Hedging in an illiquid binomial market

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ABSTRACT

We analyze numerically the superreplication problem and the associated hedging strategy in an illiquid binomial market. We prove the existence of an optimal feedback strategy for European and barrier options and compute it numerically by means of a dynamic programming principle. We exhibit that the optimal strategy is not equal to the discrete-delta strategy or to the strategy that minimizes the value function. The optimal strategy shows less variability than the discrete-delta strategy or the strategy consisting of minimizers of the value function due to the effect of liquidity. The performance of these three strategies are assessed by comparing the corresponding wealth processes with the payoff. It is shown that the discrete-delta strategy and the strategy that minimizes the value function may perform poorly, thus showing the effectiveness of the optimal feedback strategy.

1. Introduction

We study the superreplication and the hedging problem of European and barrier options in a binomial market with friction due to illiquidity. Liquidity is modeled by incorporating the impact of the transaction size and time in the price process of the underlying. Given the market observed price S_t at time t, the supply curve $\mathbf{S}(t, S_t, \nu)$ specifies the amount paid per share for an order of size ν.

These supply curve models were first introduced by [1]. We work with the binomial version of this setup following the approach in [2,3]. Here the observed price process S has a binomial lattice structure. The main focus in [3] is to compute the superreplicating cost of European claims in discrete-time and to characterize the resulting continuous-time value function. In this article, we concentrate on studying the hedging strategy for European as well as barrier options. In particular, we calculate numerically the optimal hedge in feedback form and compare to other standard strategies widely used in the literature. It turns out that due to the effect of liquidity the optimal strategy is not always equal to the discrete-delta strategy or the strategy that minimizes the value function. We illustrate this by exhibiting paths on the binomial tree, where the discrete-delta strategy or the minimizer of the value function underperform, i.e. fail to dominate the payoff, whereas the optimal hedging strategy always superreplicates the claim. Moreover, on some paths the difference between the final value of the wealth process associated with the discrete-delta strategy or the minimizer of the value function and the payoff can take large negative values. The underperformance of these two strategies is related to the large variability they exhibit on certain paths. We numerically observe that the optimal hedge computed in the feedback form exhibits less variability. So, liquidity has an effect on smoothing out the hedge. Therefore, in an illiquid market, optimal feedback policy, which depends on the portfolio position in addition to the stock value, performs much better than the strategies that ignore it. However, to achieve a "nice" hedge that performs well, we are required to pay a liquidity premium. We report on the amount of liquidity

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premium paid for a supply curve of the form $\mathbf{S}(t, s, \nu) = s + \Lambda\nu$. We observe that for sufficiently small liquidity parameters Λ, this liquidity premium is small. These numerical observations can be found in Section 6.

Computationally, one approach is to express the superreplicating cost as an optimization problem. However, with this approach one has a non-recombinant tree due to the path-dependent structure of the portfolio process. Therefore, the number of constraints grow exponentially with the time steps. Alternatively, as in [3] we use a dynamic programming principle that enables us to calculate many time steps. Here the crucial observation is to add another state variable, the initial position in the risky asset, into the value function. Without this extra state variable, it is not possible to write a dynamic programming principle for the superreplicating cost. Also this additional variable provides an optimal strategy in feedback form.

For European claims [3] study the continuous-time limit of the binomial setup for a supply curve of the form $\mathbf{S}(t, s, \nu)$. It turns out that the superreplicating cost converges locally uniformly to a partial differential equation characterized by [4]. The solution of this partial differential equation is strictly larger than the classical value of the claim for non-affine payoffs. This shows the existence of liquidity premium, the difference between the superreplicating cost and the Black–Scholes value of the claim without friction. Although we pay considerable attention to barrier options to demonstrate our numerical experiments, in this paper we do not study the continuous-time limit of the binomial model for barrier options, and characterize the resulting liquidity premium. However, a convergence argument for path dependent payoffs and path dependent supply curves based on probabilistic arguments was carried out recently by [5].

Another related problem to liquidity is the optimal execution problem. It concerns the allocation of trades to execute a buy or sell order. It is formulated as an optimization problem first by [6]. For a detailed exposition of the optimal execution problem we refer to [7] and the references therein.

A barrier option is continuously monitored, if the barrier can be breached at any time during the life of the option. On the other hand, if the knock-out or knock-in of the barrier option can occur only at discrete times it is discretely-monitored. In a Black–Scholes framework there are explicit closed-form expressions available for pricing continuously monitored barrier options, however no easy closed-form formula is known for discretely-monitored barrier options. Broadie, Glasserman and Kou [8,9] use the continuous-time exact formula to approximate the discrete barrier option value by shifting the barrier to correct for discrete monitoring. Boyle and Lau [10] use the binomial approach to price barrier options. They consider convergence of the binomial scheme to continuous-time setup and report on the irregularities that arise in the convergence. They suggest to position the grid so that the barrier always lies on a horizontal layer of nodes to improve the convergence. Rogers and Stapleton [11] introduce a random walk approximation to the logarithm of the stock price process which is independent of the positioning of the grid points with respect to the barrier. In this article we do not aim to contribute to the literature regarding the relation between the discrete and continuous-time prices of barrier options. Rather, our goal is to understand the discrete-time hedge in the presence of illiquidity. Furthermore, we want to demonstrate the smoothing out effect of illiquidity on the hedges. Another approach to price barrier options is static hedging followed in [12,13]. Since their approaches are based on the assumptions of frictionless markets, they do not apply to our setting.

The organization of the paper is as follows. After discussing in Section 2 the model, we establish the dynamic programming principle for barrier options in Section 3. In Section 4, we state the parameters we use in the rest of the paper. Then in Section 5, we outline the algorithm for European as well as barrier options. In the last Section 6, we present our numerical results.

2. Model

We suppose that our financial market consists of two assets, a risky asset and a risk-free asset. The risk-free asset is taken to be a numeraire with interest rate $r = 0$. The price of the risky asset follows the supply curve model introduced by [1] to model liquidity. In this framework, the size and time of the traded quantity of the risky asset creates a price impact, a deviation from the market observed price S_t. The agent has to pay an amount

$$\mathbf{S}(t, S_t, \nu) \tag{2.1}$$

per share, if she wants to trade ν number of shares at time t for a market observed price S_t. $\nu > 0$ represents a buy and $\nu < 0$ a sell order. When no quantity is traded, one captures the market observed price, i.e. $\mathbf{S}(t, S_t, 0) = S_t$. Intuitively, one would expect to pay more per share if one trades larger quantities. Therefore, $\mathbf{S}(t, S_t, \nu)$ is assumed to be monotone in ν.

The supply curve $\mathbf{S}(t, S_t, \nu)$ is exogenously given so that traders have no influence on the shape of $\mathbf{S}(t, S_t, \nu)$. Furthermore, only the current trade ν has an impact on the price of the underlying. The trading history of the agent has no impact on the supply curve, i.e. all investors are price takers. So the continuous-time supply curve is a temporary price impact model. However, in discrete time the price impact lasts exactly one step. This is a fundamental difference between the discrete-time and continuous-time supply curve models. On the other hand, in large trader models, [14–22], the agent owns substantial amount of shares so that all her past trading decisions have a permanent influence on the price process. [23,24] also consider a large trader model, but follow a different approach to model the price impact. In particular, instead of specifying the price impact exogenously, the dependence of the price on the strategy is determined as a result of an equilibrium.

The financial market is assumed to have finite maturity T and the agents are allowed to trade at times $t = nh$ for time spacing $h > 0$ and for $n \in \{0, 1, \ldots, N\}$, where $N := \frac{T}{h}$ is an integer. In the rest of the paper, we work under the probability

space $(\Omega, \mathcal{F}, \mathbf{P})$. Here the sample space Ω is given by

$$\Omega = \{(\omega_h, \ldots, \omega_T) : \omega_i \in \{-1, +1\}, \forall i = \{h, 2h, \ldots, T\}\}.$$

On the sample space Ω, we take the σ-algebra to be $\mathcal{F} = 2^\Omega$ and $\mathbf{P}$ to be any probability measure satisfying $\mathbf{P}(\omega) > 0$ for all $\omega \in \Omega$. Denote by π_t the projection on the tth coordinate

$$\pi_t(\omega) := \omega_t \in \{-1, +1\} \quad \forall t \in \{h, 2h, \ldots, T\}.$$

The dynamics of the marginal price process $S = \{S_t\}_{t \in \{0, h, \ldots, T\}}$ is governed by the multiplicative binomial model. In this setup, if the stock price at time t is S_t, then the stock price S_{t+h} at time $t + h$ either goes up by a factor u to uS_t or goes down by a factor d to dS_t with $d < 1 < u$. The dynamics of the stock price is expressed as

$$S_{t+h}(\omega) = S_t(\omega) \left\{ u \left(\frac{1 + \pi_{t+h}(\omega)}{2} \right) + d \left(\frac{1 - \pi_{t+h}(\omega)}{2} \right) \right\},$$

$\forall t \in \{0, \ldots, T - h\}$, $\omega \in \Omega$, $S_0(\omega) = s$. In particular, $\pi_t(\omega) := \omega_t = 1$ implies that the stock price went up and $\pi_t(\omega) := \omega_t = -1$ means the stock price went down. The stock price S can be represented on a recombinant binomial tree, where a node on the binomial tree corresponds to $(t, S_t(\omega))$ for some $t \in \{0, \ldots, T\}$ and a realization $\omega \in \Omega$. One can regard $\omega \in \Omega$ as a path on this binomial tree.

