x,\tilde{\pi}}_u\right).$$

It follows

$$\begin{aligned} v(t,x) \geq J(t,x,\tilde{z}(t,x)) &= E\left[E\left[U\left(Y^{t,x,z}_T\right)|\mathcal{F}_\tau\right]\right] \\ &\overset{(22)}{=} E\left[J\left(\tau, X^{t,x,z}_\tau, z^{\delta(t_{i(\omega)},x_{i(\omega)})}\left(\tau, X_\tau; \pi(t_{i(\omega)}, x_{i(\omega)})\right)\right)\right] \\ &\overset{(26)}{\geq} E\left[J(t_{i(\omega)}, x_{i(\omega)}, z(t_{i(\omega)}, x_{i(\omega)}))\right] \\ &\overset{(19)}{\geq} E[\varphi(t_{i(\omega)}, x_{i(\omega)})] - \frac{\varepsilon}{2} \\ &\overset{(26)}{\geq} E\left[\varphi\left(\tau, X^{t,x,z}_\tau\right)\right] - \varepsilon. \end{aligned}$$

The proof is completed by the arbitrariness of $\varepsilon > 0$ and $z \in \mathcal{A}^{(C)}(t,x)$. □

2.3 Viscosity solutions

The goal of this section is to use the notion of viscosity solutions in order to weaken the smoothness condition on the value function. We will show that the value function is a viscosity solution of the associated Hamilton–Jacobi–Bellman equation. Recall that $\mathcal{O}_T = [0,T) \times \mathcal{O}$. Since the value function may be discontinuous, we introduce

$$v^*(t,x) = \limsup_{\substack{(t',x')\to(t,x)\\ (t',x')\in\overline{\mathcal{O}}_T}} v(t',x') \quad \text{and} \quad v_*(t,x) = \liminf_{\substack{(t',x')\to(t,x)\\ (t',x')\in\overline{\mathcal{O}}_T}} v(t',x').$$

We note that the value function is bounded from below by $U(y)$. This follows immediately by taking $z = 0$. Together with Corollary 2.1 the value function is locally bounded by

$$U(y) \leq v(t,x) \leq \phi(t,x), \quad \forall\, (t,x) \in [0,T] \times \mathcal{O}.$$

Theorem 2.2 *The value function v is a viscosity supersolution on $\mathcal{O}_T$ of*

$$-\partial_t v - H\left(\cdot, D_x v, D_x^2 v\right) \geq 0.$$

Proof Let $(\bar{t}, \bar{x}) \in \mathcal{O}_T$ and $\varphi \in C^{1,2}(\mathcal{O}_T)$ be a test function such that

$$0 = (v_* - \varphi)(\bar{t}, \bar{x}) \leq (v_* - \varphi)(t, x), \quad \forall\, (t, x) \in \mathcal{O}_T, (\bar{t}, \bar{x}) \neq (t, x).$$

Assume for contradiction

$$-\partial_t \varphi(\bar{t}, \bar{x}) - H(\bar{x}, D_x \varphi(\bar{t}, \bar{x}), D_x^2 \varphi(\bar{t}, \bar{x})) < 0$$

and define $\overline{\varphi}$ by

$$\overline{\varphi}(t, x) = \varphi(t, x) - ((t - \bar{t})^2 + |x - \bar{x}|^4).$$

Since H is continuous and $(\partial_t \varphi, D_x \varphi, D_x^2 \varphi)(\bar{t}, \bar{x}) = (\partial_t \overline{\varphi}, D_x \overline{\varphi}, D_x^2 \overline{\varphi})(\bar{t}, \bar{x})$ there exists $a \in \mathbb{R}$ and $r > 0$, with $\bar{t} + r < T$ such that

$$-\partial_t \overline{\varphi}(t, x) - \mathcal{L}^a(s, \eta, D_x \overline{\varphi}(t, x), D_x^2 \overline{\varphi}(t, x)) < 0, \quad \forall\, (t, x) \in B_r(\bar{t}, \bar{x}), \tag{27}$$

where $B_r(\bar{t}, \bar{x}) \subset \mathcal{O}_T$ is a ball of radius r and center $(\bar{t}, \bar{x})$. Observe that there is a $\delta > 0$ such that

$$\varphi(t, x) \geq \overline{\varphi}(t, x) + 2\delta, \quad \forall\, (t, x) \in \mathcal{O}_T \backslash B_r(\bar{t}, \bar{x}). \tag{28}$$

Let $(t_n, x_n)_n$ be a sequence in $B_r(\bar{t}, \bar{x})$ such that

$$(t_n, x_n) \to (\bar{t}, \bar{x}) \quad \text{and} \quad v(t_n, x_n) \to v_*(\bar{t}, \bar{x}),$$

when n goes to infinity. Furthermore let $z = a$ be a constant control and $X_\cdot^{t_n, x_n, a}$ the solution of (7). Define the stopping time τ_n by $\tau_n = \inf\{s \geq t_n : (s, X_s^{t_n, x_n, a}) \notin B_r(\bar{t}, \bar{x})\}$ and note that $\tau_n < T$, since $\bar{t} + r < T$. Applying Itô's formula to $\overline{\varphi}(s, X_s^{t_n, x_n, a})$ we arrive by (27) at

$$\begin{aligned} \overline{\varphi}(t_n, x_n) &= E\left[\overline{\varphi}\left(\tau_n, X_{\tau_n}^{t_n, x_n, a}\right) - \int_{t_n}^{\tau_n} \left(\partial_t \overline{\varphi} + \mathcal{L}^a\left(\cdot, D_x \overline{\varphi}, D_x^2 \overline{\varphi}\right)\right)\left(s, X_s^{t_n, x_n, a}\right) ds\right] \\ &\leq E\left[\overline{\varphi}\left(\tau_n, X_{\tau_n}^{t_n, x_n, a}\right)\right]. \end{aligned}$$

For n large enough we have

$$v(t_n, x_n) \leq \overline{\varphi}(t_n, x_n) + \delta$$

which leads by inequality (28) to

$$v(t_n, x_n) \le E\left[\overline{\varphi}\left(\tau_n, X_{\tau_n}^{t_n,x_n,a}\right)\right] + \delta \le E\left[\varphi\left(\tau_n, X_{\tau_n}^{t_n,x_n,a}\right)\right] - \delta.$$

Since $v \ge \varphi$ the above contradicts (18). □

Next, we show that the value function v is a viscosity subsolution of the associated dynamic programming equation. We introduce the set

$$\mathcal{O}_y := \mathbb{R}_+ \times \mathbb{R} \times [0, \infty)$$

and prove the subsolution property by a contraposition argument.

Theorem 2.3 *The value function v is a viscosity subsolution on $[0, T) \times \mathcal{O}_y$ of*

$$-\partial_t v - H\left(\cdot, D_x v, D_x^2 v\right) \le 0.$$

In particular, it is also a constrained viscosity subsolution at $y = 0$.

Proof Let $(\bar{t}, \bar{x}) \in [0, T) \times \mathcal{O}_y$ and $\varphi \in C^{1,2}([0, T) \times \mathcal{O}_y)$ be a test function such that

$$0 = (v^* - \varphi)(\bar{t}, \bar{x}) \ge (v^* - \varphi)(t, x), \quad \forall\, (t, x) \in [0, T) \times \mathcal{O}_y, (\bar{t}, \bar{x}) \ne (t, x).$$

Assume on the contrary that

$$-\partial_t\varphi(\bar{t}, \bar{x}) - H\left(\bar{x}, D_x\varphi(\bar{t}, \bar{x}), D_x^2\varphi(\bar{t}, \bar{x})\right) > 0$$

and set

$$\overline{\varphi}(t, x) = \varphi(t, x) + ((t - \bar{t})^2 + |x - \bar{x}|^4).$$

By the continuity of H and since $(\partial_t\varphi, D_x\varphi, D_x^2\varphi)(\bar{t}, \bar{x}) = (\partial_t\overline{\varphi}, D_x\overline{\varphi}, D_x^2\overline{\varphi})(\bar{t}, \bar{x})$ there exists $r > 0$ such that $\bar{t} + r < T$ and

$$\begin{aligned} &-\partial_t\overline{\varphi}(t, x) - H\left(x, D_x\overline{\varphi}(t, x), D_x^2\overline{\varphi}(t, x)\right) > 0, \\ &\forall\, (t, x) \in B_r(\bar{t}, \bar{x}) \cap ([0, T) \times \mathcal{O}_y). \end{aligned} \tag{29}$$

