JOURNAL OF DIFFERENTIAL EQUATIONS 119, 54–108 (1995)

Anisotropic Motion of an Interface Relaxed by the Formation of Infinitesimal Wrinkles

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Received January 4, 1993; revised October 25, 1993

1. INTRODUCTION

A. *Mathematical Theory*

In this paper we discuss the motion, in the plane, of a region $\Omega(t)$ whose boundary-curve evolves from a given region Ω_0 according to the equation

$$B(\theta)\, V = G(\theta)\, K - U \tag{1.1}$$

with V the *normal velocity* and K the *curvature*. (Our sign convention is such that the positive normal-direction is outward from $\partial\Omega = \partial\Omega(t)$, and $K < 0$ when $\partial\Omega$ is a circle.) Here $B(\theta)$ and $G(\theta)$ are given functions of the *normal-angle* θ, which is the counterclockwise angle from a fixed axis to the outward normal of $\partial\Omega$, and U is a given constant.

For $B(\theta)$ and $G(\theta)$ continuous and strictly positive, (1.1) is a parabolic equation that is well understood, with fairly well-behaved solutions.[1] There are, however, situations of physical importance for which $G(\theta) = 0$ over certain angle-intervals and for which $G(\theta)$ need not be continuous

[1] Cf. Angenent [Ag], Chen *et al.* [CGG], Soner [So], and Barles *et al.* [BSS].

0022-0396/95 $6.00

(cf. Section 1.2). Here we will develop a fairly complete theory of (1.1) under the following assumptions:

G is piecewise continuous and $\geqslant 0$, and continuous on any interval of strict positivity; (1.2a)

B is continuous and >0. (1.2b)

In some instances we will add the hypothesis

B has polar diagram a straight line on any angle interval for which $G=0$, (1.3)

which is based on the underlying physics.

Because of the lack of continuity of G as well as the degeneracy of (1.1) when $G=0$, it is convenient to discuss this equation within the weak framework of viscosity solutions. This approach to geometric equations, initiated by Evans and Spruck [ES1] and Chen *et al.* [CGG], is based on the use of level sets to characterize evolving curves, an idea due to Sethian [Se], Osher and Sethian [OS], and Barles [Ba]. Here—to study (1.1)—we will use this approach as well as an intrinsic approach given by Soner [So] and Barles *et al.* [BSS]. The difficulties concerning (1.1) result from the *discontinuous* nature of G; the degeneracy of the equation at angles θ with $G(\theta)=0$ causes no great difficulty; were G continuous, most of our results would follow from those in [CGG].

Our main results, for evolution from a given compact region Ω_0, consist of a theorem of existence and local uniqueness and a global comparison theorem[2] for level-set solutions.

B. *Physical Background*

There are situations of interest in which the motion of a phase interface is essentially independent of the behavior of the corresponding bulk phases. One of the first models of such phenomena was proposed by Mullins [Mu] to study the *planar* motion of grain boundaries; the resulting evolution equation has the form[3]

$$V=K \tag{1.4}$$

[2] This comparison theorem was established independently by Ohnuma and Sato [OS], whose proof is different (and more concise) than ours.

[3] Allen and Cahn [AC] and Rubinstein *et al.* [RSK] deduce the equation $V=K$ as a formal approximation to the Landau–Ginzburg equation, a result established rigorously in [BK], [BSS, ESS, Ch, Sol, DS]. See also [ORS, OwS, RS].

after an appropriate scaling. Equation (1.4) is a parabolic PDE with a large literature;[4] its major consequence [GH, Gr] is that all such boundary curves, irrespective of their initial shape, shrink to a point in finite time, with asymptotic shape a circle.

Mullins's theory was generalized in [G1, Sect. 8; AG1] to include anisotropy and the possibility of a difference in bulk energies between phases. The resulting equation is

$$b(\theta)\, V = g(\theta)\, K - U, \tag{1.5}$$

where $g(\theta)$, the *energy modulus*, is given by

$$g(\theta) = f(\theta) + f''(\theta) \tag{1.6}$$

with $f(\theta) > 0$ the *interfacial energy*, U is the *relative energy* of the material in Ω, and $b(\theta) > 0$, the *kinetic modulus*, is a material function. The presence of the angle θ reflects anisotropy, and the particular form in which f appears in (1.6) is a consequence of thermodynamics. In fact, a consequence of (1.5) and (1.6) is the thermodynamic inequality

$$(d/dt)\left\{\int_{\partial\Omega(t)} f(\theta)\, ds + U\,\mathrm{area}(\Omega(t))\right\} = -\int_{\partial\Omega(t)} b(\theta)\, V^2\, ds. \tag{1.7}$$

When

$$g(\theta) > 0 \tag{1.8}$$

evolution according to (1.5) is governed by a parabolic PDE and the underlying problem is not much different than that for equation $V = K$. What makes (1.5) nonstandard is the possibility of interfacial energies that satisfy

$$g(\theta) < 0 \tag{1.9}$$

for certain ranges of the angle θ, for in these ranges the evolution equations are *backward parabolic*.[5]

Let

$$\mathbf{N}(\theta) = (\cos\theta, \sin\theta), \qquad \mathbf{T}(\theta) = (\sin\theta, -\cos\theta), \tag{1.10}$$

[4] Brakke [Br], Sethian [Se], Abresch and Langer [AL], Gage and Hamilton [GH], Grayson [Gr], Osher and Sethian [OS], Evans and Spruck [ES1-3], Chen *et al.* [CGG], Giga and Sato [GS], Almgren *et al.* [ATW], Taylor *et al.* [TCH], and the references therein.

[5] Material scientists give strong arguments in support of interfacial energies that satisfy $g(\theta) < 0$ for some values of θ (cf., e.g., Gjostein [Gj] and Cahn and Hoffman [CH]).

so that $\mathbf{T}$ and $\mathbf{N}$ represent a unit tangent and normal to the interface when θ is its normal-angle. The conditions (1.8) and (1.9) may be displayed graphically using the *Frank diagram* $\mathscr{F}$ which is the polar diagram of $f(\theta)^{-1}$ (i.e., the locus in $\mathbb{R}^2$ of all vectors $f(\theta)^{-1}\,\mathbf{N}(\theta)$): $\mathscr{F}$ is locally strictly convex where (1.8) is satisfied, locally strictly concave where (1.9) is satisfied.[6]

One method of overcoming (1.9) is to allow the interface to contain corners corresponding to jumps in angle that exclude the backward-parabolic ranges of θ [AG1]; a limitation of this method is that the initial curve $\partial\Omega(0)$ must also have such corners.

In the presence of a corner the evolution equation (1.5) does not by itself characterize the motion of the interface; there is an additional relation corresponding to the requirement that the *capillary force*

$$\mathbf{C}(\theta)=f(\theta)\,\mathbf{T}(\theta)+f'(\theta)\qquad \mathbf{N}(\theta) \text{ be continuous.} \tag{1.11}$$

Thus for a corner corresponding to an angle jump from θ_1 to θ_2 we must have $\mathbf{C}(\theta_1)=\mathbf{C}(\theta_2)$, which has an important consequence: the tangent line to $\mathscr{F}$ at θ_1 must also be a tangent line to $\mathscr{F}$ at θ_2; i.e., θ_1 and θ_2 must be *angles of bitangency* for the Frank diagram [AG1].

If the initial data $\partial\Omega(0)$ has normal angles corresponding to backward-parabolic behavior, another method of attack must be found. A possibility is to allow the interface to infinitesimally wrinkle on such backward-parabolic sections. Formally, consider an energy $f(\theta)$ with Frank diagram $\mathscr{F}$; let $\mathscr{C}(\mathscr{F})$ denote the convexification of $\mathscr{F}$, and let $F(\theta)$ denote the energy whose Frank diagram is $\mathscr{C}(\mathscr{F})$, so that $f(\theta)\geqslant F(\theta)$. Then angles θ with $f(\theta)=F(\theta)$ satisfy $g(\theta)\geqslant 0$; we refer to such angles as *globally stable* (GS), to angles θ with $f(\theta)>F(\theta)$ as *globally unstable* (GUS), and to each maximal interval (θ_1,θ_2) of angles θ with $f(\theta)>F(\theta)$ as a GUS *angle-interval.* Then each GUS angle-interval (θ_1,θ_2) has θ_1 and θ_2 as angles of bitangency for $\mathscr{F}$ and hence as admissible angles for a corner. Thus if Γ is a section of $\partial\Omega(0)$ with normal angles between θ_1 and θ_2, we can at least formally consider Γ as being infinitesimally wrinkled, with each infinitesimal facet having either θ_1 or θ_2 as normal angle.[7] The expansion

$$\mathbf{N}(\theta)=\mu_1(\theta)\,\mathbf{N}(\theta_1)+\mu_2(\theta)\,\mathbf{N}(\theta_2),\qquad \theta\in(\theta_1,\theta_2) \tag{1.12}$$

[6] The importance of the Frank diagram $\mathscr{F}$ becomes evident when one consider the homogeneous extension $\tilde{f}$ of f to $\mathbb{R}^2$: $\tilde{f}(\alpha\mathbf{N}(\theta))=\alpha f(\theta)$ for all angles θ and all $\alpha>0$. Then $\mathscr{F}$ is the one-level set of $\tilde{f}$, so that the convexity properties of $\tilde{f}$ are related to those of $\mathscr{F}$. (In particular, $\tilde{f}$ is convex if and only if $\mathscr{F}$ is convex.) Further $g(\theta)=T(\theta)\cdot[\nabla^2\tilde{f}(\mathbf{N}(\theta))]\,\mathbf{T}(\theta)$ ($\nabla^2=\nabla\nabla$), which, to some extent, explains the form of (1.3).

[7] This idea is due to Cahn and Taylor (private communication with Gurtin in 1990). Cf. Taylor [Ta] and Almgren and Taylor [AT].

then defines, for each i, the *density* $\mu_i(\theta)$ of θ_i-facets at any point of Γ with normal angle θ, with $\mu_i(\theta)$ measured per unit length of Γ.

The use of infinitesimal wrinklings is formally equivalent to replacing the interfacial energy $f(\theta)$ by the energy $F(\theta)$ corresponding to $\mathscr{C}(\mathscr{F})$, since[8]

$$F(\theta) = \mu_1(\theta)\, f(\theta_1) + \mu_2(\theta)\, f(\theta_2), \qquad \theta \in (\theta_1, \theta_2). \tag{1.13}$$

A further motivation for the use of such wrinklings is furnished by the fact that the initially wrinkled curve is more stable than the original curve:

$$\int_{\partial\Omega(0)} F(\theta)\, ds \leqslant \int_{\partial\Omega(0)} f(\theta)\, ds. \tag{1.14}$$

If we allow $\partial\Omega(t)$ to infinitesimally wrinkle in the same manner, we are led to the requirement that the effective interfacial energy for the evolution be $F(\theta)$, so that the *effective energy modulus* is given by

$$G(\theta) = F(\theta) + F''(\theta). \tag{1.15}$$

The next question we must answer is what is an appropriate kinetic modulus for the infinitesimally wrinkled curve If $\Gamma(t)$ is a finite wrinkling whose facets have θ_1 and θ_2 as normal angles, then $\Gamma(t)$ evolves as a rigid body with constant velocity $\boldsymbol{\omega}$ defined by [AG1]

$$\boldsymbol{\omega} \cdot \mathbf{N}(\theta_1) = -b(\theta_1)^{-1}\, U, \qquad \boldsymbol{\omega} \cdot \mathbf{N}(\theta_2) = -b(\theta_2)^{-1}\, U \tag{1.16}$$

(although $\Gamma(t)$ is allowed to shrink or grow tangentially). Since $\boldsymbol{\omega}$ depends on the particular wrinkling only through θ_1 and θ_2, it seems reasonable to suppose that *infinitesimal wrinklings* with θ_1 and θ_2 as normal angles also evolve with rigid velocity $\boldsymbol{\omega}$, and this is equivalent to replacing the kinetic modulus $b(\theta)$ between θ_1 and θ_2 by an effective modulus $B(\theta)$ that agrees with $b(\theta)$ at θ_1 and θ_2 and has polar diagram between θ_1 and θ_2 a straight line:

$$B(\theta)^{-1} = \mu_1(\theta)\, b(\theta_1)^{-1} + \mu_2(\theta)\, b(\theta_2)^{-1} \tag{1.17}$$

This proceedure defines an *effective kinetic modulus* $B(\theta)$ for all θ [G2]: $B(\theta) > 0$ is continuous; $B(\theta) = b(\theta)$ for all GS angles θ; the polar diagram of $B(\theta)$ is a straight line over normal-angle intervals with $f(\theta) > F(\theta)$.

We will refer to G and B derived in this manner as the *effective moduli* corresponding to f and g.