We take the filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t \in \{0, \ldots, T\}}$ to be the σ-algebra generated by S, i.e. $\mathcal{F}_t = \sigma(S_0, S_h, \ldots, S_t)$ for $t = \{0, \ldots, T\}$. Then $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_t = \sigma(\pi_h, \ldots, \pi_t)$ for $t \in \{h, \ldots, T\}$. The portfolio process is denoted by $Z = \{Z_t\}_{t \in \{0, h, \ldots, T\}}$ and the money invested in the risk-free asset by $X = \{X_t\}_{t \in \{0, h, \ldots, T\}}$. The portfolio process Z is adapted with respect to the filtration $\mathbb{F}$. All these processes are assumed to be constant on the intervals $[nh, (n + 1)h)$ for all $n \in \{0, \ldots, N - 1\}$. This choice of RCLL strategies differs from the convention in literature to use LCRL strategies. However, this change makes the dynamic programming principle easier to state.

Although there are many possibilities to measure the value of a portfolio in this supply curve model, in this article we use the marked to market value $Y = \{Y_t\}_{t \in \{0, h, \ldots, T\}}$

$$Y_t = X_t + Z_t S_t \tag{2.2}$$

as our wealth process following the approach in [1,4,3]. The self-financing condition in this setup is given by

$$X_{t+h} = X_t - \Delta Z_{t+h} \mathbf{S}(t + h, S_{t+h}, \Delta Z_{t+h}), \tag{2.3}$$

where $\Delta Z_{t+h} = Z_{t+h} - Z_t$. The change in money market account at time $t + h$ is due to trading ΔZ_{t+h} shares at a price $\mathbf{S}(t + h, S_{t+h}, \Delta Z_{t+h})$ per share. We note that ΔZ_{t+h} is a random variable to be decided at time $t + h$ and not known at time t.

The above Eq. (2.3) implies the following dynamics for wealth process,

$$Y_{t+h} = Y_t + Z_t (S_{t+h} - S_t) - \Delta Z_{t+h} \theta_{t+h} (\Delta Z_{t+h}), \tag{2.4}$$

where the loss function θ is given by

$$\theta_t(v) := [\mathbf{S}(t, S_t, v) - S_t]. \tag{2.5}$$

Note that for a perfect liquid market, i.e. $\mathbf{S}(t, S_t, v) = S_t$, we recover the usual wealth dynamics. By monotonicity of $\mathbf{S}$ in v, we see that liquidity is a positive penalty to the wealth process. An adapted portfolio process Z with an initial wealth y at time t generates a wealth process $Y^{t,s,y,Z} = \left\{ Y_u^{t,s,y,Z} : u \in \{t, \ldots, T\} \right\}$ adapted to the filtration $\mathbb{F}$.

Barrier options are characterized by a map $g : [0, T] \times \mathbb{R}_+ \to \mathbb{R}$ and an open set $\mathcal{O} \subset \mathbb{R}$ with the property

$$g(t, s) \geq 0, \quad \forall (t, s) \in [0, T] \times \overline{O}.$$

Example 2.1 (*Up-and-Out Call*). An up-and-out call with strike K and barrier B is characterized by $\mathcal{O} = (-\infty, B)$ and

$$g(t, s) = (s - K)^+ \mathbf{1}_{\{t=T, s<B\}}.$$

Let $\tau_{t,s}$ be the stopping time

$$\tau_{t,s} = \inf \left\{ u \in [t, T] : S_u^{t,s} \notin \mathcal{O} \right\} \wedge T$$

with the convention that $\inf \emptyset = \infty$. Define the superreplicating cost $\phi^h(t, s)$ of the barrier option with payoff g at time t and $S_t = s$ by

$$\phi^h(t, s) = \inf \left\{ y : Y_{\tau_{t,s}}^{t,s,y,Z} \geq g\left(\tau_{t,s}, S_{\tau_{t,s}}^{t,s}\right) \text{ a.s. for some } \mathbb{F}\text{-adapted } Z \right\}. \tag{2.6}$$

The path dependent structure of the portfolio process Z makes it impossible to express a dynamic programming principle for $\phi^h(t, s)$ on a recombinant binomial tree. For a nonrecombinant tree one can view (2.6) as an optimization problem, where

the objective is to minimize y subject to the set of constraints $Y_{\tau_{t,s}}^{t,s,y,Z} \geq g\left(\tau_{t,s}, S_{\tau_{t,s}}^{t,s}\right)$. The values of the adapted portfolio process Z on nodes of the tree are the decision variables. So the optimization problem is

$$\phi^h(t,s) = \inf_{Z \text{ adapted}} y$$

$$Y_{\tau_{t,s(\omega)}}^{t,s,y,Z}(\omega) \geq g\left(\tau_{t,s}(\omega), S_{\tau_{t,s(\omega)}}^{t,s}(\omega)\right) \quad \forall \omega \in \Omega.$$

This approach has the drawback that the constraints grow exponentially with the number of steps to maturity. Thus it is quite hard to compute $\phi^h(t,s)$ for markets with many steps. Instead we introduce the dependence of the initial portfolio position $Z_t = z$ into the minimal super-replication cost $v^h(t,s,z)$

$$v^h(t,s,z) = \inf\left\{y : Y_{\tau_{t,s}}^{t,y,Z} \geq g\left(\tau_{t,s}, S_{\tau_{t,s}}^{t,s}\right) \text{ a.s. for some } \mathbb{F}\text{-adapted } Z \text{ satisfying } Z_t = z\right\}. \tag{2.7}$$

We formulate a dynamic programming principle (DPP) for $v^h(t,s,z)$ on a recombinant binomial tree and develop an efficient algorithm utilizing the DPP. This numerical scheme allows us to consider markets with much larger time steps despite the introduction of an extra state variable $Z_t = z$. To obtain $\phi^h(t,s)$ from $v^h(t,s,z)$ we note that

$$\phi^h(t,s) = \inf_z v^h(t,s,z). \tag{2.8}$$

Remark 2.1. Observe that if $s \notin \mathcal{O}$, then $\tau_{t,s} = t$. So in this case $v^h(t,s,z)$ as well as $\phi^h(t,s)$ are equal to $g(t,s)$.

3. Properties of the value function and dynamic programming

Proposition 3.1. *There exists an optimal portfolio Z for the superreplication problem* (2.6). *The same statement holds for* (2.7).

Proof. We only prove the statement for (2.6), since (2.7) follows similarly. Set $\alpha := \phi(t,s)$.