Note that for some $\delta > 0$

$$\overline{\varphi}(t, x) \ge \varphi(t, x) + 2\delta, \quad \forall\, (t, x) \in [0, T) \times \mathcal{O}_y \backslash B_r(\bar{t}, \bar{x}).$$

By the definition of v^* there exists a sequence $(t_n, x_n) \in B_r(\bar{t}, \bar{x})$ such that

$$(t_n, x_n) \to (\bar{t}, \bar{x}) \quad \text{and} \quad v(t_n, x_n) \to v^*(\bar{t}, \bar{x}),$$

as n goes to infinity. Thus, for n large enough

$$v(t_n, x_n) \geq \overline{\varphi}(t_n, x_n) - \delta. \tag{30}$$

For an arbitrary $z \in \mathcal{A}^{(C)}(t_n, x_n)$ define a stopping time $\theta_n := \inf\{s \geq t_n : (s, X_s^{t_n,x_n,z}) \notin B_r(\bar{t}, \bar{x})\}$. Note that $\theta_n < T$, since $\bar{t} + r < T$. Applying Itô's formula to $\overline{\varphi}(s, X_s^{t_n,x_n,z})$ and by (29) we arrive at

$$\begin{aligned}\overline{\varphi}(t_n, x_n) &= E\left[\overline{\varphi}\left(\theta_n, X_{\theta_n}^{t_n,x_n,z}\right) - \int_{t_n}^{\theta_n} \left(\partial_t\overline{\varphi} + \mathcal{L}^z\overline{\varphi}\left(X_u^{t_n,x_n,z}\right) du\right)\right] \\ &\geq E\left[\overline{\varphi}\left(\theta_n, X_{\theta_n}^{t_n,x_n,z}\right)\right].\end{aligned}$$

Note that $(\theta_n, X_{\theta_n}^{t_n,x_n,z}) \notin B_r(\bar{t}, \bar{x})$, hence

$$\overline{\varphi}(t_n, x_n) \geq E\left[\varphi\left(\theta_n, X_{\theta_n}^{t_n,x_n,z}\right)\right] + 2\delta$$

which by (30) leads to

$$v(t_n, x_n) \geq E\left[\varphi\left(\theta_n, X_{\theta_n}^{t_n,x_n,z}\right)\right] + \delta,$$

for every $z \in \mathcal{A}^{(C)}(t_n, x_n)$. Since $\varphi \geq v$, the above contradicts (17). □

2.4 Comparison

In this section we prove the comparison principle. We write $\phi^{(p)}$ instead of only ϕ to indicate the exponent p. Notice that $U(y) \leq v(t, x) \leq \phi^{(p)}(t, x)$ for all $(t, x) \in [0, T] \times \mathcal{O}$ and for $p < p' < 1$ we have $\phi^{(p)}(\cdot) < \phi^{(p')}(\cdot)$. Let $\alpha \geq 1$ be a constant and $\omega : [0, \infty] \to [0, \infty]$ be such that $\omega(0+) = 0$. Following Crandall et al. (1992) the Hamiltonian needs to satisfy

$$H(\bar{x}, \alpha(x - \bar{x}), N) - H(x, \alpha(x - \bar{x}), \bar{N}) \leq \omega(\alpha|x - \bar{x}|^2 + |x - \bar{x}|),$$

for all $x, \bar{x} \in \mathcal{O}_T$, $\bar{N}$ and $N \in \mathcal{S}_3$, when

$$-3\alpha \begin{pmatrix} I_3 & 0 \\ 0 & I_3 \end{pmatrix} \leq \begin{pmatrix} \bar{N} & 0 \\ 0 & -N \end{pmatrix} \leq 3\alpha \begin{pmatrix} I_3 & -I_3 \\ -I_3 & I_3 \end{pmatrix},$$

where I_3 is the 3×3 identity matrix. This immediately implies

$$tr(\sigma(s, z)\sigma'(s, z)\bar{N}) - tr(\sigma(\bar{s}, z)\sigma'(\bar{s}, z)N) \leq 3\alpha|\sigma(s, z) - \sigma(\bar{s}, z)|^2, \tag{31}$$

where $\sigma(s, z)'$ is the transpose of matrix $\sigma(s, z)$.

Lemma 2.4 *Let B_r be an open ball with radius r, $\alpha \geq 1$ and $q = \alpha(x - \bar{x}) := \alpha((s-\bar{s}), (\eta-\bar{\eta}), (y-\bar{y})) \in B_r$ such that $y \leq \bar{y}$. Furthermore, let $N, \bar{N} \in \mathcal{S}_3$ satisfy* (31). *Then, there is a constant $c_r^* > 0$ such that*

$$H\left(x, q, \bar{N}\right) - H(\bar{x}, q, N) \leq c_r^* \alpha |x - \bar{x}|^2.$$

Proof Let B_r be an open ball with radius r. Define for all $x \in B_r$

$$m_r^* := \sup_{x \in B_r} C(y).$$

Then, for all $x, \bar{x} \in B_r$ with $\bar{y} \geq y$ we have by the monotonicity of the function C and by (31)

$$\begin{aligned}
H(x, q, \bar{N}) - H(\bar{x}, q, N) &= \sup_{|z| \leq C(y)} \left\{ b(s, \eta, z)q + \frac{1}{2} tr(\sigma(s, z)\sigma'(s, z)\bar{N}) \right\} \\
&\quad - \sup_{|z| \leq C(\bar{y})} \left\{ b(\bar{s}, \bar{\eta}, z)q + \frac{1}{2} tr(\sigma(\bar{s}, z)\sigma'(\bar{s}, z)N) \right\} \\
&\leq \sup_{|z| \leq C(\bar{y})} \{ |b(s, \eta, z) - b(\bar{s}, \bar{\eta}, z)||q| \\
&\quad + \frac{1}{2} tr(\sigma(s, z)\sigma'(s, z)\bar{N}) - \frac{1}{2} tr(\sigma(\bar{s}, z)\sigma'(\bar{s}, z)N) \Big\} \\
&\leq \sup_{|z| \leq m_r^*} \left\{ |b(s, \eta, z) - b(\bar{s}, \bar{\eta}, z)||q| + 3\alpha |\sigma(s, z) - \sigma(\bar{s}, z)|^2 \right\} \\
&\leq K_r \sqrt{(s - \bar{s})^2 + (\eta - \bar{\eta})^2} |q| + 3\alpha K_r^2 (s - \bar{s})^2 \\
&\leq K_r \alpha (2(s - \bar{s})^2 + 2(\eta - \bar{\eta})^2 + (y - \bar{y})^2) + 3\alpha K_r^2 (s - \bar{s})^2 \\
&\leq c_r^* \alpha |x - \bar{x}|^2,
\end{aligned}$$

where K_r is the Lipschitz constant and c_r^* a sufficiently large constant. □

Theorem 2.4 *Let $\bar{\varepsilon} > 0$. Furthermore, let u (resp. v) be an upper semicontinuous viscosity subsolution (resp. lower semicontinuous viscosity supersolution) of* (10) *satisfying*

$$U(y) \leq u(t, x), v(t, x) \leq \phi(t, x), \quad \forall\ (t, x) \in [0, T] \times \mathcal{O}.$$

Assume $u(T, x) \leq v(T, s, \eta, y + 2\bar{\varepsilon})$ on $\mathcal{O}_y$. Then, for all $\bar{\varepsilon} > 0$,

$$u(t, x) \leq v\left(t, s, \eta, y + 2\bar{\varepsilon}\right), \quad \text{on } [0, T) \times \mathcal{O}_y.$$

Proof 1. Let u be a viscosity subsolution on $[0, T) \times \mathcal{O}_y$ and v a viscosity supersolution on $[0, T) \times \mathcal{O}$ of (10) and $\bar{u}(t, x) = e^{\lambda t} u(t, x)$ resp. $\bar{v}(t, x) = e^{\lambda t} v(t, x)$, where $\lambda > 0$. Then direct calculation shows that $\bar{u}$ (resp. $\bar{v}$) is a viscosity subsolution (resp. supersolution) of

$$-\partial_t w + \lambda w - H\left(\cdot, D_x w, D_x^2 w\right) = 0.$$

2. Let $\varepsilon, \delta \in (0, 1]$ and p be the exponent in the utility function. Then, fix $p' \in (0, 1)$ so that $p < p'$. The function

$$u^{\varepsilon,\delta}(t, x) := \left(u - \varepsilon\phi^{(p')}\right)(t, x) - \delta\left(2\log(s+1) - \log s + c(T-t)\right)$$

is a viscosity subsolution to (10). Indeed, let φ be in $C^{1,2}([0, T) \times \mathcal{O}_y)$ and $(\tilde{t}, \tilde{x}) \in [0, T) \times \mathcal{O}_y$ be such that $(u^{\varepsilon,\delta} - \varphi)(\tilde{t}, \tilde{x}) = \max(u^{\varepsilon,\delta} - \varphi)$. Furthermore, define

$$\varphi^{\varepsilon,\delta}(t, x) := (\varphi + \varepsilon\phi^{(p')})(t, x) + \Delta^c(t, s),$$

where $\Delta^c(t, s) = \delta(2\log(s+1) - \log s + c(T-t))$. Then, since u is a viscosity subsolution of (10) and $(u - \varphi^{\varepsilon,\delta})$ has a maximum at $(\tilde{t}, \tilde{x})$ we see that

$$-\partial_t\varphi^{\varepsilon,\delta}(\tilde{t}, \tilde{x}) - H\left(\tilde{x}, D_x\varphi^{\varepsilon,\delta}(\tilde{t}, \tilde{x}), D_x^2\varphi^{\varepsilon,\delta}(\tilde{t}, \tilde{x})\right) \leq 0. \tag{32}$$