We are therefore led to the *relaxed evolution equation* (1.1) with B and G the effective moduli corresponding to f and g [G2]. It is important to note that this relaxed equation coincides with our original system (1.5) at

[8] Indeed, $\mathscr{F}$ is the locus of the vector $\mathbf{q}(\theta) = \mathbf{N}(\theta)/F(\theta)$ and $\mathscr{F}$ is flat between θ_1 and θ_2; thus $\mathbf{q}(\theta) = (1-\alpha)\, \mathbf{q}(\theta_1) + \alpha\mathbf{q}(\theta_2)$, and since $F(\theta_i) = f(\theta_i)$, (1.9) yields (1.10).

all GS angles θ. Note also that, because of the construction of $G(\theta)$, no matter how smooth $f(\theta)$ is,

$$G(\theta) \text{ will generally be discontinuous} \tag{1.18}$$

whenever the angle θ changes from GS to GUS; this property of $G(\theta)$ renders the relaxed evolution equation nonstandard. In addition, $G(\theta)=0$ whenever θ is GUS, so that (1.1) *degenerates to hyperbolic* at GUS angles.

Our main result of physical interest are:

1° Viscosity solutions of (1.1) not only satisfy (1.5) away from corners, but, what is most interesting, such solutions automatically satisfy the force balance (1.11) across corners.

2° If (θ_1, θ_2) is a GUS angle-interval, then a wedge whose two sides have normal angles θ_1 and θ_2 and evolve according to $b(\theta_1)\, V = -U$ and $b(\theta_2)\, V = -U$, respectively, is a solution of the basic equations (1.5) and (1.11) [AG1, Sect. 9]. We show that our choice of the effective moduli G and B is the only possible choice if all such wedges are to be viscosity solutions of (1.1). What makes this result so interesting is that $G(\theta)$ and $B(\theta)$ differ from $g(\theta)$ and $b(\theta)$ only at angles θ that are not globally stable, and wedges by definition do not involve such θ.

3° For $U<0$ and Ω_0 large enough, $t^{-1}\Omega(t)$ converges to a dilation of the Wulff region for $1/B(\theta)$.[9]

2. Classical Evolution: Wrinklings and Wedges

Throughout the paper we restrict attention to energies $f(\theta)$ and kinetic moduli $b(\theta)$ that are consistent with following hypotheses:

f is C^2 and >0; (2.1a)

each convexifying tangent to the Frank diagram $\mathscr{F}$ intersects $\mathscr{F}$ at most at two angles, and there are at most a finite number of such tangents; (2.1b)

$g(\theta)>0$ at each GS angle θ; (2.1c)

b is continuous and strictly positive. (2.1d)

We begin with a discussion of regions whose boundaries evolve according to (1.5), but with normal angles constrained to be GS, so that (1.5) is

[9] This result, conjectured by Angenent and Gurtin [AG1], was proved by Soner [So] for $G>0$ and B with a convex polar diagram, and extended in [AG2] to general $B>0$. Cf. Frank [Fr].

parabolic. Such boundaries will generally contain corners which are consistent with (1.11) and for which the jump in normal angle removes angles of backward parabolicity of (1.5).[10] Not all initial data are consistent with evolutions of this type; in particular, the initial region Λ must be *admissible* in the sense that[11]

$$\Lambda \text{ is closed with } \partial\Lambda \text{ piecewise } C^2\text{, and at each point of smoothness the (outward) normal angle } \theta \text{ is GS (so that } g(\theta) > 0); \tag{2.2a}$$

$$(1.11) \text{ is satisfied.} \tag{2.2b}$$

For each $t \in (0, T)$, let $\Omega(t) \subset \mathbb{R}^2$ be given. Then $\Omega(t)$ is a *classical evolution* in $(0, T)$ if:

$$\Omega(t) \text{ is admissible at each } t \in (0, T); \tag{2.3a}$$

$$\text{the evolution equation (1.5) is satisfied on each interval of smoothness of } \partial\Omega(t) \text{ (up to the endpoints).} \tag{2.3b}$$

If, in addition,

$$\Omega(0^+) = \Omega_0, \tag{2.4}$$

then $\Omega(t)$ is a *classical evolution from* Ω_0.

THEOREM 2.1 (Existence and Uniqueness of Classical Evolutions [AG2]). *Let Ω_0 be bounded and admissible. Then there is a unique maximal classical evolution $\Omega(t)$, $t \in [0, T_{\max})$; from Ω_0. Moreover, $\partial\Omega(t)$ is piecewise C^∞ at each $t \in (0, T_{\max})$.*

By definition, if the boundary curve $\partial\Omega(t)$ of a classical evolution $\Omega(t)$ has a corner corresponding to an angle jump from θ_1 to θ_2, then (θ_1, θ_2) is a GUS angle-interval and $\mathbf{C}(\theta_1) = \mathbf{C}(\theta_2)$. Suppose that θ_1, θ_2 is such a pair. Then we can construct classical evolutions, called *(θ_1, θ_2)-wrinklings*, whose normal angles jump back and forth between θ_1 and θ_2 [AG1, Sect. 9]; the flat portions of the wrinkling with angle θ_i $(i = 1, 2)$ are then called *θ_i-facets*. By (1.5), each θ_i-facet evolves according to

$$V = -b(\theta_i)\, U, \tag{2.5}$$

[10] The motivation for considering such regions can be found in [AG1, Sect. 9; AG2, Sect. 2; G2, Sect. 11].

[11] An assumption of piecewise smoothness for a boundary curve Γ will always contain the tacit assumption that Γ is locally graphlike, so that, e.g., sets with a "figure 8" boundary are ruled out.

and from this we may conclude that the wrinkling itself evolves as a rigid body with velocity $\boldsymbol{\omega}$ given by (1.16). A (θ_1, θ_2)-wrinkling with a *single corner* is called a (θ_1, θ_2)-*wedge*. A (θ_1, θ_2)-wedge $\Omega(t)$ is prescribed by specifying: (i) whether $\Omega(t)$ is convex or concave; (ii) the position of the corner at some time.

Suppose that $\partial\Omega_0$ is a piecewise flat curve whose normal angle jumps back and forth between θ_1 and θ_2, with $\partial\Omega_0$ the c-level set of an auxiliary function Φ_0; i.e.,

$$\partial\Omega_0 = \{\mathbf{x} \in \mathbb{R}^2 \colon \Phi_0(\mathbf{x}) = c\}, \qquad \Omega_0 = \{\mathbf{x} \in \mathbb{R}^2 \colon \Phi_0(\mathbf{x}) \geqslant c\}. \tag{2.6}$$

Then Ω_0 is the initial set of a (θ_1, θ_2)-wrinkling $\Omega(t)$ if and only if

$$\Omega(t) = \Omega_0 + t\boldsymbol{\omega} = \{\mathbf{x} \in \mathbb{R}^2 \colon \Phi(t, \mathbf{x}) \geqslant c\}, \qquad \Phi(t, \mathbf{x}) = \Phi_0(\mathbf{x} - t\boldsymbol{\omega}). \tag{2.7}$$

3. Viscosity Solutions: Relaxed Evolutions

We will use the relaxed equation (1.1) to discuss evolution from an initial region that has normal-angles θ with $g(\theta) < 0$. In the derivation of (1.1), G and B are the effective moduli for f and b, but we will generally require only that G and B satisfy (1.2).

A. *Definitions*

We are interested in the relaxed evolution problem defined by the relaxed equation (1.1) supplemented by the initial condition (2.4)

$$B(\theta)\, V = G(\theta)\, K - U, \qquad \Omega(0^+) = \Omega_0. \tag{E}$$

Suppose that B, G, and Ω_0 are such that (E) has a smooth solution $\Omega(t)$ with $\partial\Omega(t)$ the c-level set of an auxiliary function Φ:

$$\partial\Omega(t) = \{\mathbf{x} \in \mathbb{R}^2 \colon \Phi(t, \mathbf{x}) = c\}, \qquad \Omega(t) = \{\mathbf{x} \in \mathbb{R}^2 \colon \Phi(t, \mathbf{x}) \geqslant c\}. \tag{3.1}$$

Assume further that Φ is a smooth function whose spatial gradient $\nabla\Phi$ has $|\nabla\Phi(t, \mathbf{x})|$ never zero on $\partial\Omega(t)$. Then Φ satisfies the PDE

$$\Phi_t = \mathscr{F}(\nabla\Phi, \nabla^2\Phi), \tag{3.2}$$

where $\nabla^2\Phi$ is the Hessian matrix of second spatial derivatives of Φ, while

$$\begin{aligned} \mathscr{F}(\mathbf{p}, \mathbf{A}) &= B(\theta)^{-1}\,\{G(\theta)\,\mathbf{T}(\theta)\cdot\mathbf{A}\mathbf{T}(\theta) - U\,|\mathbf{p}|\} \\ &= B(\theta)^{-1}\,\{G(\theta)\,\mathrm{tr}[(I - \bar{\mathbf{p}}\otimes\bar{\mathbf{p}})\,\mathbf{A}] - U\,|\mathbf{p}|\}, \qquad (3.3) \\ \theta &= \sin^{-1}(-\bar{p}_2), \qquad \bar{\mathbf{p}} = \mathbf{p}/|\mathbf{p}| \end{aligned}$$

for all vectors $\mathbf{p} \neq \mathbf{0}$ and all symmetric matrices $\mathbf{A}$. Thus solving (E) at least formally reduces to solving (3.2) subject to an initial condition

$$\Phi(\mathbf{x}, 0) = \Phi_0(\mathbf{x}) \tag{3.4}$$

for all $\mathbf{x} \in \mathbb{R}^2$, where Φ_0 is an auxiliary function satisfying

$$\partial\Omega_0 = \{\mathbf{x} \in \mathbb{R}^2 \colon \Phi_0(\mathbf{x}) = c\}, \qquad \Omega_0 = \{\mathbf{x} \in \mathbb{R}^2 \colon \Phi_0(\mathbf{x}) \geqslant c\}. \tag{3.5}$$

$\mathscr{F}$ defined by (3.3) has two chief properties upon which much of the level-set theory of (3.2) is based: the *geometric property*

$$\mathscr{F}(\lambda\mathbf{p}, \lambda\mathbf{A} + \nu\mathbf{p} \otimes \mathbf{p}) = \lambda\mathscr{F}(\mathbf{p}, \mathbf{A}) \tag{3.6}$$

for all $\lambda \geqslant 0$, $\nu \in \mathbb{R}$; and the *elliptic property*

$$\mathscr{F}(\mathbf{p}, \mathbf{A} + \mathbf{B}) \geqslant \mathscr{F}(\mathbf{p}, \mathbf{A}) \tag{3.7}$$

whenever $\mathbf{B}$ is symmetric and positive semi-definite.

The level-set method is not intrinsic, since it requires data irrelevant to the problem: namely, the values of Φ_0 away from an arbitrary small neighborhood of $\partial\Omega_0$. A method of circumventing this is to work with the characteristic function[12]

$$u(t, \mathbf{x}) = \chi_{\Omega(t)}(\mathbf{x}) \tag{3.8}$$

of the region $\Omega(t)$. It is reasonable to expect that u should, in some sense, satisfy (3.2), an expectation motivated by viewing u as the limit of a sequence $\{\Phi_k\}$ of functions Φ_k consistent with (3.3) for, say, $c = 1/2$. We will use the theory of viscosity solutions[13] to define the sense in which u satisfies (3.2).

Let h be a bounded scalar function on a subset $\mathscr{H}$ of $\mathbb{R}^n$; then h^* and h_*, respectively, denote the *upper* and *lower semicontinuous envelopes* of h defined on cl $\mathscr{H}$ by

$$h^*(\mathbf{z}) = \limsup_{\mathbf{q} \to \mathbf{z}} h(\mathbf{q}), \qquad h_*(\mathbf{z}) = \liminf_{\mathbf{q} \to \mathbf{z}} h(\mathbf{q}), \qquad \mathbf{q} \in \mathscr{H}. \tag{3.9}$$

Let u be a bounded function on $[0, \infty) \times \mathbb{R}^2$. Then u is a *viscosity subsolution* of (3.4) if, for every (scalar) *test function* $w \in C^{1,2}((0, \infty) \times \mathbb{R}^2)$,

$$w_t(t_0, \mathbf{x}_0) \leqslant \mathscr{F}^*(\nabla w(t_0, \mathbf{x}_0), \nabla^2 w(t_0, \mathbf{x}_0)) \tag{3.10}$$

[12] Cf. [BSS].