Choose $y = \max\left\{g\left(\tau_{t,s}, S_{\tau_{t,s}}^{t,s}\right)(\omega) : \omega \in \Omega\right\}$ and $Z \equiv 0$ to see that $\alpha < \infty$. To shorten the notation define

$$\mathcal{A}(t,s,y) := \left\{\mathbb{F}\text{-adapted } Z : Y_{\tau_{t,s}}^{t,s,y,Z} \geq g\left(\tau_{t,s}, S_{\tau_{t,s}}^{t,s}\right) \text{ a.s.}\right\}.$$

We note that $\mathcal{A}(t,s,y)$ satisfies the monotonicity property in y, i.e. $\mathcal{A}(t,s,y_1) \subseteq \mathcal{A}(t,s,y_2)$ for $y_1 < y_2$. Clearly, $\mathcal{A}(t,s,y)$ is non-empty for every $y > \alpha$, and our aim is to show that $\mathcal{A}(t,s,\alpha)$ is non-empty. Take any decreasing sequence $\{\lambda_n\}_{n \in \mathbb{N}}$ with $\lambda_n \downarrow \alpha$. It is clear that

$$\mathcal{A}(t,s,\alpha) = \bigcap_{\lambda_n > \alpha} \mathcal{A}(t,s,\lambda_n).$$

Indeed, it follows from the monotonicity of $\mathcal{A}(t,s,y)$ that $\mathcal{A}(t,s,\alpha) \subset \mathcal{A}(t,s,\lambda_n)$ for $\lambda_n > \alpha$, which proves one side of the equality. On the other hand, if $Z \in \mathcal{A}(t,s,\lambda_n)$ for all $\lambda_n > \alpha$, then by continuity of $Y^{t,s,y,Z}$ with respect to y, it is straightforward to see that $Z \in \mathcal{A}(t,s,\alpha)$, since $\lambda_n \downarrow \alpha$. $\mathcal{A}(t,s,\lambda_n)$ is a non-empty closed set. We now claim that $\mathcal{A}(t,s,\lambda_n)$ is bounded, hence compact. For a clear and simple exposition of the boundedness argument, we assume that $\tau_{t,s} \equiv T$. Take the equivalent martingale measure $\mathbf{Q}$ of the observed stock price process S. First we show that Z_t is bounded. Consider the inequality obtained by conditioning on $\mathcal{F}_{t+h}$

$$0 \leq E_{\mathbf{Q}}\left[g\left(T, S_T\right) \big| \mathcal{F}_{t+h}\right] \leq E_{\mathbf{Q}}\left[Y_T^{t,s,y,Z} \big| \mathcal{F}_{t+h}\right] \leq y + Z_t(S_{t+h} - S_t),$$

since $\nu\theta(\nu) \geq 0$ for all $\nu \in \mathbb{R}$. We see that Z_t is bounded, for otherwise it violates the non-negativity of $y + Z_t(S_{t+h} - S_t)$, because the up and down factors satisfy $d < 1 < u$. By induction, assume that $\{Z_r\}_{r \in \{t,\ldots,u-h\}}$ is bounded. Then by conditioning on $\mathcal{F}_{u+h}$, we obtain similarly

$$0 \leq E_{\mathbf{Q}}\left[Y_T^{t,s,y,Z} \big| \mathcal{F}_{u+h}\right] \leq y + \sum_{k=t}^{u-h} Z_k(S_{k+h} - S_k) + Z_u(S_{u+h} - S_u),$$

which shows the boundedness of Z_u. Via this procedure, we can show that $\{Z_r\}_{r \in \{t,\ldots,T-h\}}$ is bounded. To prove the boundedness of Z_T, we observe conditioning on $\mathcal{F}_T$ that

$$0 \leq y + \sum_{k=t}^{T-h} Z_k(S_{k+h} - S_k) - \Delta Z_T \theta(\Delta Z_T)$$

by adaptedness of Z. Since the supply curve is $\mathbf{S}(t,s,\nu)$ monotone in ν, we see that Z_T has to be bounded. To adapt the proof for the case $\tau_{t,s} \not\equiv T$, we condition on $\mathcal{F}_{\tau_{t,s} \wedge u}$ instead of $\mathcal{F}_u$.

By monotonicity of $\mathcal{A}(t, s, y)$ in y, any finite collection of $\{\mathcal{A}(t, s, \lambda_n)\}_{n=1}^{\infty}$ has non-empty intersection. Since $\mathcal{A}(t, s, \lambda_n)$ is compact for every $n \in \mathbb{N}$, their intersection $\mathcal{A}(t, s, \alpha)$ is non-empty and compact. $\quad\square$

Proposition 3.2. *Assume that the function $\nu \in \mathbb{R} \mapsto \nu\theta(\nu) \in \mathbb{R}_+$ is convex. Then for fixed $(t, s) \in [0, T) \times \mathbb{R}_+$, the value function $v^h(t, s, z)$ is convex in the portfolio variable z.*

Proof. Fix $(t, s) \in [0, T) \times \mathbb{R}_+$. Let $\lambda \in [0, 1]$, and $z_1, z_2 \in \mathbb{R}$. We want to show that

$$v^h(t, s, \lambda z_1 + (1 - \lambda)z_2) \leq \lambda v^h(t, s, z_1) + (1 - \lambda)v^h(t, s, z_2).$$

For $i = 1, 2$, set $y^i := v(t, s, z_i)$. Then by Proposition 3.1, there exists $Z^{i,t,s,z} \in \mathbb{F}$ such that $Z_t^i = z_i$ and

$$Y_{\tau_{t,s}}^{t,s,y^i,Z^i} \geq g\left(\tau_{t,s}, S_{\tau_{t,s}}\right) \quad \text{a.s. for } i = 1, 2.$$

Then define the control $\hat{Z} := \lambda Z^1 + (1 - \lambda)Z^2$. Clearly $\hat{Z} \in \mathbb{F}$ and $\hat{Z}_t = \lambda z_1 + (1 - \lambda)z_2$. Moreover, convexity of $\nu \mapsto \nu\theta(\nu)$ implies that

$$Y_{\tau_{t,s}}^{t,s,\lambda y^1 + (1-\lambda)y^2, \hat{Z}} \geq \lambda Y_{\tau_{t,s}}^{t,s,y^1,Z^1} + (1 - \lambda)Y_{\tau_{t,s}}^{t,s,y^2,Z^2} \geq g\left(\tau_{t,s}, S_{\tau_{t,s}}\right) \quad \text{a.s.}$$

We conclude that $\lambda y^1 + (1 - \lambda)y^2$ is superreplicating. Hence, we prove the claim. $\quad\square$

Theorem 3.1. *For any stopping time η taking values in $\{t, t+h, \ldots, T\}$ and $s \in \mathcal{O}$, the minimal super-replicating cost $v^h(t, s, z)$ satisfies*

$$v^h(t, s, z) = \inf\left\{y : \ Y_{\eta \wedge \tau_{t,s}}^{t,s,y,Z} \geq g\left(\tau_{t,s}, S_{\tau_{t,s}}^{t,s}\right) \mathbf{1}_{\{\tau_{t,s} \leq \eta\}} + v^h\left(\eta, S_\eta^{t,s}, Z_\eta^{t,s,z}\right) \mathbf{1}_{\{\tau_{t,s} > \eta\}}\right.$$

$$\left. \text{a.s. for some } \mathbb{F}\text{-adapted } Z \text{ satisfying } Z_t = z\right\}.$$

Proof. Let $(t, s, z) \in [0, T) \times \mathbb{R}_+ \times \mathbb{R}$. Set

$$W^h(t, s, z) := \inf\left\{y : \ Y_{\eta \wedge \tau_{t,s}}^{t,s,y,Z} \geq g\left(\tau_{t,s}, S_{\tau_{t,s}}^{t,s}\right) \mathbf{1}_{\{\tau_{t,s} \leq \eta\}} + v^h\left(\eta, S_\eta^{t,s}, Z_\eta^{t,s,z}\right) \mathbf{1}_{\{\tau_{t,s} > \eta\}}\right.$$

$$\left. \text{a.s. for some } \mathbb{F}\text{-adapted } Z \text{ satisfying } Z_t = z\right\}.$$

Note that by convexity, v^h is continuous in z and we make a similar argument as in Proposition 3.1 to show that the infimum in $W^h(t, s, z)$ is attained. First, we claim that $v^h(t, s, z) \leq W^h(t, s, z)$. Set $y := W^h(t, s, z)$. Then there exists $Z^{t,s,z} \in \mathbb{F}$ such that $Z_t = z$ and

$$Y_{\eta \wedge \tau_{t,s}}^{t,s,y,Z} \geq g\left(\tau_{t,s}, S_{\tau_{t,s}}^{t,s}\right) \mathbf{1}_{\{\tau_{t,s} \leq \eta\}} + v^h\left(\eta, S_\eta^{t,s}, Z_\eta^{t,s,z}\right) \mathbf{1}_{\{\tau_{t,s} > \eta\}} \quad \text{a.s.}$$

Let $\omega \in \Omega$ be arbitrary. If $\eta(\omega) \geq \tau_{t,s}(\omega)$, we have

$$Y_{\tau_{t,s}}^{t,y,Z}(\omega) \geq g\left(\tau_{t,s}, S_{\tau_{t,s}}^{t,s}\right)(\omega).$$

Otherwise, $\eta(\omega) < \tau_{t,s}(\omega)$ implies that

$$Y_{\eta_{t,s}}^{t,y,Z}(\omega) \geq v\left(\eta, S_\eta^{t,s}, Z_\eta^{t,z}\right)(\omega).$$

Then there exists $\hat{Z}^{\eta(\omega),S_\eta(\omega)} \in \mathbb{F}$ such that $\hat{Z}_{\eta(\omega)} = Z_{\eta(\omega)}^{t,s,z}$ such that

$$Y_{\tau_{\eta(\omega),S_\eta(\omega)}}^{\eta(\omega),S_\eta(\omega),Y_\eta^{t,s,y,Z}(\omega),\hat{Z}} \geq g\left(\tau_{\eta(\omega),S_\eta(\omega)}, S_{\tau_{\eta(\omega),S_\eta(\omega)}}\right).$$

We define

$$\bar{Z}_s = \begin{cases} Z_s & \in s \in [t, \eta] \\ \hat{Z}_s & s \in [\eta, \tau_{t,s}]. \end{cases}$$

By construction $\bar{Z} \in \mathbb{F}$ and $\hat{Z}_t = z$. Moreover,

$$Y_{\tau_{t,s}}^{t,y,\bar{Z}} \geq g\left(\tau_{t,s}, S_{\tau_{t,s}}\right)$$

so that y is superreplicating. Hence, $v^h(t, s, z) \leq W^h(t, s, z)$.