Note that

$$\begin{aligned} &H\left(\tilde{x}, D_x\left(\varepsilon\phi^{(p')}(\tilde{t}, \tilde{x}) + \Delta^c(\tilde{t}, \tilde{s})\right), D_x^2\left(\varepsilon\phi^{(p')}(\tilde{t}, \tilde{x}) + \Delta^c(\tilde{t}, \tilde{s})\right)\right) \\ &\quad = H\left(\tilde{x}, D_x\left(\varepsilon\phi^{(p')}(\tilde{t}, \tilde{x})\right), D_x^2\left(\varepsilon\phi^{(p')}(\tilde{t}, \tilde{x})\right)\right) + \delta K, \end{aligned} \tag{33}$$

where $K = \left(\mu\tilde{s}\left(\frac{2}{\tilde{s}+1} - \frac{1}{\tilde{s}}\right) - \frac{\sigma^2\tilde{s}^2}{2}\left(\frac{2}{(\tilde{s}+1)^2} - \frac{1}{\tilde{s}^2}\right)\right)$. For $c \geq K$ and by Lemma 2.2, equations (32), (33) and the fact that $-\sup\{\mathcal{L}^z\phi\} - \sup\{\mathcal{L}^z\psi\} \leq -\sup\{\mathcal{L}^z\phi + \mathcal{L}^z\psi\}$ we arrive at

$$\begin{aligned} &-\partial_t\varphi(\tilde{t}, \tilde{x}) - H(\tilde{x}, D_x\varphi(\tilde{t}, \tilde{x}), D_x^2\varphi(\tilde{t}, \tilde{x})) \\ &\quad = \left(-\partial_t\varphi^{\varepsilon,\delta} + \varepsilon\partial_t\phi^{(p')}\right)(\tilde{t}, \tilde{x}) - \delta c - H\left(\tilde{x}, D_x\varphi(\tilde{t}, \tilde{x}), D_x^2\varphi(\tilde{t}, \tilde{x})\right) \\ &\qquad - H\left(\tilde{x}, D_x(\varepsilon\phi^{(p')}(\tilde{t}, \tilde{x}) + \Delta^c(\tilde{t}, \tilde{s})), D_x^2(\varepsilon\phi^{(p')}(\tilde{t}, \tilde{x}) + \Delta^c(\tilde{t}, \tilde{s}))\right) \\ &\qquad + H\left(\tilde{x}, D_x(\varepsilon\phi^{(p')}(\tilde{t}, \tilde{x}) + \Delta^c(\tilde{t}, \tilde{s})), D_x^2(\varepsilon\phi^{(p')}(\tilde{t}, \tilde{x}) + \Delta^c(\tilde{t}, \tilde{s}))\right) \\ &\quad \leq \left(-\partial_t\varphi^{\varepsilon,\delta} + \varepsilon\partial_t\phi^{(p')}\right)(\tilde{t}, \tilde{x}) - \delta c - H\left(\tilde{x}, D_x\varphi^{\varepsilon,\delta}(\tilde{t}, \tilde{x}), D_x^2\varphi^{\varepsilon,\delta}(\tilde{t}, \tilde{x})\right) \\ &\qquad + H\left(\tilde{x}, D_x(\varepsilon\phi^{(p')}(\tilde{t}, \tilde{x}) + \Delta^c(\tilde{t}, \tilde{s})), D_x^2(\varepsilon\phi^{(p')}(\tilde{t}, \tilde{x}) + \Delta^c(\tilde{t}, \tilde{s}))\right) \\ &\quad \leq H\left(\tilde{x}, D_x(\varepsilon\phi^{(p')}(\tilde{t}, \tilde{x}) + \Delta^c(\tilde{t}, \tilde{s})), D_x^2(\varepsilon\phi^{(p')}(\tilde{t}, \tilde{x}) + \Delta^c(\tilde{t}, \tilde{s}))\right) \end{aligned}$$

$$
\begin{aligned}
&\quad - H\left(\tilde{x}, D_x\varepsilon\phi^{(p')}(\tilde{t},\tilde{x}), D_x^2\varepsilon\phi^{(p')}(\tilde{t},\tilde{x})\right)\\
&\quad + H\left(\tilde{x}, D_x\varepsilon\phi^{(p')}(\tilde{t},\tilde{x}), D_x^2\varepsilon\phi^{(p')}(\tilde{t},\tilde{x})\right)\\
&\quad + \varepsilon\partial_t\phi^{(p')}(\tilde{t},\tilde{x}) - \delta c\\
&\le H\left(\tilde{x}, D_x(\varepsilon\phi^{(p')}(\tilde{t},\tilde{x}) + \Delta^c(\tilde{t},\tilde{s})), D_x^2(\varepsilon\phi^{(p')}(\tilde{t},\tilde{x}) + \Delta^c(\tilde{t},\tilde{s}))\right)\\
&\quad - H\left(\tilde{x}, D_x\varepsilon\phi^{(p')}(\tilde{t},\tilde{x}), D_x^2\varepsilon\phi^{(p')}(\tilde{t},\tilde{x})\right) - \delta c\\
&= \delta(K-c) \le 0.
\end{aligned}
$$

3. Let $\varepsilon, \bar{\varepsilon}, \delta \in (0,1]$ and fix $p' \in (0,1)$ such that $p < p'$. Recall, that p is the exponent in the utility function. Set $x^{(2\bar{\varepsilon})} := (s, \eta, y + 2\bar{\varepsilon})$ and notice that

$$U(y) \le u(t,x), v(t,x) \le \phi^{(p)}(t,x)$$

implies

$$-v\left(t, x^{(2\bar{\varepsilon})}\right) \le -U\left(y + 2\bar{\varepsilon}\right) \le -U\left(2\bar{\varepsilon}\right) < \infty.$$

Observe that, for $p' \in (0,1)$ such that $p < p'$, there exists a positive constant $c(\varepsilon, p, p')$ satisfying

$$\phi^{(p)}(t,x) - \varepsilon\phi^{(p')}(t,x) \le c(\varepsilon, p, p'), \quad \forall\, (t,x) \in [0,T] \times \mathcal{O}.$$

The function $u^{\varepsilon,\delta}(t,x) - v\left(t, x^{(2\bar{\varepsilon})}\right)$ is upper semicontinuous and in view of

$$
\begin{aligned}
&\lim_{|x|\to\infty} \sup_{t\in[0,T]} u^{\varepsilon,\delta}(t,x) - v\left(t, x^{(2\bar{\varepsilon})}\right)\\
&\quad \le \lim_{|x|\to\infty} \sup_{t\in[0,T]} c(\varepsilon, p, p') - U\left(2\bar{\varepsilon}\right)\\
&\qquad - \delta(2\log(s+1) - \log s + c(T-t)) = -\infty
\end{aligned}
$$

attains a maximum $(\tilde{t}, \tilde{x}) \in \overline{\mathcal{O}}_T$. Set

$$\max_{\overline{\mathcal{O}}_T} u^{\varepsilon,\delta}(t,x) - v\left(t, x^{(2\bar{\varepsilon})}\right) =: m.$$

If $m \le 0$, then for all $(t,x) \in \overline{\mathcal{O}}_T$ we have

$$u(t,x) - v\left(t, x^{(2\bar{\varepsilon})}\right) = \lim_{\varepsilon,\delta\to 0} u^{\varepsilon,\delta}(t,x) - v\left(t, x^{(2\bar{\varepsilon})}\right) \le 0$$

and we can conclude. Next, assume on the contrary

$$\max_{\overline{\mathcal{O}}_T} u^{\varepsilon,\delta}(t,x) - v\left(t, x^{(2\bar{\varepsilon})}\right) = m > 0 \tag{34}$$

and consider the following cases:

(i) if $\tilde{t} = T$, then $u(\tilde{t},\tilde{x}) \le v\left(\tilde{t},\tilde{x}^{(2\bar{\varepsilon})}\right)$ contradicts (34),

(ii) if $\tilde{s} = 0$, then $u^{\varepsilon,\delta}(\tilde{t},\tilde{x}) - v\left(\tilde{t},\tilde{x}^{(2\bar{\varepsilon})}\right) = -\infty$ which contradicts $m < \infty$,

(iii) $\tilde{y} = 0$ is possible, but by assumption u is a viscosity subsolution on $[0,T)\times\mathcal{O}_y$.