[13] Crandall and Lions [CL], Crandall *et al.* [CEL], and Jensen [Je]. A recent article of Crandall *et al.* [CIL] provides an excellent survey of the subject.

at every local maximum of $u^* - w$; u is a *viscosity supersolution* of (3.4) if, for every such w,

$$w_t(t_0, \mathbf{x}_0) \geqslant \mathscr{F}_*(\nabla w(t_0, \mathbf{x}_0), \nabla^2 w(t_0, \mathbf{x}_0)) \tag{3.11}$$

at every local minimum of $u_* - w$; u is a *viscosity solution* of (3.4) if u is both a viscosity subsolution and a viscosity supersolution of (3.4) [CGG]. We will also use viscosity subsolutions, supersolutions, and solutions on *finite* time intervals $(0, T)$.

Let $\Omega(t)$, $t \geqslant 0$, be given, and define $u(t, \mathbf{x}) = \chi_{\Omega(t)}(\mathbf{x})$. Then $\Omega(t)$, $t \geqslant 0$, is a *χ-subsolution* or a *χ-supersolution* of (1.1) according as u is a viscosity subsolution or viscosity supersolution of (3.4) and $\Omega(t)$ is uniformly bounded on compact time intervals; $\Omega(t)$, $t \geqslant 0$, is a *relaxed evolution* if it is both a χ-subsolution and a χ-supersolution of (1.1).

Let

$$\Omega^*(0^+) = \{\mathbf{x} \in \mathbb{R}^2 \colon \limsup_{t \to 0^+, \mathbf{y} \to \mathbf{x}} u^*(t, \mathbf{y}) = 1\},$$

$$\Omega_*(0^+) = \{\mathbf{x} \in \mathbb{R}^2 \colon \liminf_{t \to 0^+, \mathbf{y} \to \mathbf{x}} u_*(t, \mathbf{y}) = 1\},$$

so that $\Omega^*(0^+)$ is closed, while $\Omega_*(0^+)$ is open. Then $\Omega(t)$, $t \geqslant 0$, is:

(a) *χ-subsolution* of (1.1) *compatible with* Ω_0 if it is a χ-subsolution and $\Omega^*(0^+) \subseteq \operatorname{cl} \Omega_0$;

(b) a *χ-supersolution* of (1.1) *compatible with* Ω_0 if it is a χ-supersolution and $\Omega_*(0^+) \supseteq \operatorname{int} \Omega_0$;

(c) a *relaxed evolution from* Ω_0 if it is a χ-subsolution of (1.1) compatible with Ω_0 as well as a χ-supersolution of (1.1) compatible with Ω_0.

Note that if a relaxed evolution is to take on initial data Ω_0 in a classical sense, then Ω_0 must be regular (i.e., $\operatorname{cl} \Omega = \operatorname{cl}(\operatorname{int} \Omega_0)$).

One should expect lack of (global) uniqueness for relaxed evolutions from a given initial set;[14] with this in mind, we introduce the following definitions: the *upper* and *lower envelopes* $\mathscr{U}(t)$ and $\mathscr{L}(t)$ for relaxed evolutions from an initial set Ω_0 are defined at each $t \geqslant 0$ by[15]

[14] For G continuous there are conditions that guarantee the uniqueness of solutions [BSS, So]. For motion by mean curvature and smooth initial data ($V = K$) uniqueness holds generically [ES3].

[15] Cf. [So, Sect. 11].

$$\mathscr{U}(t) = \mathrm{cl}\{\text{union of all values at } t \text{ of } \chi\text{-subsolutions of (1.1) compatible with } \Omega_0\},$$

$$\mathscr{L}(t) = \mathrm{int}\{\text{intersection of all values at } t \text{ of } \chi\text{-supersolutions of (1.1) compatible with } \Omega_0\};$$

the *graph up to time* T of a time-dependent set $A(t)$ is defined by

$$\mathrm{graph}_T A = \bigcup_{0 \leqslant t \leqslant T} [A(t) \times \{t\}];$$

the time

$$T_{\mathrm{uniq}} = \sup\{T\colon \mathrm{graph}_T \mathscr{U} = \mathrm{cl}(\mathrm{graph}_T \mathscr{L}) \text{ and } \mathrm{int}(\mathrm{graph}_T \mathscr{U}) = \mathrm{graph}_T \mathscr{L}\}$$

is the *uniqueness time* for relaxed evolutions from Ω_0 and, for $T_{\mathrm{uniq}} > 0$,

$$\Omega_{\mathrm{uniq}}(t) = \mathscr{U}(t) = \mathrm{cl}\, \mathscr{L}(t), \qquad t \in [0, T_{\mathrm{uniq}})$$

is the *unique relaxed evolution* from Ω_0.

B. *Existence and Uniqueness*

We assume throughout this subsection that

G and B satisfy (1.2);

Ω_0 is a prescribed initial domain, assumed compact.

(Note that we do not require the consistency of B with (1.3)).

THEOREM 3.1 (Existence and Local Uniqueness of Relaxed Evolutions).

(a) *there is at least one relaxed evolution from* Ω_0;

(b) *the upper and lower envelopes are relaxed evolutions from* Ω_0.

If, in addition, $\partial\Omega_0$ *is* C^3, *then*

(c) *the uniqueness time for relaxed evolutions from* Ω_0 *is strictly positive*;[16]

We postpone, until Section 8, the proof of this theorem and the next.

Let $\hat{M}([0, T] \times \mathbb{R}^2)$ denote the set of all bounded functions on $[0, T] \times \mathbb{R}^2$ that are equal to a constant outside of a large ball; i.e., $\varphi \in \hat{M}([0, T] \times \mathbb{R}^2)$ if and only if there are constants α and R such that

[16] If Ω_0 is strictly star-shaped, then $T_{\mathrm{uniq}} = \infty$ (cf. [So, Sect. 9]; a more general condition is given by [BSS, Sect. 4]).

$\varphi(t, \mathbf{x}) = \alpha$ for $|\mathbf{x}| \geqslant R$; here α and R may depend on φ. We define $\hat{M}(\mathbb{R}^2)$ similarly. Finally,

$$\hat{M}([0, \infty) \times \mathbb{R}^2) = \bigcap_{T>0} \hat{M}([0, T] \times \mathbb{R}^2);$$

i.e., $\varphi \in \hat{M}([0, \infty) \times \mathbb{R}^2)$ if and only if for every T there are constants α_T and R_T satisfying $\varphi(t, \mathbf{x}) = \alpha_T$ for $|\mathbf{x}| \geqslant R_T$ and $t \in [0, T]$.

Let Φ_0 be an *auxiliary function* for the initial set Ω_0; that is, a continuous function $\Phi_0 \in \hat{M}(\mathbb{R}^2)$ satisfying (3.5). Then $\Phi \in \hat{M}([0, \infty) \times \mathbb{R}^2)$ is a *level-set solution* of (1.1) if Φ is a *continuous* viscosity solution of (3.2); if, in addition, Φ satisfies the initial condition (3.4), then Φ *corresponds to* Φ_0.

THEOREM 3.2 (Existence and Uniqueness of Level-Set Solutions).[17] *There is a unique level-set solution of* (1.1) *corresponding to any given choice of auxiliary function* Φ_0 *for* Ω_0. *Moreover, the upper and lower envelopes for relaxed evolutions from* Ω_0 *are given by*

$$\mathscr{U}(t) = \{\mathbf{x} \colon \Phi(t, \mathbf{x}) \geqslant c\}, \qquad \mathscr{L}(t) = \{\mathbf{x} \colon \Phi(t, \mathbf{x}) > c\}. \tag{3.12}$$

Thus the sets $\{\mathbf{x} \colon \Phi(t, \mathbf{x}) = c\}$, $\{\mathbf{x} \colon \Phi(t, \mathbf{x}) \geqslant c\}$, and $\{\mathbf{x} \colon \Phi(t, \mathbf{x}) > c\}$ are independent of the choice of auxiliary function Φ_0.[18]

C. *Comparison*

In this subsection, we state comparison theorems related to weak solutions of (1.1) and (3.2). We assume throughout that

$$G \text{ and } B \text{ satisfy } (1.2);$$

we do not require (1.3). The next theorem is the key technical result of the paper.

THEOREM 3.3.[19] *Let* $\varphi \in \hat{M}([0, T] \times \mathbb{R}^2)$ *be a viscosity subsolution and* $\psi \in \hat{M}([0, T] \times \mathbb{R}^2)$ *a viscosity supersolution, both of* (3.2) *on* $(0, T) \times \mathbb{R}^2$. *Then*

$$\sup_{[0, T] \times \mathbb{R}^2} (\varphi^* - \psi_*) = \sup_{\mathbf{y} \in \mathbb{R}^2} [\varphi^*(0, \mathbf{y}) - \psi_*(0, \mathbf{y})]. \tag{3.13}$$

[17] For G continuous and nonnegative, uniqueness and existence follow from Theorem 6.8 of [CGG].

[18] Cf. Theorem 7.1 of [CGG] for the case in which G is continuous and nonnegative.

[19] Ohnuma and Sato [OS] have independently established this theorem using a completely different method of proof.

Suppose that $\Omega_1(t)$ and $\Omega_2(t)$ are respectively a χ-subsolution and a χ-supersolution of (1.1) and set

$$u_i(t, \mathbf{x}) = \chi_{\Omega_i(t)}(\mathbf{x}).$$

Since χ-sub- and supersolutions are assumed to be uniformly bounded on compact time intervals, $u_i \in \hat{M}([0, \infty) \times \mathbb{R}^2)$. Then (3.13) with $\varphi = u_1$ and $\psi = u_2$ yields

$$(u_1)^*(t, \mathbf{x}) - (u_2)_*(t, \mathbf{x}) \leqslant \sup_{\mathbf{y} \in \mathbb{R}^2} [(u_1)^*(0, \mathbf{y}) - (u_2)_*(0, \mathbf{y})] \tag{3.14}$$

and we have

COROLLARY 3.1 (Weak Comparison). *Let $\Omega_1(t)$ be a χ-subsolution and $\Omega_2(t)$ a χ-supersolution of (1.1). Suppose that, for all* $\mathbf{x}$,

$$(u_1)^*(0, \mathbf{x}) \leqslant (u_2)_*(0, \mathbf{x}). \tag{3.15}$$

Then for all $t \geqslant 0$,

$$\mathrm{cl}\, \Omega_1(t) \subseteq \mathrm{int}\, \Omega_2(t). \tag{3.16}$$

Condition (3.14) follows if (3.15) is satisfied at $t = 0^+$. Unfortunately, (3.14) is stronger then the requirement: $\Omega_1(0) \subseteq \Omega_2(0)$.

We say that (1.1) with initial data Ω_0 has *strong comparison* in $(0, T)$ if

$$\mathrm{graph}_T\, \Omega_1 = \mathrm{cl}(\mathrm{int}(\mathrm{graph}_T\, \Omega_2)) \tag{3.17}$$

for all $t \in (0, T)$ for every χ-subsolution $\Omega_1(t)$ of (1.1) compatible with Ω_0 and χ-supersolution $\Omega_2(t)$ of (1.1) compatible with Ω_0.

The next result follows from the definitions of the upper and lower envelopes $\mathscr{U}(t)$ and $\mathscr{L}(t)$ and the uniqueness time T_{uniq} for relaxed evolutions from Ω_0.

THEOREM 3.4. *Let $\Omega_1(t)$ be a χ-subsolution and $\Omega_2(t)$ a χ-supersolution, both of (1.1) and both compatible with Ω_0. Then for all $t \geqslant 0$,*

$$\mathrm{cl}\, \Omega_1(t) \subseteq \mathscr{U}(t), \qquad \mathrm{int}\, \Omega_2(t) \supseteq \mathscr{L}(t).$$

Thus (1.1) with initial data Ω_0 has strong comparison in $(0, T_{\mathrm{uniq}})$.

4. RELATION BETWEEN CLASSICAL AND RELAXED EVOLUTIONS

Our next theorem shows that our choice of effective moduli G and B for the relaxed problem is the only possible choice, at least if wedges are to be relaxed evolutions; what makes this result so interesting is that $G(\theta)$ and $B(\theta)$ differ from $g(\theta)$ and $b(\theta)$ only at angles θ that are not globally stable, and wedges by definition do not involve such θ.

THEOREM 4.1 (Effective Moduli are Canonical). *Let f and b be consistent with* (2.1), *let G and B be consistent with* (1.2), *and let $G(\theta) = f(\theta) + f''(\theta)$ and $B(\theta) = b(\theta)$ for all GS angles θ. Then all wedges are relaxed evolutions only if G and B are the effective moduli for f and b.*

Proof. It suffices to show that $G(\theta) = 0$ and $B(\theta)$ satisfies (1.17) on any GUS angle-interval (θ_1, θ_2). Choose such an angle-interval (θ_1, θ_2). Consider a (θ_1, θ_2)-wedge $\Omega(t)$ with corner at the origin at $t = 1$, and let $\boldsymbol{\omega}$ be the corresponding rigid velocity defined by (1.16). Assume that $\Omega(t)$ is a relaxed evolution, so that $u(t, \mathbf{x}) = \chi_{\Omega(t)}(\mathbf{x})$ is a viscosity solution of (3.2).