To show the claim, it remains to prove $W^h(t, s, z) \leq v^h(t, s, z)$. Set $y := v^h(t, s, z)$. Then there exists $Z^{t,s,z} \in \mathbb{F}$ such that $Z_t = z$ and

$$Y^{t,s,y,Z}_{\tau_{t,s}} \geq g\left(\tau_{t,s}, S_{\tau_{t,s}}\right) \quad \text{a.s.}$$

Let $\omega \in \Omega$. If $\tau_{t,s}(\omega) \leq \eta(\omega)$, then

$$Y^{t,s,y,Z}_{\tau_{t,s}}(\omega) \geq g\left(\tau_{t,s}, S_{\tau_{t,s}}\right)(\omega).$$

Otherwise if $\eta(\omega) < \tau_{t,s}(\omega)$, then

$$Y^{\eta(\omega), S_\eta(\omega), Y^{t,s,y,Z}_{\eta(\omega)}, Z}_{\tau_{\eta(\omega), S_\eta(\omega)}} = Y^{t,s,y,Z}_{\tau_{\eta(\omega), S_\eta(\omega)}} \geq g\left(\tau_{\eta(\omega), S_\eta(\omega)}, S_{\tau_{\eta(\omega), S_\eta(\omega)}}\right).$$

Therefore,

$$Y^{t,s,y,Z}_{\eta(\omega)} \geq v^h(\eta(\omega), S_\eta(\omega), Z^{t,s,z}(\omega))$$

so that $v^h(t, s, z) \geq W^h(t, s, z)$. □

This theorem is an extension of the dynamic programming established in [3] for European claims. In the case of a European option with payoff f, we have $\tau_{t,s} = T$ so that for $\eta = t + h$ we have the following representation of 3.1

$$v^h(t, s, z) = \min y$$
$$\text{s.t. } y + zs(u - 1) - (z_u - z)\theta_{up}(z_u - z) \geq v^h(t + h, su, z_u),$$
$$y + zs(d - 1) - (z_d - z)\theta_{down}(z_d - z) \geq v^h(t + h, sd, z_d),$$

where z_u and z_d are the values Z_{t+h} takes depending on whether the stock price is up or down. By abuse of notation we will suppress the dependence of θ on the up or down state. We reformulate the above representation of the dynamic programming as

$$v^h(t, s, z) = \max\Bigg(\min_a \left\{ v^h(t + h, su, z + a) - zs(u - 1) + a\theta(a) \right\},$$

$$\min_b \left\{ v^h(t + h, sd, z + b) - zs(d - 1) + b\theta(b) \right\} \Bigg), \tag{3.1}$$

by using the notation $z_{up} = z + a$ and $z_{down} = z + b$. The difference equation (3.1) is called the dynamic programming equation. It is complemented by the terminal data

$$v^h(T, z) = f(s) \quad \forall s \in (0, \infty), \ z \in \mathbb{R}. \tag{3.2}$$

Next consider a barrier option characterized by the open set $\mathcal{O}$ and a payoff function g. Observe that by Remark 2.1 $v^h(t, s, z) = g(t, s)$ for $s \notin \mathcal{O}$. If $s \in \mathcal{O}$, then $\tau_{t,s} > t$. Again we take $\eta = t + h$ in Theorem 3.1. So the stock price either goes up to su or goes down to sd. Without loss of generality consider only the case that the stock price goes up. For $su \notin \mathcal{O}$ by Remark 2.1 we get that $v(t + h, su, z) = g(t + h, su)$. This implies

$$y + zs(u - 1) - (z_u - z)\theta_{up}(z_u - z) \geq g(t + h, su) = v^h(t + h, su, z_u).$$

On the other hand, if $su \in \mathcal{O}$ according to Theorem 3.1 we also obtain

$$y + zs(u - 1) - (z_u - z)\theta_{up}(z_u - z) \geq v^h(t + h, su, z_u).$$

Similar analysis can be carried if the price goes down to sd. Therefore, we obtain the same dynamic programming equation (3.1) as for the European options. However, we note that $v^h(t + h, S_{t+h}, Z_{t+h})$ is equal to $g(t, s)$, if $S_{t+h} \notin \mathcal{O}$.

Remark 3.1. The initial portfolio dependence z of the value function $v^h(t, s, z)$ has also the advantage that we can work with payoffs that depend on the portfolio value at expiry. So for instance we could consider a barrier function characterized by the payoff $g(\tau, S_\tau, Z_\tau)$ and the open set $\mathcal{O}$, where τ is the first exit time from the set $\mathcal{O}$. This enables us to consider liquidation costs at time τ. This cost at τ depends on the settlement of the option.

A portfolio (x, z) is said to dominate $(\bar{x}, \bar{z})$ if

$$x - (\bar{z} - z)(s + \Lambda(\bar{z} - z))^+ \geq \bar{x}. \tag{3.3}$$

Under a scenario ω if the option settles at time $\tau(\omega)$ with $\bar{x}$ money in the bank account and $\bar{z}$ number of shares in the stock, then we require that $(X_\tau(\omega), Z_\tau(\omega))$ dominates $(\bar{x}, \bar{z})$.

For example, on a path ω the call option settles with $\bar{x}$ money in the bank account and $\bar{z}$ number of shares, where

$$(\bar{x}, \bar{z}) = \begin{cases} (-K, 1) & S_T(\omega) > K \\ (0, 0) & S_T(\omega) \le K. \end{cases} \tag{3.4}$$

Because of $Y = X + ZS$ and (3.3), we obtain the boundary condition for the value function

$$v^h(T, s, z) = \begin{cases} zs + (1 - z)(s + \Lambda(1 - z))^+ - K & s > K \\ zs + (0 - z)(s + \Lambda(0 - z))^+ & s \le K. \end{cases} \tag{3.5}$$

4. Parameters

Although we can adapt our numerical algorithm for any loss function θ such that $\nu \mapsto \nu\theta(\nu)$ is convex, we restrict ourselves for a linear supply curve with liquidity parameter Λ, i.e.

$$\mathbf{S}(t, S_t, \nu) = S_t + \Lambda\nu. \tag{4.1}$$

This supply curve may take negative values, which is undesirable for a price. However, we work with (4.1) instead of $(S_t + \Lambda\nu)^+$. The same approach was followed in [3], because the change in optimal portfolios are expected to be small so that one never has a negative price.

We emphasize that we can work with any up and down factors, u and d, as well as with any time step N. In particular, we take

$$u = \exp\left(\sigma\sqrt{h}\right), \quad \text{and} \quad d = \exp\left(-\sigma\sqrt{h}\right), \quad \text{where} \quad h = \sqrt{\frac{T}{N}}. \tag{4.2}$$

Here σ is the volatility of the price process and $h > 0$ is the time step. With this choice of parameters $ud = 1$.

5. Algorithm

In this section, we develop an algorithm to calculate the superreplicating cost $v^h(t, s, z)$, therefore $\phi^h(t, s)$ as a result of (2.8). In the algorithm, we also compute the optimal hedge $z^*(t, s, z)$ at the node (t, s), if one has an initial position of z shares. Solving for $\phi^h(t, s)$ by means of the optimization problem discussed in Section 2 is computationally expensive for many time steps. This is because one has to work on a non-recombinant tree due to the path dependent structure of the portfolio process. Therefore, the number of nodes in the tree grows exponentially with the number of steps to maturity. As an alternative we use the dynamic programming equation (3.1) to calculate $v^h(t, s, z)$ by marching backwards in time on a recombinant tree. Despite the fact that we are adding another state variable, the initial portfolio dependence $Z_t = z$, to the value function $v^h(t, s, z)$ our numerical scheme allows us to compute larger time steps efficiently. Once we computed $v^h(t, s, z)$ for all z, we can find $\phi^h(t, s)$ by Eq. (2.8).