Thus, the maximizer $(\tilde{t},\tilde{x})$ is in $[0,T)\times\mathcal{O}_y$.

4. Let $\varepsilon, \bar{\varepsilon}, \delta \in (0,1]$, $\alpha \ge 1$ and define the region

$$\mathcal{C}(\bar{\varepsilon}) = \{(t,x),(\bar{t},\bar{x}) \mid (t,x) \in [0,T]\times\overline{\mathcal{O}}, (\bar{t},\bar{s},\bar{\eta},\bar{y}-\bar{\varepsilon}) \in [0,T]\times\overline{\mathcal{O}}\}.$$

If $(t,x),(\bar{t},\bar{x}) \in \mathcal{C}(\bar{\varepsilon})$, then $\bar{y} \ge \bar{\varepsilon}$ and

$$-v(\bar{t},\bar{x}) \le -U(\bar{y}) \le -U(\bar{\varepsilon}).$$

Consider the upper semicontinuous function

$$\psi^{\varepsilon,\bar{\varepsilon},\delta,\alpha}(t,\bar{t},x,\bar{x}) = u^{\varepsilon,\delta}(t,x) - v(\bar{t},\bar{x}) - \frac{\alpha}{2}\overline{\phi}(t,\bar{t},x,\bar{x})$$
$$\overline{\phi}(t,\bar{t},x,\bar{x}) = \left(|t-\bar{t}|^2 + |s-\bar{s}|^2 + |\eta-\bar{\eta}|^2 + |y-\bar{y}+2\bar{\varepsilon}|^2\right).$$

In view of

$$\lim_{|x|,|\bar{x}|\to\infty}\ \sup_{t,\bar{t}\in[0,T]} \psi^{\varepsilon,\bar{\varepsilon},\delta,\alpha}(t,\bar{t},x,\bar{x}) = -\infty$$

we claim that $\psi^{\varepsilon,\bar{\varepsilon},\delta,\alpha}$ attains a maximum at $(t_\alpha,\bar{t}_\alpha,x_\alpha,\bar{x}_\alpha)$ with $(t_\alpha,x_\alpha),(\bar{t}_\alpha,\bar{x}_\alpha) \in \mathcal{C}(\bar{\varepsilon})$ so that

$$\max_{(t,x),(\bar{t},\bar{x})\in\mathcal{C}(\bar{\varepsilon})} \psi^{\varepsilon,\bar{\varepsilon},\delta,\alpha}(t,\bar{t},x,\bar{x}) =: m_\alpha.$$

More precisely, we assume without loss of generality that $m_\alpha > 0$ and for all $\varepsilon,\bar{\varepsilon},\delta \in (0,1]$ and $\alpha \ge 1$ there exists $\alpha^* := \alpha(\varepsilon,\bar{\varepsilon},\delta,p,p')$ such that for all $\alpha > \alpha^*$ the maximizer $(t_\alpha,x_\alpha),(\bar{t}_\alpha,\bar{x}_\alpha) \in [0,T]\times\mathbb{R}_+\times\mathbb{R}\times[\bar{\varepsilon},\infty)$. Indeed,

(i) For any $(t,x),(\bar{t},\bar{x}) \in \mathcal{C}(\bar{\varepsilon})$

$$\psi^{\varepsilon,\bar{\varepsilon},\delta,\alpha}(t,\bar{t},x,\bar{x}) \le c(\varepsilon,p,p') - U(\bar{\varepsilon}) - \delta(2\log(s+1) - \log s + c(T-t)).$$

Hence, there exists $s_1 := s_1(\varepsilon,\bar{\varepsilon},\delta,p,p')$ and $s_2 := s_2(\varepsilon,\bar{\varepsilon},\delta,p,p')$ such that $0 < s_1 \le s_2 < \infty$ and

$$\psi^{\varepsilon,\bar{\varepsilon},\delta,\alpha}(t,\bar{t},x,\bar{x}) \le 0,$$

if $s \notin [2s_1, \frac{1}{2}s_2]$.

(ii) Again for any $(t, x), (\bar{t}, \bar{x}) \in \mathcal{C}(\bar{\varepsilon})$

$$\psi^{\varepsilon,\bar{\varepsilon},\delta,\alpha}(t, \bar{t}, x, \bar{x}) \le c(\varepsilon, p, p') - U(\bar{\varepsilon}) - \frac{\alpha}{2}\overline{\phi}(t, \bar{t}, x, \bar{x}).$$

Hence, the maximum of $\psi^{\varepsilon,\bar{\varepsilon},\delta,\alpha}$ is achieved in the region

$$\overline{\phi}(t, \bar{t}, x, \bar{x}) < \frac{2}{\alpha}\left(c(\varepsilon, p, p') - U(\bar{\varepsilon})\right).$$

Suppose that

$$\alpha > \alpha^* = \frac{(c(\varepsilon, p, p') - U(\bar{\varepsilon}))}{2s_1^2}.$$

Then,

$$\overline{\phi}(t, \bar{t}, x, \bar{x}) < \frac{2}{\alpha}\left(c(\varepsilon, p, p') - U(\bar{\varepsilon})\right) \quad \Rightarrow \quad \overline{\phi}(t, \bar{t}, x, \bar{x}) < s_1^2/4.$$

In view of the definition of $\overline{\phi}$ we conclude that $|s - \bar{s}| \le s_1/2$. Since, $s_1 \le s_2$, using step i), we can restrict $s, \bar{s}$ in $[s_1, s_2]$ provided that $\alpha > \alpha^*$.

(iii) For any $(t, x), (\bar{t}, \bar{x}) \in \mathcal{C}(\bar{\varepsilon})$ and $s, \bar{s} \in [s_1, s_2]$

$$\psi^{\varepsilon,\bar{\varepsilon},\delta,\alpha}(t, \bar{t}, x, \bar{x}) \le \phi^{(p)}(t, x) - \varepsilon\phi^{(p')}(t, x) - U(\bar{\varepsilon}).$$

Hence, there exists $c_1 := c_1(\varepsilon, \bar{\varepsilon}, p, p') < \infty$ so that if $|\eta| > c_1$ or $y > c_1$, then $\psi^{\varepsilon,\bar{\varepsilon},\delta,\alpha} < 0$. Therefore, we can restrict the maximization of $\psi^{\varepsilon,\bar{\varepsilon},\delta,\alpha}$ to $|\eta| < c_1$ and $y < c_1$. Notice that $y_\alpha = 0$ and $\bar{y}_\alpha \ge \bar{\varepsilon}$ is possible.

Next, we claim that

$$\lim_{\alpha\to\infty} \frac{\alpha}{2}\overline{\phi}\left(t_\alpha, \bar{t}_\alpha, x_\alpha, \bar{x}_\alpha\right) = 0 \quad \text{and} \quad \lim_{\alpha\to\infty} m_\alpha = m.$$

Indeed, since $\overline{\phi}$ is positive, m_α is decreasing when α increases. Furthermore,

$$\begin{aligned} m_{\frac{\alpha}{2}} &\ge u^{\varepsilon,\delta}(t_\alpha, x_\alpha) - v(\bar{t}_\alpha, \bar{x}_\alpha) - \frac{\alpha}{4}\overline{\phi}(t_\alpha, \bar{t}_\alpha, x_\alpha, \bar{x}_\alpha) \\ &= m_\alpha + \frac{\alpha}{4}\overline{\phi}(t_\alpha, \bar{t}_\alpha, x_\alpha, \bar{x}_\alpha). \end{aligned}$$