Let $\Omega(t)$ be convex, and let

$$w(t, \mathbf{x}, \theta) = 1 - [\mathbf{x} - (t-1)\,\boldsymbol{\omega}] \cdot \mathbf{N}(\theta)$$

for all $(t, \mathbf{x})$ and all $\theta \in (\theta_1, \theta_2)$. Then by (2.7),

$$\Omega(t) = \{\mathbf{x} \in \mathbb{R}^2 \colon w(t, \mathbf{x}, \theta) \geqslant 1,\ \theta = \theta_1, \theta_2\}.$$

Fix $\theta \in (\theta_2, \theta_2)$. Then,

$$u^*(t, \mathbf{x}) - w(t, \mathbf{x}, \theta) \leqslant u^*(1, \mathbf{0} - w(1, \mathbf{0}, \theta) = 0$$

for all $(t, \mathbf{x})$ near $(1, \mathbf{0})$. Thus, since u is a viscosity solution of (3.2),

$$w_t(1, \mathbf{0}, \theta) \leqslant \mathscr{F}^*(\nabla w(1, \mathbf{0}, \theta), \nabla^2 w(1, \mathbf{0}, \theta)).$$

Further,

$$w_t(1, \mathbf{0}, \theta) = \boldsymbol{\omega} \cdot \mathbf{N}(\theta), \qquad \nabla w(1, \mathbf{0}, \theta) = -\mathbf{N}(\theta), \qquad \nabla^2 w(1, \mathbf{0}, \theta) = \mathbf{0},$$

and (3.3) yields

$$\mathscr{F}^*(\nabla w(1, \mathbf{0}, \theta), \nabla^2 w(1, \mathbf{0}, \theta)) = -U/B(\theta);$$

hence

$$B(\theta) \leqslant -U[\boldsymbol{\omega} \cdot \mathbf{N}(\theta)]^{-1}. \tag{4.1}$$

Now let $\Omega(t)$ be *concave*, and let

$$w(t, \mathbf{x}) = w(t, \mathbf{x}, \theta) = -[\mathbf{x} - (t-1)\,\boldsymbol{\omega}] \cdot \mathbf{N}(\theta)$$

with θ fixed. Then $u_* - w$ has a local minimum at $(t, \mathbf{x}) = (1, \mathbf{0})$, so that, arguing as before,

$$B(\theta) \geqslant -U[\boldsymbol{\omega} \cdot \mathbf{N}(\theta)]^{-1}.$$

Thus, appealing to (4.1),

$$B(\theta) = -U[\boldsymbol{\omega} \cdot \mathbf{N}(\theta)]^{-1} \qquad \text{for all} \quad \theta \in (\theta_1, \theta_2), \tag{4.2}$$

and (1.17) follows from (1.16), (4.2), and (1.12).

Next, to show that $G(\theta) = 0$ on (θ_1, θ_2), we again take $\Omega(t)$ to be convex, and let

$$\hat{w}(t, \mathbf{x}, \theta) = 1 - [\mathbf{x} - (t-1)\,\boldsymbol{\omega}] \cdot \mathbf{N}(\theta) + \beta[\mathbf{x} - (t-1)\,\boldsymbol{\omega}]^2$$

for all $(t, \mathbf{x})$ and all $\theta \in (\theta_1, \theta_2)$. Fix θ and write $\hat{w}(t, \mathbf{x}) = \hat{w}(t, \mathbf{x}, \theta)$.

We first show that, given any $\beta \in \mathbb{R}$, $u^* - \hat{w}$ has a local maximum at $(t, \mathbf{x}) = (1, 0)$. Choose $(\tau, \mathbf{y})$ with $u^*(\tau, \mathbf{y}) = 1$. Then

$$[\mathbf{y} - (\tau - 1)\,\boldsymbol{\omega}] \cdot \mathbf{N}(\theta_i) \leqslant 0,$$

$i = 1, 2$, and, since $\theta \in (\theta_1, \theta_2)$,

$$[\mathbf{y} - (\tau - 1)\,\boldsymbol{\omega}] \cdot \mathbf{N}(\theta) \leqslant -\alpha < 0,$$

$\alpha = \alpha(\theta)$. Therefore if $(\tau, \mathbf{y})$ is close enough to $(1, \mathbf{0})$ that $|\mathbf{y} - (\tau - 1)\boldsymbol{\omega}| \leqslant \alpha/2\beta$, then

$$\begin{aligned} 1 = u^*(\tau, \mathbf{y}) &\leqslant 1 + \tfrac{1}{2}\alpha\, |\mathbf{y} - (\tau - 1)\,\boldsymbol{\omega}| \\ &\leqslant 1 - [\mathbf{x} - (t-1)\,\boldsymbol{\omega}] \cdot \mathbf{N}(\theta) + \beta[\mathbf{x} - (t-1)\,\boldsymbol{\omega}]^2 = \hat{w}(\tau, \mathbf{y}). \end{aligned}$$

Further,

$$\hat{w}(\tau, \mathbf{y}) \geqslant 0 \qquad \text{if} \quad |\mathbf{y} - (\tau - 1)\,\boldsymbol{\omega}| + |\beta|\,[\mathbf{y} - (\tau - 1)\,\boldsymbol{\omega}]^2 \leqslant 1,$$

and hence

$$u^*(\tau, \mathbf{y}) - \hat{w}(\tau, \mathbf{y}) \leqslant 0 = u^*(1, \mathbf{0}) - \hat{w}(1, \mathbf{0})$$

for all $(\tau, \mathbf{y})$ sufficiently close to $(1, \mathbf{0})$; thus

$$\hat{w}_t(1, \mathbf{0}) \leqslant \mathscr{F}^*(\nabla\hat{w}(1, \mathbf{0}), \nabla^2\hat{w}(1, \mathbf{0})).$$

Further,

$$\hat{w}_t(1, 0) = \boldsymbol{\omega} \cdot \mathbf{N}(\theta), \qquad \nabla\hat{w}(1, \mathbf{0}) = -\mathbf{N}(\theta), \qquad \nabla^2\hat{w}(1, \mathbf{0}) = 2\beta I,$$

$$\mathscr{F}^*(\nabla\hat{w}(1, \mathbf{0}), \nabla^2\hat{w}(1, \mathbf{0})) = (2\beta G(\theta) - U)/B(\theta);$$

hence

$$\boldsymbol{\omega} \cdot \mathbf{N}(\theta) \leqslant (2\beta G(\theta) - U)/B(\theta), \tag{4.3}$$

and this must hold for all $\beta \in \mathbb{R}$ and $\theta \in (\theta_1, \theta_2)$. On the other hand, by (4.2), $\boldsymbol{\omega} \cdot \mathbf{N}(\theta) = -U/B(\theta)$, and (4.3) can hold for all β only if $G(\theta) = 0$. ∎

THEOREM 4.2 (Classical Evolutions are Relaxed Evolutions). *Let Ω_0 be bounded and admissible. Let G and B be the effective moduli corresponding to f and b, with f and b consistent with (2.1). Let $\Omega(t)$, $t \in [0, T_{\max})$, be the maximal classical evolution from Ω_0. Then the uniqueness time T_{uniq} for relaxed evolutions from Ω_0 satisfies $T_{\text{uniq}} \geqslant T_{\max}$ and $\Omega(t)$ coincides with the unique relaxed evolution $\Omega_{\text{uniq}}(t)$ for all $t \in [0, T_{\max})$.*

Proof. Let $\Omega(t)$, $0 < t < T_{\max}$ be a classical evolution. We will show only that $\Omega(t)$ is a χ-subsolution; the proof that $\Omega(t)$ is a χ-supersolution is analogous. Let $u(t, \mathbf{x}) = \chi_{\Omega(t)}(\mathbf{x})$. Suppose that for a test function w

$$u^*(t, \mathbf{x}) - w(t, \mathbf{x}) \leqslant u^*(t_0, \mathbf{x}_0) - w(t_0, \mathbf{x}_0) = 0$$

for all $(t, \mathbf{x})$ near $(t_0, \mathbf{x}_0)$.

Case 1. $\mathbf{x}_0 \in \operatorname{int} \Omega(t_0)$. Then $u^*(t, \mathbf{x}) = 1$ for all $(t, \mathbf{x})$ near $(t_0, \mathbf{x}_0)$ and

$$w_t(t_0, \mathbf{x}_0) = 0, \qquad \nabla w(t_0, \mathbf{x}_0) = \mathbf{0}, \qquad \nabla^2 w(t_0, \mathbf{x}_0) \geqslant 0.$$

Hence

$$\mathscr{F}^*(\nabla w(t_0, \mathbf{x}_0), \nabla^2 w(t_0, \mathbf{x}_0)) \geqslant 0$$

and (3.10) is satisfied.

Case 2. $\mathbf{x}_0 \in \mathbb{R}^2 \backslash \bar{\Omega}(t_0)$. Then $u^*(t, \mathbf{x}) = 0$ for all $(t, \mathbf{x})$ near $(t_0, \mathbf{x}_0)$ and an analysis similar to that of Case 1 yields (3.10).

Case 3. $\mathbf{x}_0 \in \partial\Omega(t_0)$ and $\nabla w(t_0, \mathbf{x}_0) = \mathbf{0}$. Then $w_t(t_0, \mathbf{x}_0) = 0$, since the normal velocity V of $\partial\Omega(t)$ is finite. Moreover, the definition of the upper semicontinuous envelope yields

$$\mathscr{F}^*(\nabla w(t_0, \mathbf{x}_0), \nabla^2 w(t_0, \mathbf{x}_0)) = G(\theta)\, B(\theta)^{-1} \max\{\mathbf{q} \cdot \nabla^2 w(t_0, \mathbf{x}_0)\, \mathbf{q} \colon |\mathbf{q}| = 1\}.$$

We claim that the quantity $\max\{\cdots\}$ is nonnegative. Indeed, $\mathbf{x}_0 \in \partial\Omega(t_0)$ and $t_0 < T_{\max}$; hence $\mathbf{x}_0$ is not an isolated point of $\Omega(t_0)$ and there is a sequence $\{\mathbf{x}_n\}$ with $\mathbf{x}_n \in \partial\Omega(t_0)$, $\mathbf{x}_n \neq \mathbf{x}_0$, and $\mathbf{x}_n \to \mathbf{x}_0$. By choosing a subsequence, if necessary, we may assume that $(\mathbf{x}_n - \mathbf{x}_0)/|\mathbf{x}_n - \mathbf{x}_0|$ is convergent, say to $\mathbf{e}$. Then

$$w(t_0, \mathbf{x}_n) \geqslant u^*(t_0, \mathbf{x}_n) = 1 = w(t_0, \mathbf{x}_0),$$
$$\mathbf{e} \cdot \nabla^2 w(t_0, \mathbf{x}_0)\, \mathbf{e} = 2 \lim_{n \to \infty} [w(t_0, \mathbf{x}_n) - w(t_0, \mathbf{x}_0)]/|\mathbf{x}_n - \mathbf{x}_0|^2, .$$

so that

$$0 = w_t(t_0, \mathbf{x}_0) \leqslant \mathscr{F}^*(\nabla w(t_0, \mathbf{x}_0), \nabla^2(t_0, \mathbf{x}_0)).$$

Case 4. $\mathbf{x}_0$ belongs to a smooth part of $\partial\Omega(t_0)$ and $|\nabla w(t_0, \mathbf{x}_0)| \neq 0$. Then the normal angle θ, the curvature K, and the normal velocity V of $\partial\Omega(t)$ at $t = t_0$ and $\mathbf{x}_0$ satisfy

$$N(\theta) = -\nabla w(t_0, \mathbf{x}_0)/|\nabla w(t_0, \mathbf{x}_0)|,$$
$$K \leqslant \operatorname{div}[\nabla w/|\nabla w|](t_0, \mathbf{x}_0),$$
$$V = w_t(t_0, \mathbf{x}_0)/|\nabla w(t_0, \mathbf{x}_0)|,$$

and (3.10) follows from (1.15).