Next we describe the algorithm for a barrier option characterized by the open set $\mathcal{O}$ and the payoff function g. We start by discretizing the continuous state variable $Z_t = z$ with a grid size of Δz. As described above, we march backwards in time on the recombinant tree using the dynamic programming equation (3.1). Therefore to start the algorithm, first we set $v^h(T, s, z) = g(T, s)$ for all discretized z values. In the recursion step if $s \notin \mathcal{O}$, then we set $v^h(t, s, z) = g(t, s)$ for all z. Otherwise, we note that the dynamic programming equation (3.1) consists of taking the maximum of two minimization problems. Hence, it is sufficient enough to develop an algorithm to solve one of them. To this aim, we look for the optimizer $z^*(t, s, z)$

$$z^*(t, s, z) = \arg\min_\xi v^h(t, s, \xi) + \Lambda(\xi - z)^2, \tag{5.1}$$

where in the recursion step we calculated $v^h(t, s, \xi)$ for all discretized ξ and the initial portfolio value z is known. Formally, we force the first order condition and obtain that

$$z = \hat{G}(t, s, z^*(t, s, z)) \quad \text{for } \hat{G}(t, s, \xi) = \frac{1}{2\Lambda}v_\xi(t, s, \xi) + \xi. \tag{5.2}$$

The convexity of $v^h(t, s, z)$ in z implies that

$$\hat{G}_\xi(t, s, \xi) = \frac{1}{2\Lambda}v_{\xi\xi}(t, s, \xi) + 1 > 0.$$

So $\hat{G}(t, s, \xi)$ is an increasing function in ξ. To find $z^*(t, s, z)$ we employ a search algorithm for a dummy variable $\bar{\bar{\xi}}_z$. We start from an initial value $\bar{\bar{\xi}}_z$ and increase incrementally by Δz until $\bar{\bar{\xi}}_z$ satisfies

$$\hat{G}(t, s, \bar{\bar{\xi}}_z) \le z < \hat{G}(t, s, \bar{\bar{\xi}}_z + \Delta z).$$

We denote by $z^*(t, s, z)$ the $\overline{\overline{\xi}}_z$ satisfying the above condition. Let us assume that the initial portfolio variable z takes values in an interval $\{-A, -A + \Delta z, \ldots, B\}$. To find $z^*(t, s, -A)$ we need to start the search from an initial value $\overline{\overline{\xi}}_{-A}$. This can be chosen to be $-A$ as well. However, the strength of the algorithm relies on the fact that to compute $z^*(t, s, z)$ for $z > -A$, we do not need to start from a fixed portfolio value. This saves considerable amount of time and computational effort. In fact once we computed $z^*(t, s, z)$ we can start the search for $z^*(t, s, z + \Delta z)$ from $z^*(t, s, z)$. The previous observation is based on the monotonicity property of $\hat{G}(t, s, \xi)$ in ξ. After we calculate $z^*(t, s, z)$, we store it and use it to evaluate

$$\min_{\xi} v^h (t + h, S_{t+h}, \xi) - z (S_{t+h} - S_t) + \Lambda (\xi - z)^2,$$

since the minimizer of the above equation is $z^*(t + h, S_{t+h}, z)$. Then we can compute $v^h(t, s, z)$ via (3.1) and store it for all (s, z) and complete the recursion step.

6. Numerical results

In this section, we are concerned with computing the optimal hedge in feedback form and compare it to other standard strategies. In particular, we consider the three different strategies given in feedback form

$$z^*(t, s, z) = \arg \min_{\xi} v^h(t, s, \xi) + \Lambda(\xi - z)^2, \tag{6.1}$$

$$z^\Delta(t, s) = \frac{\phi^h(t + h, su) - \phi^h(t + h, sd)}{s(u - d)}, \tag{6.2}$$

$$z^m(t, s) = \arg \min_z v^h(t, s, z). \tag{6.3}$$

The first portfolio value (6.1) is the optimal hedge at node (t, s) if one has an initial position of z shares. In (6.1) it is crucial to keep track of the initial portfolio position z because of the path dependent structure of the problem. The second portfolio value (6.2) is the discrete version of the delta hedge. The third portfolio value (6.3) gives us the portfolio z for which the value function $v^h(t, s, z)$ takes its minimum value. Observe that for the second (6.2) and the third (6.3) portfolio values the initial portfolio variable z dependence disappears.

The discrete-delta strategy and the minimizer of the value function are path-independent strategies, i.e. they depend on the node of the binomial lattice but not on the path followed up to that node. However, we observe that the optimal hedge is path-dependent. In fact, to compute the optimal strategy at a node of the binomial tree, one needs to know the initial position in the risky asset before coming to that node. Intuitively, if the initial number of shares is known at a node, then one knows which path is followed up to that node. This supports the introduction of the extra state variable, the initial position in the risky asset, to the value function.

Let us assume we are initially at node (t, s) and consider a path $\omega \in \Omega$. Also suppose at this node the initial portfolio value is $Z_t = z$ and the initial wealth value is $Y_t = y$. For this path ω on the binomial tree the values of the wealth process corresponding to a strategy $Z^{t,s,z}$ are given by

$$Y_{u+h}^{t,s,y,Z}(\omega) = Y_u^{t,s,y,Z}(\omega) + Z_u^{t,s,z}(\omega) \left(S_{u+h}^{t,s}(\omega) - S_u^{t,s}(\omega)\right) - \Lambda \left(Z_{u+h}^{t,s,z}(\omega) - Z_u^{t,s,z}(\omega)\right)^2$$

for $u \in \{t, \ldots, \tau_{t,s}(\omega) - h\}$.

Choose $y := Y_t = v^h(t, s, z)$ for $Z_t = z$ and compare the wealth values and the superreplicating values on a given path ω by looking at the difference

$$Y_u^{t,s,y,Z}(\omega) - v^h \left(u, S_u^{t,s}(\omega), Z_u^{t,s,z}(\omega)\right), \quad \forall u \in \{t, \ldots, \tau_{t,s}(\omega)\} \tag{6.4}$$

for the three different strategies: First for the hedge defined recursively by

$$Z_t(\omega) = z \quad \text{and} \quad Z_{u+h}^{t,s,z}(\omega) = z^*(u + h, S_{u+h}(\omega), Z_u^{t,s,z}(\omega)) \quad \forall u \in \{t, \ldots, \tau_{t,s}(\omega) - h\}.$$

Second for the strategy

$$Z_t(\omega) = z^\Delta(t, s) \quad \text{and} \quad Z_u^{t,s,z}(\omega) = z^\Delta(u, S_u^{t,s}(\omega)) \quad \forall u \in \{t, \ldots, \tau_{t,s}(\omega)\}.$$

If $\tau_{t,s}(\omega) = T$, then by convention we set $Z_T^{t,s,z}(\omega) = Z_{T-h}^{t,s,z}(\omega)$. Third for the strategy

$$Z_t(\omega) = z^m(t, s) \quad \text{and} \quad Z_u^{t,s,z}(\omega) = z^m(u, S_u^{t,s}(\omega)) \quad \forall u \in \{t, \ldots, \tau_{t,s}(\omega)\}.$$

We choose (6.4) as a criterion to evaluate the performance of different strategies. If $Z^{t,s,z}$ is a superreplicating strategy, it should satisfy

$$Y_u^{t,s,y,Z}(\omega) \geq v^h \left(u, S_u^{t,s}(\omega), Z_u^{t,s,z}(\omega)\right), \quad \forall u \in \{t, \ldots, \tau_{t,s}(\omega)\}, \ \forall \omega \in \Omega.$$

Assume this is not the case, then there exists a node $(u, S_u(\omega))$ on some path ω starting from (t, s) such that

$$Y_u^{t,s,y,Z}(\omega) < v^h(u, S_u(\omega), Z_u(\omega)). \tag{6.5}$$

This implies that for any strategy $\overline{Z}^{u,S_u(\omega)} \in \mathbb{F}$ with $\overline{Z}_u^{u,S_u(\omega)} = Z_u^{t,s,z}$, we can find a path $\overline{\omega}$ starting from $(u, S_u(\omega))$ such that the wealth value at time $\tau_{u,S_u(\omega)}(\overline{\omega})$ fails to dominate the payoff, i.e.

$$Y_{\tau_{u,S_u(\omega)}}^{u,S_u(\omega),Y_u^{t,s,y,Z},\overline{Z}}(\overline{\omega}) < g\left(\tau_{u,S_u(\omega)}(\overline{\omega}), S_{\tau_{u,S_u(\omega)}}(\overline{\omega})\right).$$

In particular, choosing $\overline{Z} = Z$ one can see that Z is not superreplicating.