Thus, $0 \le \frac{\alpha}{2}\overline{\phi}(t_\alpha, \bar{t}_\alpha, x_\alpha, \bar{x}_\alpha) \le 2(m_{\frac{\alpha}{2}} - m_\alpha)$. Since m_α is a non-increasing function of α and since it is bounded from below by zero, it has a limit as α tends to zero. Hence, the difference $m_{\frac{\alpha}{2}} - m_\alpha$ converges zero as α approaches to zero. This in turn implies $\frac{\alpha}{2}\overline{\phi}(t_\alpha, \bar{t}_\alpha, x_\alpha, \bar{x}_\alpha) \to 0$. Assume now $\alpha_n \to \infty$ and $(t_{\alpha_n}, \bar{t}_{\alpha_n}, x_{\alpha_n}, \bar{x}_{\alpha_n}) \to (t^*, \bar{t}^*, x^*, \bar{x}^*)$. Then,

$$\overline{\phi}\left(t_{\alpha_n}, \bar{t}_{\alpha_n}, x_{\alpha_n}, \bar{x}_{\alpha_n}\right) \to 0$$

which by continuity leads to $\overline{\phi}(t^*, \bar{t}^*, x^*, \bar{x}^*) = 0$. This implies $t_{\alpha_n}, \bar{t}_{\alpha_n} \to t^*$, $s_{\alpha_n}, \bar{s}_{\alpha_n} \to s^*$, $\eta_{\alpha_n}, \bar{\eta}_{\alpha_n} \to \eta^*$, $y_{\alpha_n} \to y^*$ and $\bar{y}_{\alpha_n} \to y^* + 2\bar{\varepsilon}$. Note that since $y^* \geq 0$, we obtain $\bar{y}_{\alpha_n} > \bar{\varepsilon}$. Therefore, $\bar{y}_{\alpha_n}$ is an interior point. Next, we prove $\lim_{\alpha\to\infty} m_\alpha = m$. Set $x_{\alpha_n}^{(2\bar{\varepsilon})} := (s_{\alpha_n}, \eta_{\alpha_n}, y_{\alpha_n} + 2\bar{\varepsilon})$ and $x^{*(2\bar{\varepsilon})} = (s^*, \eta^*, y^* + 2\bar{\varepsilon})$. Notice that

$$m_{\alpha_n} \geq \sup \psi^{\varepsilon,\bar{\varepsilon},\delta,\alpha_n}\left(t, t, x, x^{(2\bar{\varepsilon})}\right) = m$$

which implies $m \leq m_{\alpha_n}$. Moreover, since $u^{\varepsilon,\delta} - v$ is upper semicontinuous we see that

$$m \leq m_{\alpha_n} \leq u^{\varepsilon,\delta}(t_{\alpha_n}, x_{\alpha_n}) - v(\bar{t}_{\alpha_n}, \bar{x}_{\alpha_n})$$

and

$$m \leq \limsup_{\alpha_n\to\infty} m_{\alpha_n} \leq u^{\varepsilon,\delta}(t^*, x^*) - v\left(\bar{t}^*, \bar{x}^{*(2\bar{\varepsilon})}\right) \leq m.$$

5. Since $(t_\alpha, \bar{t}_\alpha, x_\alpha, \bar{x}_\alpha)_\alpha$ converges to $\left(t^*, t^*, x^*, x^{*(2\bar{\varepsilon})}\right)$ with $(t^*, x^*) \in [0, T) \times \mathcal{O}_y$, for all α sufficiently large $(t_\alpha, x_\alpha) \in [0, T) \times \mathcal{O}_y$ and $(\bar{t}_\alpha, \bar{x}_\alpha) \in \mathcal{O}_T$. Moreover, since $(t_\alpha, x_\alpha, \bar{t}_\alpha, x, \bar{x}_\alpha)$ is a maximum of $u^{\varepsilon,\delta} - \phi_\alpha - v$, by freezing the variables $(\bar{t}_\alpha, x, \bar{x}_\alpha)$, we conclude that (t_α, x_α) is a maximum of

$$(t, x) \to u^{\varepsilon,\delta}(t, x) - \frac{\alpha}{2}\overline{\phi}(t, \bar{t}_\alpha, x, \bar{x}_\alpha)$$

and $(\bar{t}_\alpha, \bar{x}_\alpha)$ is a local minimum of

$$(\bar{t}, \bar{x}) \to v(\bar{t}, \bar{x}) + \frac{\alpha}{2}\overline{\phi}(t_\alpha, \bar{t}, x_\alpha, \bar{x}).$$

Set $\frac{\alpha}{2}\overline{\phi} =: \phi_\alpha$. Following Crandall et al. (1992) by Ishii's lemma for all $\iota > 0$ there exist $M, N \in \mathcal{S}_3$ such that

$$\left(\partial_t \phi_\alpha(t_\alpha, \bar{t}_\alpha, x_\alpha, \bar{x}_\alpha), D_x \phi_\alpha(t_\alpha, \bar{t}_\alpha, x_\alpha, \bar{x}_\alpha), M\right) \in \bar{\mathcal{P}}^{2,+} u^{\varepsilon,\delta}(t_\alpha, x_\alpha)$$
$$\left(- \partial_{\bar{t}} \phi_\alpha(t_\alpha, \bar{t}_\alpha, x_\alpha, \bar{x}_\alpha), -D_{\bar{x}} \phi_\alpha(t_\alpha, \bar{t}_\alpha, x_\alpha, \bar{x}_\alpha), N\right) \in \bar{\mathcal{P}}^{2,-} v(\bar{t}_\alpha, \bar{x}_\alpha)$$

and

$$\begin{pmatrix} M & 0 \\ 0 & -N \end{pmatrix} \leq D^2_{x,\bar{x}}\phi_\alpha((t_\alpha, \bar{t}_\alpha, x_\alpha, \bar{x}_\alpha) + \iota D^2_{x,\bar{x}}\phi_\alpha((t_\alpha, \bar{t}_\alpha, x_\alpha, \bar{x}_\alpha))^2.$$

Choosing $\iota = \frac{1}{\alpha}$ we arrive at

$$-3\alpha \begin{pmatrix} I_3 & 0 \\ 0 & I_3 \end{pmatrix} \leq \begin{pmatrix} M & 0 \\ 0 & -N \end{pmatrix} \leq 3\alpha \begin{pmatrix} I_3 & -I_3 \\ -I_3 & I_3 \end{pmatrix}.$$

Note that

$$tr(\sigma(s_\alpha, z)\sigma(s_\alpha, z)' M - \sigma(\bar{s}_\alpha, z)\sigma(\bar{s}_\alpha, z)' N) \leq 3\alpha |\sigma(s_\alpha, z) - \sigma(\bar{s}_\alpha, z)|^2.$$

The viscosity subsolution property of $u^{\varepsilon,\delta}$ resp. viscosity supersolution property of v in terms of superjets and subjets lead to

$$\begin{aligned} -\alpha(t_\alpha - \bar{t}_\alpha) + \lambda u^{\varepsilon,\delta}(t_\alpha, x_\alpha) - H(x_\alpha, \alpha(x_\alpha - \bar{x}_\alpha), M) \leq 0 \\ -\alpha(t_\alpha - \bar{t}_\alpha) + \lambda v(\bar{t}_\alpha, \bar{x}_\alpha) - H(\bar{x}_\alpha, \alpha(x_\alpha - \bar{x}_\alpha), N) \geq 0. \end{aligned}$$

Set $p_\alpha = \alpha((s_\alpha - \bar{s}_\alpha), (\eta_\alpha - \bar{\eta}_\alpha), (y_\alpha - \bar{y}_\alpha + 2\bar{\varepsilon}))$. Subtracting the above two inequalities and by Lemma 2.4 we arrive at

$$\begin{aligned} &\lambda(u^{\varepsilon,\delta}(t_\alpha, x_\alpha) - v(\bar{t}_\alpha, \bar{x}_\alpha)) \\ &\quad \leq H(x_\alpha, p_\alpha, M) - H(\bar{x}_\alpha, p_\alpha, N) \\ &\quad \leq \sup_{|z| \leq C(\bar{y}_\alpha + 2\bar{\varepsilon})} \Big\{ (b(s_\alpha, \eta_\alpha, z) - b(\bar{s}_\alpha, \bar{\eta}_\alpha, z)) p_\alpha \\ &\qquad + \frac{1}{2} tr(\sigma(s_\alpha, z)\sigma(s_\alpha, z)' M - \sigma(\bar{s}_\alpha, z)\sigma(\bar{s}_\alpha, z) N) \Big\} \\ &\quad \leq c_r^* \alpha \left[|s_\alpha - \bar{s}_\alpha|^2 + |\eta_\alpha - \bar{\eta}_\alpha|^2 \right] \\ &\quad \leq c_r^* \alpha \overline{\phi}(t_\alpha, x_\alpha, \bar{t}_\alpha, \bar{x}_\alpha) \to 0, \end{aligned}$$

when $\alpha \to \infty$. This contradicts (34). Thus $u^{\varepsilon,\delta}(\tilde{t}, \tilde{x}) - v\left(\tilde{t}, \tilde{x}^{(2\bar{\varepsilon})}\right) \leq 0$. We conclude the proof by noting

$$u(\tilde{t}, \tilde{x}) - v\left(\tilde{t}, \tilde{x}^{(2\bar{\varepsilon})}\right) = \lim_{\varepsilon,\delta \to 0} u^{\varepsilon,\delta}(\tilde{t}, \tilde{x}) - v\left(\tilde{t}, \tilde{x}^{(2\bar{\varepsilon})}\right) \leq 0.$$