Case 5. $\mathbf{x}_0$ is a corner point of $\partial\Omega(t_0)$ and $|\nabla w(t_0, \mathbf{x}_0)| \neq 0$. Let (θ_1, θ_2) be the GUS angle-interval that defines the corner, and let $\mathbf{z}(t)$ with $\mathbf{z}(t_0) = \mathbf{x}_0$ denote the trajectory of the corner for t near t_0. Then $\partial\Omega(t)$ must have a "convex-type" corner of the type shown in Figure 4.1 near $\mathbf{z}(t)$, with curvature $K(\mathbf{x}, t) \leqslant 0$ for $\mathbf{x}$ near but not equal to $\mathbf{z}(t)$. Thus and by (1.5), for such $\mathbf{x}$ the normal velocity of $\partial\Omega(t)$ must satisfy

$$V(\mathbf{x}, t) \leqslant -U/b(\theta(\mathbf{x}, t)). \tag{4.4}$$

On the other hand, consider the simple (θ_1, θ_2)-wedge $\partial A(t)$ which has $A(t)$ convex and has corner at $\mathbf{x}_0$ at time t_0, and let $\boldsymbol{\omega}$ be the rigid velocity of the wedge as defined by (1.16). Then, since this wedge must have normal velocity $V = -U/b(\theta_i)$ on each of its facets, we may conclude from (4.4) that there is a ball $\mathscr{B}$ centered at $\mathbf{x}_0$ such that $A(t) \cap \mathscr{B} \subseteq \Omega(t) \cap \mathscr{B}$ for all t near t_0 with $t \leqslant t_0$; thus

$$A(t) \cap \mathscr{B} \subseteq \{\mathbf{x}\colon w(t, \mathbf{x}) \geqslant 1\}.$$

Further, since $A(t)$ moves with rigid velocity $\boldsymbol{\omega}$ and $\mathbf{x}_0$ is the corner point of $A(t_0)$,

$$\mathbf{x}_0 + (t - t_0)[\boldsymbol{\omega} \cdot \mathbf{N}(\alpha)]\, \mathbf{N}(\alpha) \in A(t)$$

for all t and all $\alpha \in [\theta_1, \theta_2]$. Thus for such α and for t near t_0 with $t \leqslant t_0$,

$$w(t, \mathbf{x}_0 + (t - t_0)[\boldsymbol{\omega} \cdot \mathbf{N}(\alpha)]\, \mathbf{N}(\alpha)) \geqslant 1 = w(t_0, \mathbf{x}_0)$$

and it follows that

$$w_t(t_0, \mathbf{x}_0) + [\boldsymbol{\omega} \cdot \mathbf{N}(\alpha)][\nabla w(t_0, \mathbf{x}_0) \cdot \mathbf{N}(\alpha)] \leqslant 0 \tag{4.5}$$

for all $\alpha \in [\theta_1, \theta_2]$.

Now let $\alpha \in [\theta_1, \theta_2]$ be the angle defined by

$$\nabla w(t_0, \mathbf{x}_0)/|\nabla w(t_0, \mathbf{x}_0)| = -\mathbf{N}(\alpha);$$

then (4.5) yields

$$w_t(t_0, \mathbf{x}_0) \leqslant \boldsymbol{\omega} \cdot \mathbf{N}(\alpha)\, |\nabla w(t_0, \mathbf{x}_0)|, \tag{4.6}$$

and, by (4.2),

$$w_t(t_0, \mathbf{x}_0) \leqslant -U\, |\nabla w(t_0, \mathbf{x}_0)|/B(\alpha). \tag{4.7}$$

If $\alpha \in (\theta_1, \theta_2)$, then $G(\alpha) = 0$ and (3.3) yields

$$\mathscr{F}^*(\nabla w(t_0, \mathbf{x}_0), \nabla^2 w(t_0, \mathbf{x}_0)) = -U\, |\nabla w(t_0, \mathbf{x}_0)|/B(\alpha), \tag{4.8}$$

so that, by (4.7), (3.10) is satisfied. If α equals θ_1 or θ_2, then $G(\alpha)$ is generally nonzero, but the definition of $\mathscr{F}^*$ yields

$$\mathscr{F}^*(\mathbf{p}, \mathbf{A}) \geqslant \limsup_{n \to \infty} \mathscr{F}^*(\mathbf{p}_n, \mathbf{A})$$

for any sequence $\mathbf{p}_n \to \mathbf{p}$. Let $\mathbf{p} = \nabla w(t_0, \mathbf{x}_0)$ and choose a sequence so that $G(\theta_n) = 0$ for all n, where θ_n is defined by $\mathbf{N}(\theta_n) = \mathbf{p}_n/|\mathbf{p}_n|$. Then (4.8) is replaced by

$$\mathscr{F}^*(\nabla w(t_0, \mathbf{x}_0), \nabla^2 w(t_0, \mathbf{x}_0)) \geqslant -U\, |\nabla w(t_0, \mathbf{x}_0)|/B(\alpha), \tag{4.9}$$

which, with (4.7), yields (3.10).

We have only to show that

$$T_{\text{uniq}} \geqslant T_{\max}. \tag{4.10}$$

Given Ω_0 we can construct a one-parameter family of *admissible* initial domains $\Omega_0(\delta)$ ($|\delta| \leqslant \delta_0$ for some $\delta_0 > 0$) satisfying

$$\Omega_0(\delta) \subset \Omega_0(\delta') \qquad \text{if} \quad \delta > \delta', \tag{4.11a}$$

$$\lim_{\delta' \to \delta} \Omega_0(\delta') = \Omega_0(\delta), \tag{4.11b}$$

the limit being in the *Hausdorf metric*.[20] Let $\Omega(t; \delta)$, ($t \in [0, T_{\max}(\delta))$, $|\delta| \leqslant \delta_0$) be the unique maximal evolution from the initial data $\Omega_0(\delta)$

[20] For bounded sets the Hausdorf metric $d_H(A, B)$ is the largest of the distances $\sup\{\operatorname{dist}(\mathbf{x}, A)\colon \mathbf{x} \in B\}$ and $\sup\{\operatorname{dist}(\mathbf{x}, B)\colon \mathbf{x} \in A\}$.

(cf. Theorem 2.1). Since for $t \in (0, T_{\max}(\delta))$, $\partial\Omega(t; \delta)$ is piecewise smooth, we may use the compactness lemma [AG2, Lemma 8.3] and the uniqueness of classical solutions to show that: (i) $T_{\max}(\delta)$ is lower semicontinuous in δ (cf. also the proof of [AG2, Lemma 8.2]); (ii) by Corollary 3.1, $\Omega(t; \delta) \subset \Omega(t; \delta')$ if $\delta \geqslant \delta'$; (iii) for fixed t, the map $\delta \mapsto \Omega(t; \delta)$ is continuous in the Hausdorf metric. (Assertion (iii) is proved by showing that any limit point of $\Omega(t; \delta)$ as $\delta \to \delta'$ is a classical solution with initial data $\Omega_0(\delta')$ and hence by uniqueness is equal to $\Omega(t; \delta')$.

Fix $T < T_{\max} = T_{\max}(0)$. By the lower semicontinuity of $T_{\max}(\delta)$ there is a $\delta(T) > 0$ satisfying

$$T_{\max}(\delta) \geqslant T \qquad \text{for all} \quad |\delta| \leqslant \delta(T).$$

For $(t, \mathbf{x}) \in [0, T] \times \mathbb{R}^2$ define

$$\Phi(t, \mathbf{x}) = \begin{cases} \inf\{\delta\colon |\delta| \leqslant \delta(T),\ \mathbf{x} \in \Omega(t; \delta)\}, \\ -\delta(T) \qquad \text{if the set above is empty.} \end{cases}$$

Since $\Omega(t; \delta) \subseteq \Omega(t; \delta')$ for $\delta \geqslant \delta'$,

$$\{\mathbf{x}\colon \Phi(t, \mathbf{x}) \geqslant \delta\} = \Omega(t; \delta)$$

whenever $|\delta| \leqslant \delta(T)$ and $t \in [0, T]$. Moreover, for each δ, $\Omega(t; \delta)$ is a classical and therefore related evolution from $\Omega_0(\delta)$. From this one can show that Φ is a continuous viscosity solution of (1.1), so that, by Theorem 3.2,

$$\mathscr{U}(t) = \{\mathbf{x}\colon \Phi(t, \mathbf{x}) \geqslant 0\} = \Omega(t; 0) = \Omega(t), \qquad \mathscr{L}(t) = \{\mathbf{x}\colon \Phi(t, \mathbf{x}) > 0\}.$$

Since

$$\lim_{\delta \downarrow 0} \Omega(t; \delta) = \Omega(t; 0)$$

for every $\mathbf{x} \in \mathscr{U}(t)$, there are $\delta_n > 0$ and $\mathbf{x}_n \to \mathbf{x}$ such that $\mathbf{x}_n \in \Omega(t; \delta_n) \subset \mathscr{L}(t)$. Hence $\operatorname{cl} \mathscr{L}(t) = \mathscr{U}(t)$ for all $t \in [0, T]$. An analogous argument shows that $\operatorname{int} \mathscr{U}(t) = \mathscr{L}(t)$ at each $t \in [0, T]$. Hence $T_{\text{uniq}} \geqslant T$, and the desired conclusion follows, since $T < T_{\max}$ was chosen arbitrarily. ∎

Remark 4.1. For bounded, admissible initial data Ω_0 there exist a maximal existence time $T_{\max}$ and a classical evolution $\hat{\Omega}(t)$, $t \in [0, T_{\max})$, from Ω_0. There is also (at least one) relaxed evolution $\Omega(t)$, $t \in [0, \infty)$, from Ω_0, and, by Theorem 4.2,

$$\hat{\Omega}(t) = \Omega(t) \qquad \text{for all} \quad t \in [0, T_{\max}).$$

Hence the relaxed evolution represents a weak extension of the classical evolution $\hat{\Omega}(t)$ after $\hat{\Omega}(t)$ develops a singularity at $t = T_{\max}$.

5. CONVERGENCE

Throughout this section

G and B satisfy (1.2);

Ω_0 is a prescribed initial domain, assumed compact;

T_{uniq} is the uniqueness time for relaxed evolutions from Ω_0.

A. *General Results*

We say that a sequence $\{\Omega_0^n\}$ of compact domains *approximates* Ω_0 provided the signed distance to Ω_0^n approaches the signed distance to Ω_0, uniformly on $\mathbb{R}^2$.

THEOREM 5.1 (Convergence of Relaxed Evolutions). *Assume that* $T_{\text{uniq}} > 0$. *Let* $\{\Omega_0^n\}$ *approximate* Ω_0. *For each integer n, let* $\Omega^n(t)$, $t \in [0, \infty)$, *be a relaxed evolution from* Ω_0^n. *Then, for each* $t \in [0, T_{\text{uniq}})$, $\Omega^n(t)$ *converges, in the Hausdorf topology, to the unique relaxed evolution from* Ω_0.

We now state a result that holds for all time. The proof will be given at the end of this section, as will the proof of the theorem just stated.

THEOREM 5.2. (Convergence of Level-Set Solutions). *Let* Φ *be the unique level-set solution corresponding to an auxiliary function* Φ_0 *for* Ω_0. *Let* $\{\Phi_n\}$ *be a sequence of level-set solutions of* (1.1) *such that*

$$\lim_{n \to \infty} \Phi_n(0, \mathbf{x}) = \Phi_0(\mathbf{x})$$

uniformly on $\mathbb{R}^2$. *Then*

$$\lim_{n \to \infty} \Phi_n(t, \mathbf{x}) = \Phi(t, \mathbf{x})$$

uniformly on compact subsets of $[0, \infty) \times \mathbb{R}^2$.

B. *Infinitesimally Wrinkled Solutions as Limits of Solutions from Admissible Initial Domains*

If the initial domain Ω_0 is *admissible*, then there is a classical evolution from Ω_0 up to a maximal existence time $T_{\max}$; each relaxed evolution from Ω_0 (unique up to $T_{\text{uniq}} \geqslant T_{\max}$) supplies a weak extension of this classical solution for times greater than $T_{\max}$.

Suppose that Ω_0 is *not admissible* (for example, suppose that $\partial\Omega_0$ has normal angles θ for which $g(\theta) < 0$). Then the notion of a classical evolution from Ω_0 breaks down, since classical evolutions are required to be admissible and hence to have globally stable normal-angels. On the other hand, there is a relaxed evolution from Ω_0. The derivation of the relaxed formulation is based on allowing the boundary curve to develop infinitesimal wrinkles whenever its normal angle is not globally stable. We now use Theorem 5.1 to give a partial justification of this proceedure, under the assumption that $\partial\Omega_0$ is C^1 and piecewise C^2, and that $T_{\text{uniq}} > 0$.