We give examples of paths for European call and capped options as well as up-and-out call options, on which the standard strategies $z^m(t, s)$ and $z^\Delta(t, s)$ perform badly. Given such a path ω starting from (t, s), we observe that (6.5) holds at some node $(u, S_u(\omega))$ for these portfolio strategies. Moreover, these paths are exactly those, where the wealth processes Y fail to dominate the payoff g at node $\left(\tau_{t,s}(\omega), S_{\tau_{t,s}}(\omega)\right)$, if one uses $z^m(t, s)$ and $z^\Delta(t, s)$ as portfolio strategies. As expected, we see numerically that the optimal strategy given in feedback form $z^*(t, s, z)$ always superreplicates the claim. So at every node $(u, S_u(\omega))$ on each path ω it satisfies $Y_u^{t,s,y,Z}(\omega) \geq v^h(u, S_u, Z_u(\omega))$ and it dominates the payoff g. This indicates that the optimal strategy given in feedback form $z^*(t, s, z)$ is not equal to the strategies consisting of $z^m(t, s)$ and $z^\Delta(t, s)$. However, there are scenarios ω, where $z^*(t, s, z)$, $z^\Delta(t, s)$ and $z^m(t, s)$ agree with each other at some or at all nodes on ω. Since $z^*(t, s, z)$ is not always equal to $z^m(t, s)$ and $z^\Delta(t, s)$, the initial portfolio z dependence of the value function $v^h(t, s, z)$ turns out to be crucial to extract information about the hedge. Therefore, we note that knowing only the value function $\phi^h(t, s)$ is not sufficient to compute the hedge.

Denote by $\phi^{h,\Lambda}(0, 1)$ the discrete-time superreplicating cost with liquidity parameter Λ. Then, we take the relative difference between $\phi^{h,\Lambda}(0, 1)$ and $\phi^{h,0}(0, 1)$, i.e.

$$\frac{\phi^{h,\Lambda}(0, 1) - \phi^{h,0}(0, 1)}{\phi^{h,0}(0, 1)}$$

as the liquidity premium. It measures how much in addition has to be paid due to the liquidity parameter Λ.

If one uses a strategy consisting of $z^m(t, s)$ or $z^\Delta(t, s)$, then there exists paths, where the difference between the wealth process and the payoff take large negative values. We numerically observe this is the case, if the $z^m(t, s)$ and $z^\Delta(t, s)$ vary rapidly, in particular oscillate. However, the optimal hedge given in feedback form

$$z^*(t, s, z) = \arg\min_\xi v^h(t, s, \xi) + \Lambda(\xi - z)^2$$

is immune to rapid changes due to the penalization term $\Lambda(\xi - z)^2$. Therefore, this penalization has a "smoothing out" effect on the hedge even for a small liquidity parameter Λ. In our numerical experiments, the liquidity parameter Λ and the maturity T are chosen such that these effects of illiquidity on the portfolio strategies as well as their associated wealth processes are exaggerated. Moreover, for sufficiently small Λ, this penalization is relatively "cheap" for $z^*(t, s, z)$, since for $z^\Delta(t, s)$ and $z^m(t, s)$ the difference between the wealth process and the payoff may take negative values. We also report on the dependence of $\phi^{h,\Lambda}(0, 1)$ on various liquidity parameters Λ, and exhibit the premiums paid for these liquidity parameters. We see that for appropriate liquidity parameters these premiums are small. We suggest to use the hedge given in feedback form by $z^*(t, s, z)$ from the algorithm instead of $z^m(t, s)$ and $z^\Delta(t, s)$. For a sufficiently small $\Lambda > 0$, this portfolio choice will yield a "nice" hedge with a "small" cost.

6.1. Numerical experiments

In all the conducted numerical experiments we start from the node $(t_0, s_0) = (0, 1)$ with zero initial shares, i.e. $Z_0 = 0$. We work with a yearly maturity $T = 1$ and take a yearly volatility $\sigma = 0.25$. Also, the spacing between two consecutive portfolio values is $\Delta z = 0.0005$.

6.1.1. Call option

In this numerical example, we illustrate that there exists scenarios ω, where the three strategies given in feedback form by (6.1)–(6.3) are not only very close to each other, but also their corresponding wealth processes are almost equal. More importantly, although we start with $Z_0 = 0$ number of shares, which is not equal to $z^m(0, 1)$ and $z^\Delta(0, 1)$ in both cases, $z^*(t, s, z)$ is attracted to $z^\Delta(t, s)$ and $z^m(t, s)$ after some time.

In this example we take the strike of the call option to be $K = 0.85$, the number of time steps $N = 65$, the discretized portfolio values z lie in the interval $[0, 1]$ and the liquidity parameter Λ is 0.03. We take the final liquidation cost into account and consider the payoff at maturity to be given by (3.5).

For the path ω, Fig. 1 depicts the stock price values over time, and Fig. 2 shows how the three different strategies evolve with respect to time on this path ω. Moreover, the Fig. 3 illustrates the performance of the three different strategies by plotting the difference $Y_u^{t,s,y,Z}(\omega) - v^h\left(u, S_u^{t,s}(\omega), Z_u^{t,s,z}(\omega)\right)$ for $u \in \{0, \dots, T\}$ on the path ω.
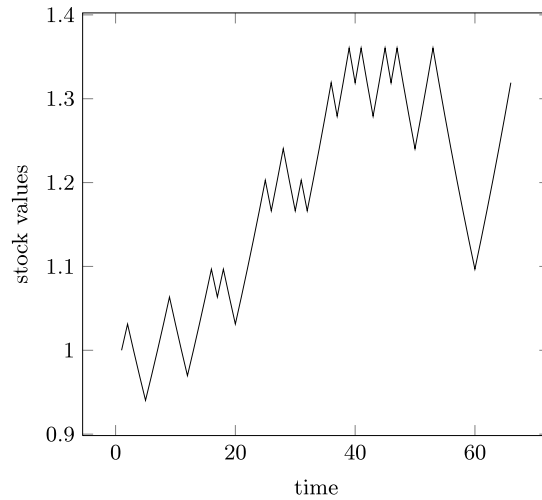
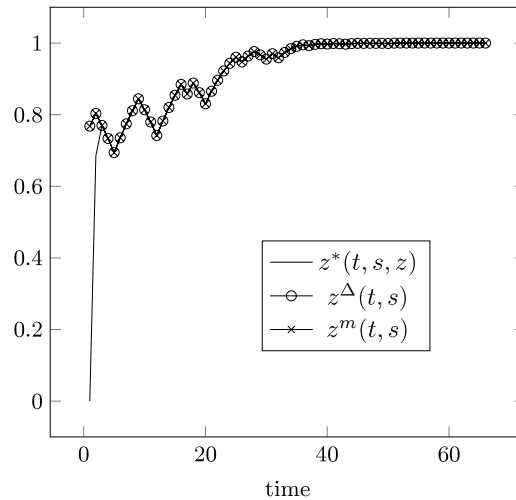**Fig. 1.** Stock price values over time for a call option.

Fig. 2. Values of different strategies over time for a call option.