□

Remark 2.3 Let $\bar{\varepsilon} > 0$. The condition $u(T, x) \leq v(T, s, \eta, y + 2\bar{\varepsilon})$ is natural. Indeed, since the utility function U is non-decreasing in y, $u(T, x) \leq U(y)$ and $U(y) \leq v(T, x)$ imply

$$u(T, x) \leq U(y) \leq U(y + 2\bar{\varepsilon}) \leq v(T, s, \eta, y + 2\bar{\varepsilon}).$$

2.5 Uniqueness: almost!

If the upper and lower semicontinuous envelopes of the value function at final time T coincide with $U(y)$, then the comparison principle immediately implies that the value function is essentially the unique discontinuous viscosity solution of (10). By this we mean that any the upper semi-continuous envelope of any other solution agrees with the semi-continuous envelope of the value function.

For a function $h(t, s, \eta, y)$ that is non-decreasing in the y-variable, we define

$$h^+(t, s, \eta, y) := \lim_{\epsilon \downarrow 0} h(t, s, \eta, y + \epsilon).$$

Corollary 2.3 *Let u and v be viscosity solutions of* (10) *that are non-decreasing in the y-variable. We assume further that they satisfy,*

(i) $U(y) \leq u(t, x), v(t, x) \leq \phi(t, x)$ *for all* $(t, x) \in [0, T] \times \mathcal{O}$,
(ii) $u^*(T, x) = v^*(T, x) = U(y)$ *for all* $x \in \mathcal{O}_y$.

Then $u^*(t, x) = v^*(t, x)$ *and* $u_*^{(+)}(t, x) = v_*^{(+)}(t, x)$ *for all* $(t, x) \in [0, T) \times \mathcal{O}_y$.

Proof Observe that $u^{*(+)} = u^*$. From $U(y) \leq u(t, x), v(t, x)$ on $[0, T] \times \mathcal{O}$ it immediately follows that $v_*(T, x), u_*(T, x) \geq U(y)$. Assumption (ii) implies

$$u^*(T, x) = U(y) \leq U(y + \bar{\varepsilon}) \leq v_*(T, x^{(\bar{\varepsilon})}),$$

where $x^{(\bar{\varepsilon})} = (s, \eta, y+\bar{\varepsilon})$. Thus, Theorem 2.4 implies

$$v^*(t, x) \leq u_*^{(+)}(t, x) \leq u^{*(+)}(t, x) = u^*(t, x) \leq v_*^{(+)}(t, x) \leq v^*(t, x)$$

for all $(t, x) \in [0, T) \times \mathcal{O}_y$. Hence, $v^*(t, x) = u^*(t, x)$ and $u_*^{(+)}(t, x) = v_*^{(+)}(t, x)$ for all $(t, x) \in [0, T) \times \mathcal{O}_y$. □

Theorem 2.4 together with the Lemma below implies that the value function is a unique discontinuous viscosity solution of the Hamilton–Jacobi–Bellman equation, in the sense that all non-decreasing solutions have the same upper semi-continuous envelope.

Lemma 2.5 *Let v be the value function. Then,*

$$v_*(T, x) = v^*(T, x) = U(y)$$

for all $x \in \mathcal{O}$.

Proof For any $(t, x) \in [0, T] \times \mathcal{O}$, $v(t, x) \geq U(y)$. Thus, $v_*(T, x) \geq U(y)$. Let $(t, x) \in [0, T] \times \mathcal{O}$, $z \in \mathcal{A}^{(C)}(t, x)$,

$$J(t, x, z) = E\left[U\left(Y_T^{t,x,z}\right)\right] = E\left[U\left(Y_T^{t,x,z}\right)\chi_{\{\tau^{(N)}=T\}}\right] + E\left[U\left(Y_T^{t,x,z}\right)\chi_{\{\tau^{(N)}<T\}}\right],$$

where $N \in \mathbb{N}$ and $\tau^{(N)} = \tau^{(N)}_{t,x,z}$ is defined by

$$\tau^{(N)} = \inf\{u \in [t, T] : \phi^{(\frac{1}{2})}(u, X^{t,x,z}_u) = N, P\text{-a.s.}\} \wedge T.$$

1. On the set $\{u \leq \tau^{(N)}\}$

$$\phi^{(\frac{1}{2})}\left(u, X^{t,x,z}_u\right) \leq N, \quad \forall\, u \in \left[t, \tau^{(N)}\right].$$

   This implies that there is a constant $c_1 > 0$ so that

$$Y^{t,x,z}_u \leq c_1 N^2, \quad \forall\, u \in \left[t, \tau^{(N)}\right]$$

   and

$$|z_u| \leq C\left(Y^{t,x,z}_u\right) \leq C(c_1 N^2), \quad \forall\, u \in \left[t, \tau^{(N)}\right].$$

   Furthermore, there is a constant $c_2 > 0$ so that

$$\frac{\left(\eta^{t,s,\eta,z}_u\right)^2}{S^{t,s}_u} \leq c_2 N^2, \quad \forall\, u \in \left[t, \tau^{(N)}\right].$$

   Set

$$b_u := b\left(S^{t,s}_u, \eta^{t,s,\eta,z}_u, z_u\right), \quad \sigma_u := \sigma(S^{t,s}_u, z_u).$$

   The above estimates and the definition of b and σ imply that there is a constant $c_3(N)$, depending only on N and not on t, x, z so that

$$|b_u| \leq c_3(N)\left(1 + S^{t,s}_u\right), \quad |\sigma_u| \leq c_3(N)\left(1 + S^{t,s}_u\right).$$

   Therefore, there exists another constant $c_4(N)$ so that

$$\begin{aligned} E\left[|Y^{t,x,z}_T - y|^2\right] &\leq 2E\Big[\Big(\int_t^T b_u du\Big)^2\Big] + 2E\Big[\Big(\int_t^T \sigma_u dW_u\Big)^2\Big] \\ &\leq c_4(N)\left(1 + s^2\right)\left[(T-t)^2 + (T-t)\right]. \end{aligned} \tag{35}$$

2. By Corollary 2.1 and by the definition of $\tau^{(N)}$,

$$\begin{aligned} NP[\tau^{(N)} < T] &= E\left[\phi^{\left(\frac{1}{2}\right)}\left(\tau^{(N)}, X^{t,x,z}_{\tau^{(N)}}\right)\chi_{\{\tau^{(N)} < T\}}\right] \\ &\leq E\left[\phi^{\left(\frac{1}{2}\right)}\left(\tau^{(N)}, X^{t,x,z}_{\tau^{(N)}}\right)\right] \\ &\leq \phi^{(\frac{1}{2})}(t, x). \end{aligned}$$

Therefore,

$$P[\tau^{(N)} < T] \le \frac{\phi^{(\frac{1}{2})}(t,x)}{N},$$

which implies $\lim_{N\to\infty} P[\tau^{(N)} < T] = 0$.