We first approximate Ω_0 by a sequence $\{\Omega_0^n\}$ of *admissible* bounded domains. We accomplish this by dividing $\partial\Omega_0$ into curves whose normal angles are GS, interspaced with curves whose normal angles are GUS. We approximate $\partial\Omega_0$ by leaving the GS curves unchanged, but replacing each GUS curve by a wrinkled curve. If Γ is such a GUS curve, then the normal angles of Γ lie in a GUS angle-interval (θ_1, θ_2) with θ_1 and θ_2 angles for a corner consistent with (1.11). We replace Γ by a wrinkled curve $\mathscr{W}$ such that: the endpoints of $\mathscr{W}$ coincide with those of Γ; the facet angles of $\mathscr{W}$ are θ_1 and θ_2; $\mathscr{W}$ lies in an arbitrary small neighborhood of Γ. The replacement for $\partial\Omega_0$ constructed in this manner is admissible and arbitrarily close to $\partial\Omega_0$ in the required sense.

For each n, we let $\Omega^n(t)$, $t \in [0, \infty)$, be a relaxed evolution from the admissible initial domain Ω_0^n. Then, by Theorem 5.1, for each $t \in [0, T_{\text{uniq}})$, $\Omega^n(t)$ converges, in the Hausdorf topology, to the unique relaxed evolution from Ω_0.

C. *Proofs*

Proof of Theorem 5.1. 1° Let

$$\Phi_0(\mathbf{x}) = \begin{cases} \operatorname{dist}(\mathbf{x}, \partial\Omega_0) \wedge 1, & \mathbf{x} \in \Omega_0, \\ -(\operatorname{dist}(\mathbf{x}, \partial\Omega_0) \wedge 1), & \mathbf{x} \notin \Omega_0. \end{cases}$$

Then Φ_0 is an auxiliary function for Ω_0 as defined in Section 3, and there is a unique level-set solution Φ of (1.1) corresponding to Φ_0. For each n, let $\Phi_{0n}(\mathbf{x})$ and $\Phi_n(t, \mathbf{x})$ be defined in the same manner using Ω_0^n as the initial set. Since $\{\Omega_0^n\}$ approximates Ω_0, $\Phi_{0n}(\mathbf{x})$ converges to $\Phi_0(\mathbf{x})$, uniformly for $\mathbf{x} \in \mathbb{R}^2$. Thus Theorem 5.2, which will be proved subsequently, implies that $\Phi_n(t, \mathbf{x})$ converges to $\Phi(t, \mathbf{x})$, uniformly on compact subsets of $[0, \infty) \times \mathbb{R}^2$.

2° Let $\mathscr{B}(r)$ denote the (closed) ball of radius r centered at the origin. Since Ω_0 is compact and $\{\Omega_0^n\}$ approximates Ω_0, there is an R_0 such that $\Omega_0^n \subset \mathscr{B}(R_0)$ for all n. Set

$$\mu = |U| \{\inf B(\theta)\}^{-1}.$$

Then

$$\mathscr{B}_0(t) = \mathscr{B}(R_0 + \mu t)$$

is a χ-supersolution of (1.1) compatible with Ω_0, and hence

$$\Psi(t, \mathbf{x}) = \begin{cases} -1, & \mathbf{x} \notin \operatorname{int} \mathscr{B}_0(t), \\ 1, & \mathbf{x} \in \operatorname{int} \mathscr{B}_0(t). \end{cases}$$

is a viscosity supersolution of (3.2) with $\Psi(0, \mathbf{x}) \geqslant \Phi_n(0, \mathbf{x})$. Thus Theorem 3.3 with Ψ as supersolution and Φ_n as subsolution yields $\Psi \geqslant \Phi_n$. In particular,

$$\Phi_n(t, \mathbf{x}) \leqslant -1, \qquad \mathbf{x} \notin \operatorname{int} \mathscr{B}_0(t).$$

3° By Theorem 3.4,

$$\{\mathbf{x}: \Phi_n(t, \mathbf{x}) > 0\} \subseteq \Omega^n(t) \subseteq \{\mathbf{x}: \Phi_n(t, \mathbf{x}) \geqslant 0\} \tag{5.1a}$$

for $t \geqslant 0$. Therefore

$$\Omega^n(t) \subseteq \mathscr{B}_0(t) \tag{5.1b}$$

for $t \geqslant 0$. Also, on $[0, T_{\text{uniq}})$ the unique relaxed evolution from Ω_0 is given by

$$\Omega(t) = \{\mathbf{x}: \Phi(t, \mathbf{x}) \geqslant 0\}. \tag{5.2}$$

4° For $\delta > 0$, let

$$\mathscr{U}(t; \delta) = \{\mathbf{x}: \Phi(t, \mathbf{x}) \geqslant -\delta\}, \qquad \mathscr{L}(t; \delta) = \{\mathbf{x}: \Phi(t, \mathbf{x}) > \delta\}.$$

Since $\Phi_n(t, \mathbf{x})$ converges to $\Phi(t, \mathbf{x})$ locally uniformly, we may use (5.1b) to conclude that there is an $n(\delta)$ such that, for all $n \geqslant n(\delta)$ and $t \in [0, T_{\text{uniq}})$,

$$\mathscr{U}(t; \delta) \supseteq \{\mathbf{x}: \Phi_n(t, \mathbf{x}) \geqslant 0\}, \qquad \mathscr{L}(t; \delta) \subseteq \{\mathbf{x}: \Phi_n(t, \mathbf{x}) > 0\}.$$

Hence (5.1a) yields

$$\mathscr{L}(t; \delta) \subseteq \Omega^n(t) \subseteq \mathscr{U}(t; \delta) \tag{5.3}$$

for all $n \geqslant n(\delta)$ and $t \in [0, T_{\text{uniq}})$.

5° Using the arguments of step 2°, we can show that

$$\mathscr{U}(t; \delta), \mathscr{L}(t; \delta) \subseteq \mathscr{B}_0(t).$$

6° Our next step will be to show that, for every $t \in [0, T_{\text{uniq}})$, the Hausdorf distance

$$d_\delta = d_{\mathrm{H}}(\mathscr{U}(t; \delta), \mathscr{L}(t; \delta))$$

satisfies

$$d_\delta \to 0 \qquad \text{as} \quad \delta \to 0. \tag{5.4}$$

Since $\mathscr{L}(t;\delta) \subseteq \mathscr{U}(t;\delta)$, we may use the definitions of $\mathscr{L}(t;\delta)$ and $\mathscr{U}(;\delta)$ to conclude that

$$d_\delta = \sup\{d(t, \mathbf{x}; \delta) \colon \Phi(t, \mathbf{x}) \geqslant -\delta\},$$

$$d(t, \mathbf{x}; \delta) = \inf\{|\mathbf{x} - \mathbf{y}| \colon \Phi(t, \mathbf{y}) \geqslant \delta\}.$$

Choose $\mathbf{x}(\delta)$ satisfying

$$\begin{aligned} &\Phi(t, \mathbf{x}(\delta)) \geqslant -\delta, \\ &d(t, \mathbf{x}(\delta); \delta) \geqslant d_\delta - \delta. \end{aligned} \tag{5.5}$$

Since $\mathbf{x}(\delta) \in \mathscr{U}(t;\delta) \subset \mathscr{B}_0(t)$ and $\mathscr{B}_0(t)$ is compact, there is a sequence (also denoted by δ) such that $\mathbf{x}(\delta) \to \mathbf{x}_0$ as $\delta \downarrow 0$; hence

$$\Phi(t, \mathbf{x}_0) = \lim_{\delta \downarrow 0} \Phi(t, \mathbf{x}(\delta)) \geqslant 0,$$

and $\mathbf{x}_0 \in \mathscr{U}(t)$. Also, $t \in [0, T_{\text{uniq}})$; hence we may conclude from the definition of T_{uniq} that $\mathscr{U}(t) = \operatorname{cl} \mathscr{L}(t)$, and there is a sequence $\mathbf{y}_m \to \mathbf{x}_0$, $\mathbf{y}_m \in \mathscr{L}(t)$, or equivalently, $\Phi(t, \mathbf{y}_m) > 0$. Thus, for all $\delta < \Phi(t, \mathbf{y}_m)$,

$$d(t, \mathbf{x}(\delta); \delta) \leqslant |\mathbf{x}(\delta) - \mathbf{y}_m|.$$

Now let δ tend to zero and then m to infinity to obtain

$$\lim_{\delta \downarrow 0} d(t, \mathbf{x}(\delta); \delta) = 0,$$

and this, with (5.5), implies (5.4).

7° By (5.2) and (5.3),

$$d_{\mathrm{H}}(\Omega^n(t), \Omega(t)) \leqslant d_{\mathrm{H}}(\mathscr{U}(t;\delta), \mathscr{L}(t;\delta)) = d_\delta$$

for every $t \in [0, T_{\text{uniq}})$, $\delta > 0$, and $n > n(\delta)$. Therefore, by (5.3),

$$\lim_{n \to \infty} d_{\mathrm{H}}(\Omega^n(t), \Omega(t)) = 0$$

for all $t \in [0, T_{\text{uniq}})$. ∎

Proof of Theorem 5.2. 1° Since Φ_0 is bounded and $\Phi_n(0, \mathbf{x})$ converges uniformly to $\Phi_0(\mathbf{x})$, there is a $\kappa > 0$ such that

$$|\Phi_n(0, \mathbf{x})| \leqslant \kappa \tag{5.6}$$

for all $\mathbf{x} \in \mathbb{R}^2$. Since $\psi \equiv \kappa$ is a solution of (3.2), the inequality (5.6) and the Comparison Theorem 3.3 yield $\Phi_n(t, \mathbf{x}) \leqslant \kappa$. Similarly, $\psi \equiv -\kappa$ yields $\Phi_n(t, \mathbf{x}) \geqslant -\kappa$. Hence

$$|\Phi_n(t, \mathbf{x})| \leqslant \kappa \tag{5.7}$$

for all $(t, \mathbf{x}) \in [0, \infty) \times \mathbb{R}^2$.

2° For $(t, \mathbf{x}) \in [0, \infty) \times \mathbb{R}^2$, define

$$\Phi^+(t, \mathbf{x}) = \limsup_{\substack{n \to \infty \\ (s, \mathbf{y}) \to (t, \mathbf{x})}} \Phi_n(s, \mathbf{y}),$$

$$\Phi^-(t, \mathbf{x}) = \liminf_{\substack{n \to \infty \\ (s, \mathbf{y}) \to (t, \mathbf{x})}} \Phi_n(s, \mathbf{y}).$$

Then Φ^+ is a viscosity subsolution and Φ^- a viscosity supersolution of (3.2) in $(0, \infty) \times \mathbb{R}^2$ (cf. [FS; Sect. 2.6 and 7.4]).

3° Theorem 3.3 applied to the subsolution Φ^+ and the supersolution Φ^- yields

$$\Phi^+(t, \mathbf{x}) - \Phi^-(t, \mathbf{x}) \leqslant \sup_{\mathbf{y}} [\Phi^+(0, \mathbf{y}) - \Phi^-(0, \mathbf{y})].$$

Note that Φ_n is locally uniformly convergent if and only if $\Phi^+ = \Phi^-$. Also, by construction, $\Phi^+ \geqslant \Phi_0 \geqslant \Phi^-$. Hence to prove local uniform convergence of Φ_n it suffices to show that

$$\Phi^+(0, \mathbf{x}) = \Phi_0(\mathbf{x}) = \Phi^-(0, \mathbf{x}), \tag{5.8}$$

which we shall accomplish in the next three steps.

4° Let

$$g^* = \sup\{G(\theta): \theta \in [0, 2\pi)\}, \qquad \alpha = |U|.$$

For $\mathbf{x} \in \mathbb{R}^2$ and $\delta > 0$, define (cf. (5.6))

$$\Psi(t, \mathbf{x}; \mathbf{y}, \delta) = \begin{cases} 0, & |\mathbf{x} - \mathbf{y}| \leqslant R(t; \delta), \\ -2\kappa, & |\mathbf{x} - \mathbf{y}| > R(t; \delta), \end{cases}$$

where $R(t; \delta)$ is a solution of

$$dR(t; \delta)/dt = -g^* R(t; \delta)^{-1} - \alpha, \qquad t \in (0, T(\delta)),$$

$$R(0; \delta) = \delta,$$

with $T(\delta) < +\infty$ the first time t for which $R(t; \delta) = 0$. For each $\mathbf{y} \in \mathbb{R}^2$,

$$\{\mathbf{x}: |\mathbf{x} - \mathbf{y}| \leqslant R(t; \delta)\}, \qquad t \in (0, T(\delta))$$

is a classical subsolution of the relaxed equation (1.1), and hence Ψ is a viscosity subsolution of (3.2) on $(0, T(\delta)) \times \mathbb{R}^2$.