6.1.2. Capped option

For the capped option with payoff $g(s) = s \wedge 1$ we choose a liquidity parameter $\Lambda = 0.2$, the number of time steps $N = 75$ and the discretized z values lie in the interval $[-2, 2]$. We also incorporate the liquidity cost at maturity. This option settles on the path ω with zero money in the bank account and with one share if $S_T(\omega) < 1$. On the other hand in the case $S_T(\omega) \geq 1$ the agent gets 1 unit in the bank account and zero number of shares. Therefore, the terminal data for the value function is given by

$$v^h(T, s, z) = \begin{cases} zs + (1 - z)(s + \Lambda(1 - z))^+ & s < 1 \\ zs + (0 - z)(s + \Lambda(0 - z))^+ + 1 & s \geq 1. \end{cases}$$

Fig. 4 illustrates the stock price values with respect to time on a path ω. We also can see from Fig. 5 how the three different strategies evolve over time on ω. The strategies given in feedback form by $z^m(t, s)$ and $z^\Delta(t, s)$ vary rapidly, in fact they oscillate. However, the optimal hedge $z^*(t, s, z)$ makes less rapid changes, it is a "smoothened out" version of $z^m(t, s)$ and $z^\Delta(t, s)$. Fig. 6 depicts the performance of corresponding three strategies. It plots $Y_u^{t,s,y,Z}(\omega) - v^h\left(u, S_u^{t,s}(\omega), Z_u^{t,s,z}(\omega)\right)$ for $u \in \{0, \dots, T\}$ on the path ω. We note that this difference is always positive for $z^*(t, s, z)$ indicating that the corresponding wealth dominates the payoff on ω. On the other hand, the difference $Y_u^{t,s,y,Z}(\omega) - v^h\left(u, S_u^{t,s}(\omega), Z_u^{t,s,z}(\omega)\right)$ for $z^m(t, s)$ and $z^\Delta(t, s)$ become negative on the path, in particular they fail to dominate the payoff at maturity on this path. This supports
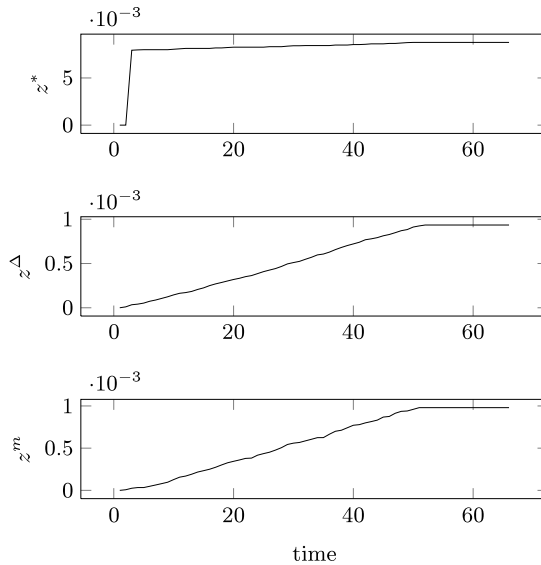
Fig. 3. The performance of different strategies over time for a call option.
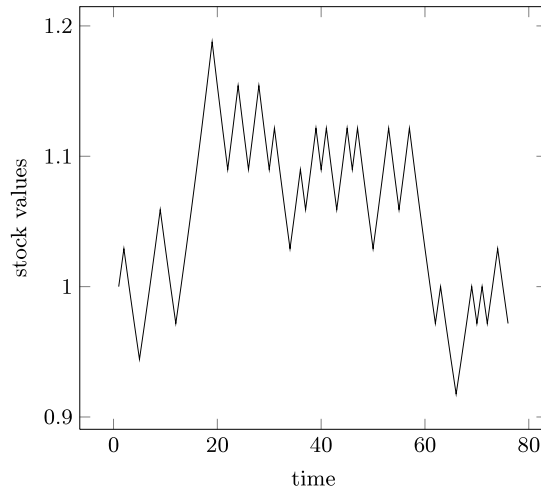
Fig. 4. Stock price values over time for a capped option.

that the optimal hedge is not always equal to the discrete delta-hedge $z^\Delta(t, s)$ or to $z^m(t, s)$. Furthermore, the optimal hedge behaves nicer than $z^\Delta(t, s)$ or $z^m(t, s)$ and it performs well in the sense that it superreplicates.

6.1.3. Up-and-out call options

We also investigate an up-and-out call option with strike $K = 0.9$, barrier $B = 1.55$ with liquidity parameter $\Lambda = 0.01$. The discretized z-values lie in the interval $[-4, 4]$ and work with a time step $N = 72$. The settlement of the barrier option at maturity resembles the one of vanilla call with the same strike. For a path ω with $\tau_{0,1}(\omega) < T$, the settlement is that the investor gets zero share and pays zero amount of money. Hence, we obtain for $S_{\tau_{0,1}(\omega)}(\omega) = s$

$$v^h(\tau_{0,1}(\omega), s, z) = \begin{cases} zs + (0 - z)(s + \Lambda(0 - z))^+ & B \le s \\ zs + (0 - z)(s + \Lambda(0 - z))^+ & s \le K, \ \tau_{0,1}(\omega) = T \\ zs + (1 - z)(s + \Lambda(1 - z))^+ - K & K < s < B, \ \tau_{0,1}(\omega) = T. \end{cases}$$

Fig. 7 shows the stock price values over time on a path ω. As in the capped option, Fig. 8 illustrates that the optimal hedge given by $z^*(t, s, z)$ shows much less variability than the strategies consisting of $z^m(t, s)$ and $z^\Delta(t, s)$. In the Fig. 9 we can see that the wealth process using the strategy composed of $z^*(t, s, z)$ dominates the payoff at maturity, whereas the wealth processes corresponding to $z^m(t, s)$ and $z^\Delta(t, s)$ perform badly.

In the next numerical experiment, we exhibit that the underperformance of the strategies $z^m(t, s)$ and $z^\Delta(t, s)$ can be severe to demonstrate the effect of liquidity on the hedges. Fig. 10 depicts the stock price values. Fig. 11 clearly illustrates
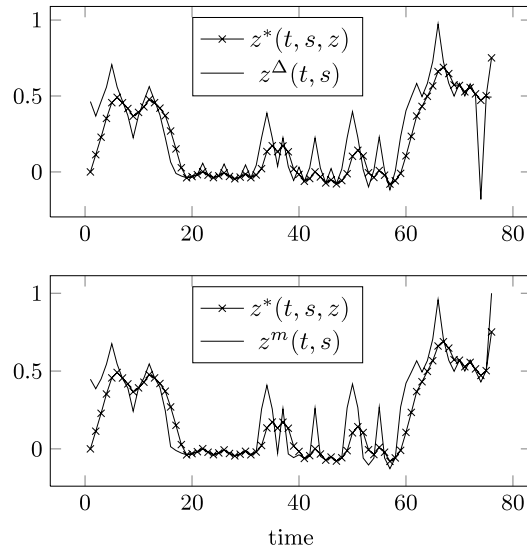
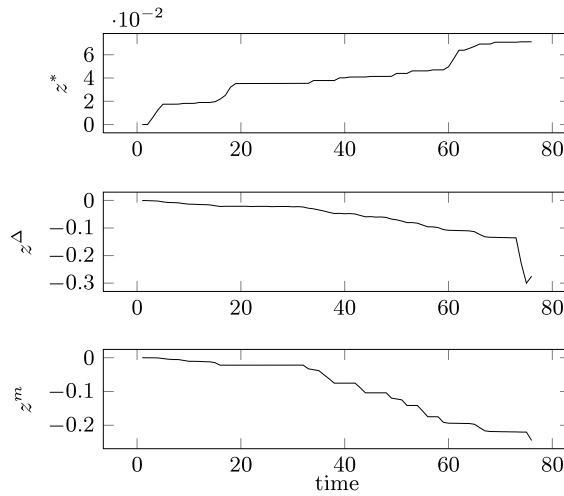**Fig. 5.** Values of different strategies over time for a capped option.

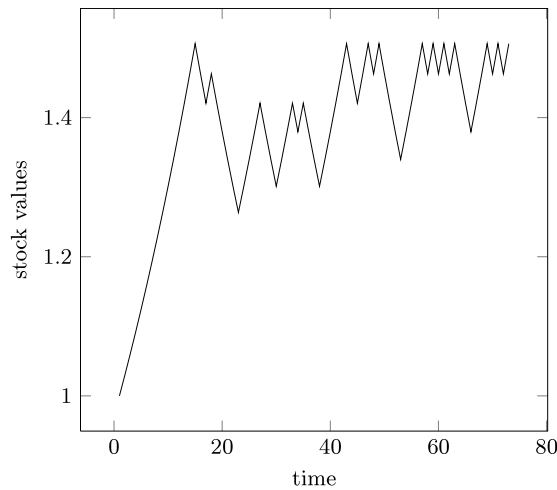**Fig. 6.** The performance of different strategies over time for a capped option.