3. By concavity of U and by (35) we arrive at

$$\begin{aligned}
&E\left[U(Y_T^{t,x,z})\chi_{\{\tau^{(N)}=T\}}\right]\\
&\quad\le E\left[(U(y)+U'(y)\left(Y_T^{t,x,z}-y\right))\chi_{\{\tau^{(N)}=T\}}\right]\\
&\quad= U(y)P[\tau^{(N)}=T]+U'(y)E\left[\left(Y_T^{t,x,z}-y\right)\chi_{\{\tau^{(N)}=T\}}\right]\\
&\quad\le U(y)P[\tau^{(N)}=T]+U'(y)E\left[|Y_T^{t,x,z}-y|^2\right]^{\frac{1}{2}}P[\tau^{(N)}=T]^{\frac{1}{2}}\\
&\quad\le U(y)-|U(y)|P[\tau^{(N)}<T]+|U'(y)|E\left[|Y_T^{t,x,z}-y|^2\right]^{\frac{1}{2}}P[\tau^{(N)}=T]^{\frac{1}{2}}\\
&\quad\le U(y)+|U'(y)|\sqrt{c_4(N)(1+s^2)\left((T-t)^2+(T-t)\right)}P[\tau^{(N)}=T]^{\frac{1}{2}}.
\end{aligned}$$

Hence, for every N,

$$\lim_{(t',x')\to(T,x)}\ \sup_{z\in\mathcal{A}^{(C)}(t',x')} E\left[U\left(Y_T^{t',x',z}\right)\chi_{\{\tau^{(N)}_{t',x',z}=T\}}\right] \le U(y). \tag{36}$$

4. Let $p'\in(0,1)$ and $p<p'$, where p is the exponent in the utility function U. Hölder's inequality with $q=\frac{p'}{p'-p}$ and Corollary 2.1 reveal

$$\begin{aligned}
E\left[U\left(Y_T^{t,x,z}\right)\chi_{\{\tau^{(N)}<T\}}\right] &\le E\left[\phi^{(p)}\left(T,X_T^{t,x,z}\right)\chi_{\{\tau^{(N)}<T\}}\right]\\
&\le c\left(p,p'\right)E\left[\phi^{(p')}\left(T,X_T^{t,x,z}\right)\right]^{\frac{p}{p'}}P\left[\tau^{(N)}<T\right]^{\frac{1}{q}}\\
&\le c\left(p,p'\right)\phi^{(p')}(t,x)^{\frac{p}{p'}}\left(\phi^{(\frac{1}{2})}(t,x)N^{-1}\right)^{\frac{1}{q}},
\end{aligned}$$

where $c(p,p')$ is a positive constant. Hence,

$$\limsup_{(t',x')\to(T,x)}\ \sup_{z\in\mathcal{A}^{(C)}(t',x')} E\left[U\left(Y_T^{t',x',z}\right)\chi_{\{\tau^{(N)}_{t',x',z}<T\}}\right] \le c(T,x,p,p')(N^{-1})^{\frac{1}{q}}, \tag{37}$$

where $c(T,x,p,p')$ is a positive constant.

5. We combine (36), (37) and let $N \to \infty$ to arrive at

$$v^*(T, x) = \limsup_{(t',x') \to (T,x)} \sup_{z \in \mathcal{A}^{(C)}(t',x')} J(t', x', z) \leq U(y).$$

$\square$

3 Numerical results

In this section we provide numerical results that were also given in the earlier paper of the authors (Soner and Vukelja 2013). In that paper the discrete-time approximation of problem (9) is used to compute the optimal strategy z and the value function w. However, the convergence of the scheme is not discussed there. Here, w is the value function of the discrete-time approximation. In the discrete-time approximation the control z is always bounded. Thus, the condition $|z_u| \leq C(Y_u^{t,x,z})$ for all $u \in [t, T]$ P-a.s. is not needed. For admissibility we only need $Y_u^{t,x,z} > 0$. Moreover, in discrete time we always work on a bounded domain and therefore the condition holds for sufficiently large functions C. More details are provided in the thesis of the second author (Vukelja 2014). In particular, the convergence of this algorithm is studied in Vukelja (2014) as well. Since the approximating scheme is the dynamic programming equation of the discretized optimal control problem, this numerical scheme is monotone. Hence, one can employ the classical (Barles and Souganidis 1991) result, proving the convergence of the value functions at the points of discontinuity. Since the value function is possibly discontinuous, one does not expect to design a numerical scheme that converges at all points. Moreover, the possible discontinuity points are expected to be small and most likely at the boundary.

Here we give a brief discussion of this result and for further details we refer to Vukelja (2014).

Let v^h be the solution of the discretized problem where $h > 0$ is the discretization parameter. Following the Barles and Souganidis procedure, we define the relaxed limits,

$$\overline{u}(x) := \limsup_{x' \to x, h \downarrow 0} v^h, \quad \underline{u}(x) := \liminf_{x' \to x, h \downarrow 0} v^h.$$

By standard techniques, one can show that $\overline{u}$ is a viscosity sub-solution and $\underline{u}$ is a viscosity super-solution of the Eq. (10). Let v be the value function for the continuous time problem. We now apply the comparison result Theorem 2.4 as in Corollary 2.3 (and Lemma 2.5) to conclude :

- Since $\overline{u}$ is an upper semi-continuous sub-solution, v_* is a super-solution of (10), $(v_*)^* = v^*$,

$$\overline{u} \leq (v_*)^* = v^*;$$

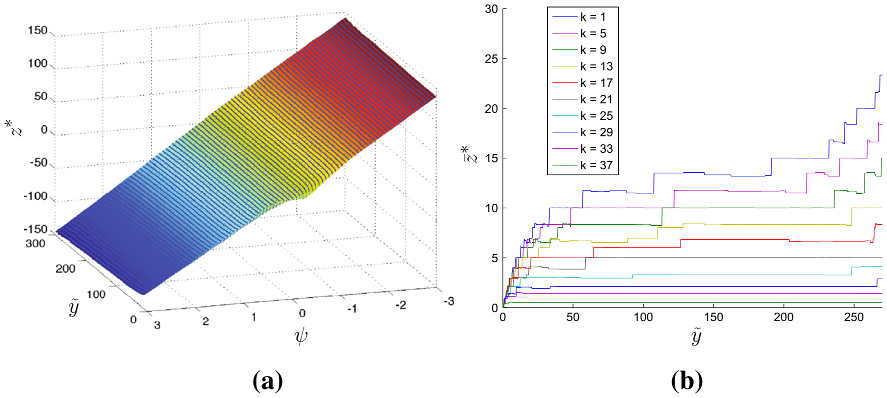

Fig. 1 Optimal trading strategy z^*. **a** At time step 1, **b** at different time steps k

- Since v^* is a sub-solution, $\underline{u}$ is a lower semi-continuous super-solution of (10),

$$v^* \leq (\underline{u})^* \leq \overline{u};$$

- The above inequalities imply that

$$\overline{u} = (v_*)^* = v^* = (\underline{u})^*.$$

Furthermore, Soner and Vukelja (2013) show that in this setting the dynamic programming principle (DPP) holds true. Thus, we can develop an efficient algorithm to compute the optimal strategy z^* and the value function w backwards in time. The optimal stochastic control problem (9) has three state variables; however, the homotheticity property of the CRRA utility function reduces the problem to two state variables $\psi_. = \frac{\eta_.}{S_.}$ and $\tilde{Y}_. = \frac{Y_.}{S_.}$. For more details on the dynamics of ψ and $\tilde{Y}$ and the (DPP) we refer to Soner and Vukelja (2013). Clearly, the optimal trading strategy depends on the initial data ψ, $\tilde{y}$ and the time step k, i.e., $z^*(k, \psi, \tilde{y})$.

We set $p = 0.25, \sigma = 0.3, \mu = 0.04, \kappa = 5, N = 40, T = 2$ and $M = 0.01$. The optimal trading strategy z^* is plotted in Fig. 1. Figure 1a shows that the optimal strategy grows almost linearly in ψ and it seems that it changes less in $\tilde{y}$. This comes from the fact, that the state variable $\psi = \frac{\eta}{s} = \frac{l}{s} - 2Mz$ depends on the strategy z. If we fix l/s, for instance set $l/s = 0$ and plug in the computed optimal strategy $z^*(k, \psi, \tilde{y})$, then $\psi = -2Mz^*(k, \psi, \tilde{y})$. Furthermore, for fixed $(k, \tilde{y})$ we can find the fixed point ψ such that $\psi = -2Mz^*(k, \psi, \tilde{y})$. Let ψ^* be that fixed point. Then, define

$$\tilde{z}^* = z^*(k, \psi^*, \tilde{y}).$$

We plot $\tilde{z}^*$ in Fig. 1b. It shows that $\tilde{z}^*$ is increasing in $\tilde{y}$.