5° Fix $\mathbf{y} \in \mathbb{R}^2$ and let $\beta_0 = \Phi_0(\mathbf{y})$. Then, for all $\beta < \beta_0$, there are $\delta > 0$ and η_0 such that

$$\Psi(0, \mathbf{x}; \mathbf{y}, \delta) + \beta \leqslant \Phi_n(0, \mathbf{x})$$

for all $\mathbf{x} \in \mathbb{R}^2$ and $n \geqslant \eta_0$. Since Ψ is a viscosity subsolution of (3.4), it is clear from the form of this equation that $\Psi + \beta$ is also a viscosity subsolution (3.4).

6° We now use Theorem 3.3 with subsolution $\Psi + \beta$ and supersolution Φ_n to obtain

$$\Psi(t, \mathbf{x}; \mathbf{y}, \delta) + \beta \leqslant \Phi_n(t, \mathbf{x})$$

for all $(t, \mathbf{x}) \in [0, T(\delta)) \times \mathbb{R}^2$; hence

$$\Psi(t, \mathbf{x}; \mathbf{y}, \delta) + \beta \leqslant \Phi^-(t, \mathbf{x})$$

for all $(t, \mathbf{x}) \in [0, T(\delta)) \times \mathbb{R}^2$. Applying this inequality at $(t, \mathbf{x}) = (0, \mathbf{y})$ yields $\beta \leqslant \Phi^-(0, \mathbf{y})$ for all $\beta < \beta_0 = \Phi_0(\mathbf{y})$. Therefore $\Phi_0(\mathbf{y}) = \Phi^-(0, \mathbf{y})$.

7° To show that $\Phi_0(\mathbf{y}) = \Phi^+(0, \mathbf{y})$, we follow the procedure of the three previous steps replacing Ψ with supersolution

$$\hat{\Psi}(t, \mathbf{x}; \mathbf{y}, \delta) = \begin{cases} 0, & |\mathbf{x} - \mathbf{y}| \leqslant R(t; \delta), \\ 2\kappa, & |\mathbf{x} - \mathbf{y}| > R(t; \delta), \end{cases}$$

of (3.2). ∎

6. Large-Time Asymptotics

In this section we discuss the large-time asymptotics of relaxed evolutions, assuming throughout that:

G and B satisfy (1.2) and (1.3);

Ω_0 is a prescribed initial domain, assumed compact.

In particular, we will prove that, for $U < 0$ and Ω_0 large enough, $t^{-1}\Omega(t)$ converges to a dilation of the Wulff region for $1/B(\theta)$. This result, conjectured by Angenent and Gurtin [AG1], was proved by Soner [So] for

$G>0$ and B with a convex polar diagram, and extended in [AG2] to general $B>0$. We here follow the ideas of [So, Sect. 12–13].

Let $\Omega(t) \subset \mathbb{R}^2$, $t \geqslant 0$, be given. Then $\Omega(t)$ *vanishes in finite time* if there is a $T>0$ such that

$$\Omega(t)=\varnothing \qquad \text{for all} \quad t>T.$$

Given a function $\varphi>0$ on $(0, \infty)$ and a set $A \subset \mathbb{R}^2$, we write

$$\Omega(t) \sim \varphi(t)\, A \qquad \text{as} \quad t \rightarrow \infty$$

if there are functions, $\varphi_1, \varphi_2>0$ on $(0, \infty)$ such that

$$\varphi_1(t)\, A \subset \Omega(t) \subset \varphi_2(t)\, A$$

for all sufficiently large t, and

$$\varphi_i(t)/\varphi(t) \rightarrow 1 \qquad \text{as} \quad t \rightarrow \infty \ (i=1,2).$$

The *Wulff region* $W(h)$ for a given function $h(\theta)$ (cf. e.g., [G2]) is the set

$$W(h)=\left\{\mathbf{x} \in \mathbb{R}^2\colon \mathbf{x} \cdot \mathbf{N}(\theta) \leqslant h(\theta),\ \theta \in[0, 2\pi]\right\}.$$

Our main result of this section is

THEOREM 6.1 (Asymptotic Behavior of Relaxed Evolutions). *Let $\Omega(t)$ be a relaxed evolution from Ω_0.*

(a) *If $U>0$, then $\Omega(t)$ vanishes in finite time.*

(b) *If $U<0$ with $|U|$ sufficiently large, then*

$$\Omega(t) \sim t\,|U|\, W(1/B) \qquad \textit{as} \quad t \rightarrow \infty.$$

Assertion (a) is a direct consequence of

LEMMA 6.1. *Let $\Omega(t)$ be a χ-subsolution of* (1.1) *compatible with Ω_0. Choose α_0 such that*

$$\operatorname{int} \Omega_0 \subset \alpha_0 W(1/B). \tag{6.1}$$

Then, for $t>0$,

$$\Omega(t) \subset(-Ut+\alpha_0)\, W(1/B). \tag{6.2}$$

Proof. The right side of (6.2), denoted by $\Lambda(t)$, is a χ-solution of

$$B(\theta)\, V=-U$$

for $t>0$ [So, Sect. 12]. Since $W(1/B)$ is convex, $A(t)$ has curvature $\leqslant 0$. Thus, since $G \geqslant 0$, $A(t)$ is a χ-supersolution of (1.1); (6.2) then follows from (5.1) and Corollary 3.1. ▌

Assertion (b) is more difficult to prove; for that reason we first give a simple proof under a more stringent hypothesis on B. To state this hypothesis, let $\mathscr{D}$ denote the differential operator defined on functions $H(\theta)$ by

$$\mathscr{D}H = H + H''.$$

Then the polar diagram of H is convex at angles θ for which $\mathscr{D}H(\theta) \geqslant 0$, strictly convex at angles with $\mathscr{D}H(\theta) > 0$. We now establish (b) under the assumption that, for some constant $C>0$,

$$G \leqslant C\mathscr{D}(1/B) \tag{6.3}$$

on $[0, 2\pi]$, so that the polar diagram of $1/B$ is convex; in fact, strictly convex at angles θ with $G(\theta)>0$. Granted (6.3), (b) follows from Lemma 6.1 and

LEMMA 6.2. *Assume that* (6.3) *is satisfied, and that* $U<0$ *and sufficiently large that*

$$\alpha_0 \operatorname{int} W(1/B) \subset \Omega_0, \qquad \alpha_0 = -2C/U. \tag{6.4}$$

Let $\alpha(t)$ *be the solution of*

$$\alpha'(t) = -U - C/\alpha(t), \qquad \alpha(0) = \alpha_0.$$

Then any relaxed evolution $\Omega(t)$ *compatible with* Ω_0 *satisfies, for* $t>0$,

$$\alpha(t)\, W(1/B) \subset \Omega(t). \tag{6.5}$$

Proof. Let $A(t) = \alpha(t)\, W(1/B)$. Then $A(t)$ is a χ-solution of (1.1) with G replaced by $C\mathscr{D}(1/B)$ [So, Sect. 12]. Since $W(1/B)$ is smooth and $0 \leqslant G \leqslant C\mathscr{D}(1/B)$, $A(t)$ is a χ-subsolution of (1.1) (with G); hence (6.5) follows from (6.4) and Corollary 3.1. ▌

Proof of Theorem 6.1. 1° Let

$$g = \sup G(\theta), \qquad b = \sup B(\theta),$$

and assume that $|U|$ is sufficiently large that

$$\alpha_0 \operatorname{int} \mathscr{B}_1 \subset \Omega_0, \qquad \alpha_0 = -2g/U,$$

where $\mathscr{B}_1$ is the unit ball in $\mathbb{R}^2$. Let $\alpha(t)$ be the solution of

$$b\alpha'(t) = -U - g/\alpha(t), \qquad \alpha(0) = \alpha_0,$$

and let $\Lambda(t) = \alpha(t)\,\mathscr{B}_1$. Then $\Lambda(t)$ is a classical solution of the isotropic equation $bV = gK - U$. Also, $V > 0$, since $\alpha'(t) > 0$ for all $t > 0$; consequently, $\Lambda(t)$ is a χ-subsolution of (1.1) compatible with Ω_0, and, by Corollary 3.1, $\Lambda(t) \subset \Omega(t)$. Further, $\alpha(t) \to \infty$ as $t \to \infty$; hence:

$$\text{as } t \to \infty,\ \Omega(t) \text{ expands to fill the entire space.} \tag{6.6}$$

2° Let $W_n \subset \mathbb{R}^2$ be a sequence of strictly convex, closed domains, with smooth boundary, satisfying

$$(1 - n^{-1})\, W(1/B) \subset W_n \subset W(1/B). \tag{6.7}$$

Further, let $\gamma_n(\theta)$ denote the support function of W_n,

$$\gamma_n(\theta) = \sup\{\mathbf{x} \cdot \mathbf{N}(\theta) \colon \mathbf{x} \in W_n\}$$

(so that $W_n = W(\gamma_n)$). Since W_n is strictly convex and ∂W_n is smooth, γ_n is smooth and $\mathscr{D}\gamma_n > 0$. Also,

$$1/B(\theta) \geqslant \sup\{\mathbf{x} \cdot \mathbf{N}(\theta) \colon \mathbf{x} \in W(1/B)\},$$

and hence, by (6.7),

$$1/B \geqslant \gamma_n. \tag{6.8}$$

3° Let

$$c_n = g\{\inf_{\theta} \mathscr{D}\gamma_n(\theta)\}^{-1}.$$

By 2°, $c_n < \infty$, but c_n may diverge to $+\infty$ as $n \to \infty$.

Choose $t_n > 0$ satisfying

$$\alpha_n = \operatorname{int} W_n \subset \Omega(t_n), \qquad \alpha_n = -2c_n/U, \tag{6.9}$$

and let $\alpha_n(t)$ be the solution of

$$\alpha_n'(t) = -U - c_n/\alpha_n(t), \qquad t > t,$$
$$\alpha_n(t_n) = \alpha_n.$$

Then $\Lambda_n(t) = \alpha_n(t)\, W_n$ is a χ-solution of

$$\gamma_n(\theta)^{-1}\, V = c_n \mathscr{D}\gamma_n(\theta)\, K - U$$

for $t > t_n$ [So, Sect. 12]. Further, (6.8), the definition of c_n, the convexity of $\Lambda_n(t)$, and the positivity of V imply that

$$B(\theta)\, V \leqslant \gamma_n(\theta)^{-1}\, V = c_n \mathscr{D}\gamma_n(\theta)\, K - U \leqslant G(\theta)\, K - U;$$

hence $\Lambda_n(t)$ is a χ-subsolution of (1.1) for $t > t_n$. By Corollary 3.1 and (6.9), $\Lambda_n(t) \subset \Omega(t)$ for all $t > t_n$, and using (6.7) we conclude that

$$\alpha_n(t)(1 - n^{-1})\, W(1/B) \subset \Omega(t), \qquad t > t_n. \tag{6.10}$$

4° For $t > t_1$ define

$$\alpha(t) = \sup\{(1 - n^{-1})\, \alpha_n(t) : n \geqslant 1,\ t_n \geqslant t\}.$$

Then $\alpha(t)\, W(1/B) \subset \Omega(t)$ for $t > t_1$. Also, for each n,

$$\alpha(t) \geqslant (1 - n^{-1})\, \alpha_n(t), \qquad t \geqslant t_n.$$

Hence

$$\liminf_{t \to \infty} \alpha(t)/t \geqslant (1 - n^{-1}) \liminf_{t \to \infty} \alpha_n(t)/t = -(1 - n^{-1})\, U$$

for every n, and consequently

$$\liminf_{t \to \infty} \alpha(t)/t \geqslant -U. \tag{6.11}$$

5° Summarizing, in 4° and Lemma 6.1 we have shown that

$$\alpha(t)\, W(1/B) \subset \Omega(t) \subset (-Ut + \alpha_0)\, W(1/B),$$

which, with (6.11), yields

$$\lim_{t \to \infty} \alpha(t)(-Ut + \alpha_0)^{-1} = 1;$$

hence $\Omega(t) \sim t\, |U|\, W(1/B)$ as $t \to \infty$. ∎

7. Proofs of the Comparison Theorems

A. *Sub- and Superdifferentials*

We recall several definitions from the theory of viscosity solutions.[21] Let φ be a bounded function on $(0, \infty) \times \mathbb{R}^2$, and let $\mathscr{S}$ denote the set of

[21] Cf [CIL, Sect. 2; C; CEL; FS Sect. 5.4].