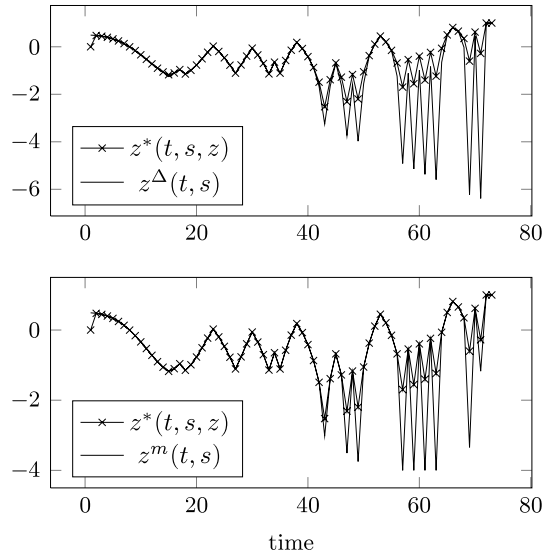**Fig. 7.** Stock price values over time for an up-and-out call option.

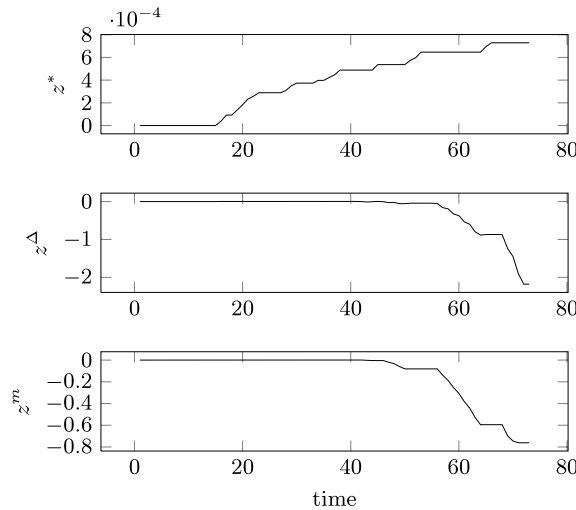**Fig. 8.** Values of different strategies over time for an up-and-out call option.

Fig. 9. The performance of different strategies over time for an up-and-out call option.

that the optimal hedge given by $z^*(t, s, z)$ shows much less variability than the strategies consisting of $z^m(t, s)$ and $z^{\Delta}(t, s)$. Fig. 12 shows the performance of the three different strategies over time on this path ω. For a liquidity parameter $\Lambda = 0.05$, we see that from Fig. 12 that the wealth processes at expiry take huge negative values.

6.2. Dependence on lambda

We end this section by demonstrating the dependence of the superreplicating value $\phi^{h,\Lambda}(0, 1)$ on the liquidity parameter Λ. We take as an example the up-and-out call option with strike $K = 0.9$, barrier $B = 1.55$ and 3-month maturity. As before, the discretized z-values lie in the interval $[-4, 4]$ and work with a time step $N = 72$.

Fig. 13 depicts the dependence of $\phi^{h,\Lambda}(0, 1)$ on the liquidity parameter Λ. Moreover, we can see in Table 1 the premium paid for the liquidity parameters.

7. Conclusion

In this article, we studied the hedging strategy of European and barrier options in a binomial illiquid market. We computed numerically the optimal strategy in feedback form by means of a dynamic programming equation (3.1). This optimal strategy is not equal to the discrete-delta strategy or to the strategy consisting of minimizers of the value function.
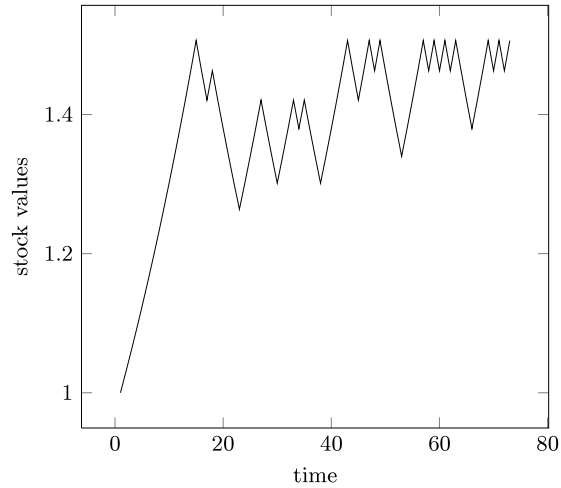
Fig. 10. Stock price values over time for an up-and-out call option on a different path.
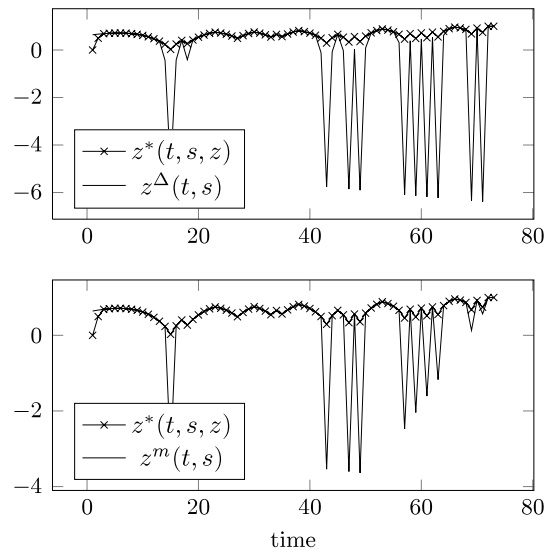

Fig. 11. Values of different strategies over time for an up-and-out call option on a different path.

Table 1
Dependence on Λ.

Λ	$\phi^h(0, 1)$	Premium	Λ	$\phi^h(0, 1)$	Premium
0	0.11306585	0%	0.11	0.12536238	10.9%
0.01	0.11501288	1.72%	0.12	0.12621952	11.6%
0.02	0.11631608	2.87%	0.13	0.12705196	12.4%
0.03	0.11753354	3.95%	0.14	0.12786791	13.1%
0.04	0.11865432	4.94%	0.15	0.12867836	13.8%
0.05	0.11973516	5.89%	0.16	0.12947439	14.5%
0.06	0.12075685	6.80%	0.17	0.13025493	15.2%
0.07	0.12174484	7.67%	0.18	0.13104049	15.9%
0.08	0.12268063	8.50%	0.19	0.13180431	16.6%
0.09	0.12361261	9.32%	0.2	0.13256691	17.3%
0.1	0.12448931	10.1%			

We demonstrate this by exhibiting paths, where the discrete-delta strategy or the minimizer of the value function fail to dominate the payoff. On such paths, these strategies show great variability, whereas the optimal hedging strategy is exposed to less variability due to the effect of liquidity. Therefore, we propose superreplicating the option using a sufficiently small liquidity parameter Λ to obtain a "nice" hedge but pay some premium to achieve it.
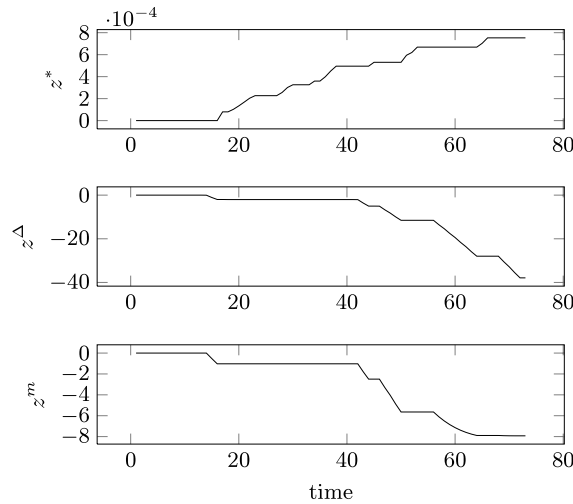
Fig. 12. The performance of different strategies over time for an up-and-out call option on a different path.
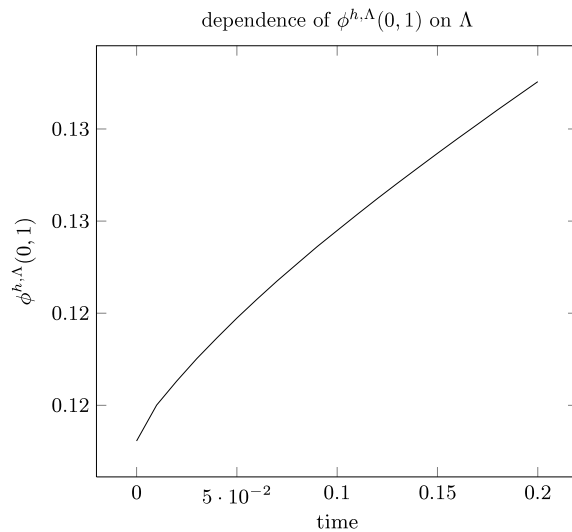

Fig. 13. The dependence of $\phi^h(0, 1)$ on Λ.

Acknowledgments

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