In Fig. 2 we plot the graph of the corresponding value function at time step 39, Fig. 2a resp. at time step 1, Fig. 2b. We see that local arbitrage is possible, i.e., the value function w is for some values of ψ larger than the Merton value function v^m

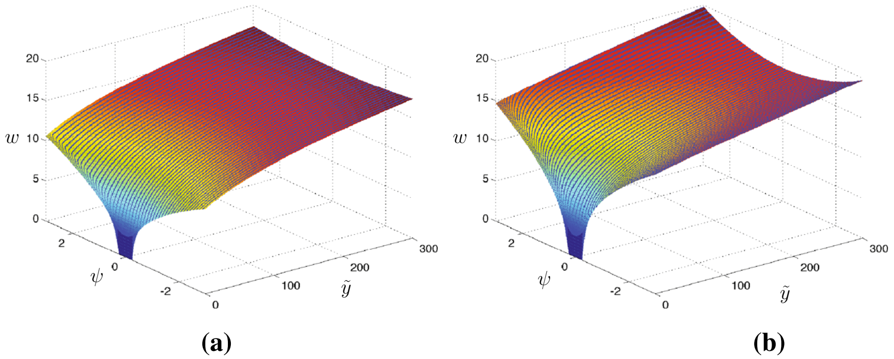

Fig. 2 Value function w at different time steps. **a** At time step 39, **b** at time step 1

Table 1 Difference of w and v^m

$M = 0.01$	$M = 0.05$	$M = 0.09$
−0.0057	−0.0157	−0.0242
−0.0056	−0.0169	−0.0243
−0.0059	−0.0176	−0.0247
−0.0067	−0.0184	−0.0254
−0.0075	−0.0193	−0.0259
−0.0083	−0.0200	−0.0264
−0.0091	−0.0208	−0.0269
−0.0099	−0.0216	−0.0273
−0.0107	−0.0224	−0.0278
−0.0114	−0.0231	−0.0283
−0.0120	−0.0238	−0.0288

with no friction. When initial liquidity premium exists in the observed stock price, then it is intuitively expected that the investor may and does use this to achieve a value larger than the Merton one. Figure 2 also shows that the value function is not concave. In Table 1 we compute the difference of the value function in the liquidity risk setup w and Merton's value function v^m for different values of M when $\psi = 0$. It shows that $w \le v^m$. For a proof of this statement we refer to Soner and Vukelja (2013). Furthermore, as expected, when the depth parameter M increases the difference $w - v^m$ decreases.

Appendix

In this appendix we provide details on the proof of Lemma 2.2. We only prove the lemma when $p < 1, p \neq 0$. The case $p = 0$ is proved analogously. For this, set $\xi := \beta y + \frac{\eta^2}{s}$ and note that $\frac{\eta^2}{s} \le \xi$. Let $c_1, \varepsilon, \varepsilon_1, \varepsilon_2 \in \mathbb{R}$ and $c_2 > 0$ be given constants. Direct computations reveal

$$-(c_1\eta z + c_2 z^2 s) = -\Big(\frac{c_1\eta}{2\sqrt{c_2 s}} + \sqrt{c_2 s}z\Big)^2 + \frac{c_1^2}{4c_2}\frac{\eta^2}{s} \le \frac{c_1^2}{4c_2}\xi \tag{38}$$

$$(z\eta)^2 = z^2\frac{\eta^2}{s}s \le \xi z^2 s$$

$$-\eta z\Big(\frac{\eta^2}{s}\Big) \le \frac{1}{4}\Big(\frac{\eta^2}{s}\Big)^2 + z^2\eta^2 \le \frac{1}{4}\xi^2 + z^2 s\xi \tag{39}$$

$$\begin{aligned}\eta^2 z(1-Mz) &= (\varepsilon_1 sz(1-Mz))^2 - \Big(\varepsilon_1 zs(1-Mz) - \frac{\eta^2}{2\varepsilon_1 s}\Big)^2 + \Big(\frac{\eta^2}{2\varepsilon_1 s}\Big)^2 \\ &\le (\varepsilon_1 sz(1-Mz))^2 + \frac{1}{4\varepsilon_1^2}\xi^2\end{aligned} \tag{40}$$

$$s\eta z^2(1-Mz) \le (\varepsilon_2 zs(1-Mz))^2 + \frac{\xi z^2 s}{4\varepsilon_2^2} \tag{41}$$

$$\xi^{p-1}\beta\mu zs(1-Mz) \le \xi^{p-2}(\varepsilon zs(1-Mz))^2 + \Big(\frac{\beta\mu}{2\varepsilon}\Big)^2\xi^p. \tag{42}$$

Set $\tilde{\phi}(s,\eta,y) = \frac{1}{p}\Big(\beta y + \frac{\eta^2}{s}\Big)^p$. Equations (38)–(42) reveal

$$\begin{aligned}\mathcal{L}^z\tilde{\phi} &= \xi^{p-1}\Big((-2\kappa-\mu)\frac{\eta^2}{s} - (4M(\kappa+\mu)+\beta\kappa)\eta z - 2M\kappa\beta z^2 s\Big) \\ &\quad + \xi^{p-1}\mu\beta zs(1-Mz) + \sigma^2\xi^{p-1}\Big(\frac{\eta^2}{s} + 4M^2z^2s\Big) - \frac{\sigma^2(1-p)}{2}\xi^{p-2}\Big(\Big(\frac{\eta^2}{s}\Big)^2 \\ &\quad + (4M\eta z)^2 + \beta^2(zs(1-Mz))^2\Big) \\ &\quad + 4M\sigma^2\xi^{p-1}\eta z + \sigma^2(1-p)\xi^{p-2} \\ &\quad \times\Big(-4Mz\eta\frac{\eta^2}{s} + \beta z\eta^2(1-Mz) + 4M\beta\eta sz^2(1-Mz)\Big) \\ &\le \Big(-2\kappa-\mu+\sigma^2\Big)\frac{\eta^2}{s}\xi^{p-1} - ((4M+\beta)\kappa + 4M(\mu-\sigma^2))\eta z\xi^{p-1} \\ &\quad + \xi^{p-1}\mu\beta zs(1-Mz) - (2M\kappa\beta - 4(M\sigma)^2)z^2s\xi^{p-1} \\ &\quad - \frac{\beta\sigma^2(1-p)}{2}(\beta zs(1-Mz))^2\xi^{p-2} - 4M\sigma^2(1-p)\xi^{p-2}z\eta\frac{\eta^2}{s} \\ &\quad + 4M\beta\sigma^2(1-p)\xi^{p-2}\eta sz^2(1-Mz) + \beta\sigma^2(1-p)\xi^{p-2}\eta^2 z(1-Mz) \\ &\le \Big(-2\kappa-\mu+\sigma^2\Big)^+\xi^p - ((4M+\beta)\kappa + 4M(\mu-\sigma^2))\eta z\xi^{p-1} \\ &\quad + \xi^{p-1}\mu\beta zs(1-Mz) - (2M\kappa\beta - 4(M\sigma)^2)z^2s\xi^{p-1} \\ &\quad - \frac{\beta\sigma^2(1-p)}{2}(\beta zs(1-Mz))^2\xi^{p-2} \\ &\quad + 4M\sigma^2(1-p)\xi^{p-2}\Big(\frac{1}{4}\xi^2 + z^2s\xi\Big)\end{aligned}$$

$$
\begin{aligned}
&+ \beta\sigma^2(1-p)\xi^{p-2}\Big((\varepsilon_1 sz(1-Mz))^2 + \frac{1}{4\varepsilon_1^2}\xi^2\Big) \\
&+ 4M\beta\sigma^2(1-p)\xi^{p-2}\Big((\varepsilon_2 zs(1-Mz))^2 + \frac{\xi z^2 s}{4\varepsilon_2^2}\Big) \\
&\leq \bar{c}\xi^p - c_1\eta z\xi^{p-1} - c_2 z^2 s\xi^{p-1} + \xi^{p-1}\mu\beta zs(1-Mz) \\
&\quad - (zs(1-Mz))^2\xi^{p-2}\varepsilon^2,
\end{aligned}
$$

Setting

$$
\begin{aligned}
\bar{c} &= (-2\kappa - \mu + \sigma^2)^+ + M\sigma^2(1-p) + \frac{\beta\sigma^2(1-p)}{4\varepsilon_1^2} \\
c_1 &= (4M+\beta)\kappa + 4M(\mu - \sigma^2) \\
c_2 &= 2M\kappa\beta - 4(M\sigma)^2 - 4M\sigma^2(1-p) - \frac{\beta M\sigma^2(1-p)}{\varepsilon_2^2} \\
\varepsilon^2 &= \sigma^2(1-p)\beta\Big(\frac{\beta}{2} - \varepsilon_1^2 - 4M\varepsilon_2^2\Big)^+.
\end{aligned}
$$

Set $\varepsilon_1 = 1$ and $\varepsilon_2^2 = \frac{\beta}{16M}$. It follows that $\varepsilon^2 > 0$ if $\beta > 4$ and $c_2 > 0$ if $\beta > \frac{\sigma^2(2M+2(1-p)+8M(1-p))}{\kappa}$. Using (42) we arrive at

$$
\mathcal{L}^z\tilde{\phi} \leq \bar{c}\xi^p + \frac{c_1^2}{4c_2}\xi^p + \Big(\frac{\beta\mu}{2\varepsilon}\Big)^2\xi^p =: \alpha\xi^p.
$$

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