symmetric 2×2 matrices. Then the subdifferential $D^+\varphi(t,\mathbf{x})$ and the super-differential $D^-\varphi(t,\mathbf{x})$ of φ at $(t,\mathbf{x})\in(0,\infty)\times\mathbb{R}^2$ are defined by

$$D^+\varphi(t,\mathbf{x})=\{(q,\mathbf{p},\mathbf{A})\in\mathbb{R}\times\mathbb{R}^2\times\mathscr{S}:\limsup_{(h,\mathbf{z})\to 0}\mathscr{D}(\varphi^*)(h,\mathbf{z})\leqslant 0\},$$

$$D^-\varphi(t,\mathbf{x})=\{(q,\mathbf{p},\mathbf{A})\in\mathbb{R}\times\mathbb{R}^2\times\mathscr{S}:\liminf_{(h,\mathbf{z})\to 0}\mathscr{D}(\varphi^*)(h,\mathbf{z})\geqslant 0\},$$

where

$$\begin{aligned}\mathscr{D}\Phi(h,\mathbf{z})&=\mathscr{D}\Phi(t,\mathbf{x};h,\mathbf{z};q,\mathbf{p},\mathbf{A})\\&=(|h|+|\mathbf{z}|^2)^{-1}\{\Phi(t+h,\mathbf{x}+\mathbf{z})-\Phi(t,\mathbf{x})-hq-\mathbf{z}\cdot\mathbf{p}-\tfrac{1}{2}\mathbf{z}\cdot\mathbf{A}\mathbf{z}\};\end{aligned}$$

we close the sets $D^\pm\varphi(t,\mathbf{x})$ as follows:

$$cD^\pm\varphi(t,\mathbf{x})=\{\lim_{n\to\infty}(q_n,\mathbf{p}_n,\mathbf{A}_n):(q_n,\mathbf{p}_n,\mathbf{A}_n)\in D^\pm\varphi(t_n,\mathbf{x}_n),\ (t_n,\mathbf{x}_n)\to(t,\mathbf{x})\}.$$

Then[22] $(q,\mathbf{p},\mathbf{A})\in D^+\varphi(t,\mathbf{x})$ if and only if there is a $w\in C^{1,2}$ satisfying

$$w_t(t,x)=q,\qquad Dw(t,x)=\mathbf{p},\qquad D^2w(t,x)=\mathbf{A},$$

and $(t,\mathbf{x})$ is a maximum of the difference (φ^*-w). Hence φ is a viscosity subsolution of (3.4) if and only if

$$q\leqslant\mathscr{F}^*(\mathbf{p},\mathbf{A})\qquad\text{for all}\quad(q,\mathbf{p},\mathbf{A})\in D^+\varphi(t,\mathbf{x})$$

and all $(t,\mathbf{x})\in(0,\infty)\times\mathbb{R}^2$. A limit argument then shows that

$$q\leqslant\mathscr{F}^*(\mathbf{p},\mathbf{A})\qquad\text{for all}\quad(q,\mathbf{p},\mathbf{A})\in cD^+\varphi(t,\mathbf{x}).\tag{7.1}$$

Similarly, φ is a viscosity supersolution of (3.2) if and only if

$$q\geqslant\mathscr{F}_*(\mathbf{p},\mathbf{A})\qquad\text{for all}\quad(q,\mathbf{p},\mathbf{A})\in cD^-\varphi(t,\mathbf{x})\tag{7.2}$$

and all $(t,\mathbf{x})\in(0,\infty)\times\mathbb{R}^2$.

B. *Semiconvex and Semiconcave Functions*

Let $C\subset\mathbb{R}^d$ be a convex set. We say that Ψ is *semiconvex* on C if there is a constant κ such that

$$\bar{\Psi}(\mathbf{Y})=\Psi(\mathbf{Y})+\kappa\,|\mathbf{Y}|^2$$

is convex on C; $\bar{\Psi}$ is *semiconcave* on C if $-\bar{\Psi}$ is semiconvex.

[22] Cf., e.g., [FS, Prop. 4.1, Sect. 5.4]; since φ is not necessarily continuous, the proof given in [FS] must be slightly modified.

Let Ψ be semiconvex. Since $\bar{\Psi}(\mathbf{Y})$ is convex, the set of subdifferentials, standard in convex analysis [C], is given by

$$\partial\bar{\Psi}(\mathbf{Y}) = \{P \in \mathbb{R}^d\colon \bar{\Psi}(\bar{\mathbf{Y}}) \geqslant \bar{\Psi}(\mathbf{Y}) + P \cdot (\bar{\mathbf{Y}} - \mathbf{Y}),\ \forall \bar{\mathbf{Y}} \in C\}.$$

In addition, w the directional derivatives of $\bar{\Psi}$ exist and are given by

$$(\partial/\partial\mathbf{Z})\ \bar{\Psi}(\mathbf{Y}) = \lim_{r \downarrow 0} r^{-1}[\bar{\Psi}(\mathbf{Y} + r\mathbf{Z}) - \bar{\Psi}(\mathbf{Y})] = \sup\{\mathbf{P} \cdot \mathbf{Z}\colon \mathbf{P} \in \partial\bar{\Psi}(Y)\} \tag{7.3}$$

for $\mathbf{Z} \in \mathbb{R}^d \backslash \{0\}$. We now define

$$\hat{\partial}\Psi(\mathbf{Y}) = \partial\bar{\Psi}(\mathbf{Y}) + \{-2\kappa\mathbf{Y}\} = \{\mathbf{P}\colon \mathbf{P} = \bar{\mathbf{P}} - 2\kappa\mathbf{Z},\ \bar{\mathbf{P}} \in \partial\bar{\Psi}(\mathbf{Y})\}.$$

Then, using the formula for the directional derivative of $\bar{\Psi}$,

$$(\partial/\partial\mathbf{Z})\ \Psi(\mathbf{Y}) = \sup\{\mathbf{P} \cdot \mathbf{Z}\colon \mathbf{P} \in \hat{\partial}\Psi(\mathbf{Y})\}$$

for $\mathbf{Z} \in \mathbb{R}^d \backslash \{\mathbf{0}\}$.

The following properties of semiconvex functions are well known:

Ψ is differentiable at $\mathbf{Y}$ if and only if $\hat{\partial}\Psi(\mathbf{Y})$ is a singleton. (7.4)

if there are sequences $\mathbf{P}_n \to \mathbf{P}$, $\mathbf{Y}_n \to \mathbf{Y}$ and convex functions $\Psi_n \to \Psi$ (uniformly on C) satisfying $\mathbf{P}_n \in \hat{\partial}\Psi(\mathbf{Y}_n)$ for all n, then $\mathbf{P} \in \hat{\partial}\Psi(\mathbf{Y})$. (7.5)

Our next result is an implicit function theorem for semiconvex functions. Let Ψ be a semiconvex function on C, and let $\mathbf{0}$ be an interior point of C. We assume that Ψ is differentiable[23] at $\mathbf{0}$ with a nonzero gradient $\mathbf{P}_0$; and, without loss in generality, we assume that $\bar{\mathbf{P}}_0 = \mathbf{P}_0/|\mathbf{P}_0|$ satisfies

$$\bar{\mathbf{P}}_0 = (0, 0, ..., 0, 1).$$

Let δ_1 be a constant satisfying

$$|\mathbf{P} - \mathbf{P}_0| \leqslant |\mathbf{P}_0|/2, \qquad \forall \mathbf{P} \in \hat{\partial}\bar{\Psi}(\mathbf{Y}), \quad |\mathbf{Y}| \leqslant 2\delta_1, \tag{7.6a}$$

$$\mathbf{P}_d \geqslant |\mathbf{P}_0|/2, \qquad \forall \mathbf{P} \in \hat{\partial}\bar{\Psi}(\mathbf{Y}), \quad |\mathbf{Y}| \leqslant 2\delta_1, \tag{7.6b}$$

$$\mathscr{B}(2\delta_1) = \{\mathbf{Y}\colon |\mathbf{Y}| \leqslant 2\delta_1\} \subset C. \tag{7.6c}$$

Note that the existence of δ_1 follows from (7.5) (with $\Psi_n = \Psi$ for all n).

For $\mathbf{Y} = (Y_1, Y_2, ..., Y_{d-1}, Y_d) \in \mathbb{R}^d$ we write

$$\mathscr{P}\mathbf{Y} = (Y_1, Y_2, ..., Y_{d-1}) \in \mathbb{R}^{d-1}.$$

[23] We make this assumption to simplify the analysis; an analogous result holds under the weaker assumption $\hat{\partial}\Psi(\mathbf{0}) \cap \{\mathbf{0}\} = \varnothing$.

THEOREM 7.1 (Implicit Function Theorem). *There is a $\delta > 0$ and a unique real-valued, Lipschitz continuous function I on $\mathscr{B}(\delta)$ such that, for all* $\mathbf{Y}$,

$$\Psi(\mathscr{P}\mathbf{Y}, I(\mathbf{Y})) = \Psi(Y_d \bar{\mathbf{P}}_0) \qquad \textit{for all} \quad \mathbf{Y} = (\mathscr{P}\mathbf{Y}, Y_d) \in \mathscr{B}(\delta). \tag{7.7}$$

Moreover, δ depends only on $|\mathbf{P}_0|$, δ_1, and the Lipschitz constant of Ψ on $\mathscr{B}(\delta_1)$.

Proof. 1° For $|\mathbf{Y}| \leqslant \delta_1$ and $\alpha \in \mathbb{R}$ with $|\alpha| \leqslant \delta_1$, we define

$$\Phi(\alpha; \mathbf{Y}) = \Psi(\mathscr{P}\mathbf{Y}, \alpha) - \Psi(Y_d \bar{\mathbf{P}}_0).$$

By (7.3),

$$\Psi(\mathscr{P}\mathbf{Y}, \alpha) = \Psi(\mathbf{Y}) + \int_{Y_d}^{\alpha} (\partial/\partial \bar{\mathbf{P}}_0)\, \Psi(\mathscr{P}\mathbf{Y}, \rho)\, d\rho,$$

$$(\partial/\partial \bar{\mathbf{P}}_0)\, \Psi(\mathscr{P}\mathbf{Y}, \rho) = \sup\{P_d \colon \mathbf{P} \in \hat{\partial}\Psi(\mathscr{P}\mathbf{Y}, \rho)\}.$$

Thus, by (7.6b), for $|\mathbf{Y}| \leqslant \delta_1$ and $|\alpha| \leqslant \delta_1$,

$$\Psi(\mathscr{P}\mathbf{Y}, \alpha) \geqslant \Psi(\mathbf{Y}) + (\alpha - Y_d)\, |\mathbf{P}_0|/2, \qquad \forall \alpha \geqslant Y_d, \tag{7.8a}$$

$$\Psi(\mathscr{P}\mathbf{Y}, \alpha) \leqslant \Psi(\mathbf{Y}) + (\alpha - Y_d)\, |\mathbf{P}_0|/2, \qquad \forall \alpha \leqslant Y_d. \tag{7.8b}$$

2° Our next step will be to show that there is a $\delta \in (0, \delta_1]$ such that

$$\Phi(-\delta_1; \mathbf{Y}) \leqslant 0 \leqslant \Phi(\delta_1; \mathbf{Y}) \tag{7.9}$$

for all $|\mathbf{Y}| \leqslant \delta$. Indeed, by (7.8a),

$$\begin{aligned}\Phi(\delta_1; \mathbf{Y}) &= \Psi(\mathscr{P}\mathbf{Y}, \delta_1) - \Psi(\mathbf{Y}) + \Psi(\mathbf{Y}) - \Psi(Y_d \bar{\mathbf{P}}_0) \\ &\geqslant (\delta_1 - Y_d)\, |\mathbf{P}_0|/2 - L\, |\mathbf{Y} - Y_d \bar{\mathbf{P}}_0|,\end{aligned}$$

where L is the Lipschitz constant of Ψ on $\mathscr{B}(\delta_1)$. (Ψ is semiconvex on C and hence Lipschitz continuous on every compact subset interior to C) Let δ be the lesser of $\delta_1/2$ and $\delta_1\, |\mathbf{P}_0|/4L$. Then for all $|\mathbf{Y}| \leqslant \delta$,

$$\begin{aligned}\Phi(\delta_1; \mathbf{Y}) &\geqslant (\delta_1 - \delta)\, |\mathbf{P}_0|/2 - L(|\mathbf{Y}| - |Y_d|), \\ &\geqslant \delta_1\, |\mathbf{P}_0|/4 - 2L\delta \geqslant 0.\end{aligned}$$

The other inequality in (7.9) is proved similarly, with the same choice for δ.

3° Fix $|\mathbf{Y}| \leqslant \delta$ and consider the map

$$\ell(\alpha) = \Phi(\alpha; \mathbf{Y}), \qquad \alpha \in [-\delta_1, \delta_1].$$

Then ℓ is continuous on $[-\delta_1, \delta_1]$ with

$$\ell(-\delta_1) \leqslant 0 \leqslant \ell(\delta_1).$$

Also, the argument leading to (7.8a) yields, for $\alpha, \beta \in [-\delta_1, \delta_1]$,

$$\ell(\alpha) \geqslant \ell(\beta) + (\alpha - \beta)\, |\mathbf{P}_0|/2 \qquad \text{for} \quad \alpha \geqslant \beta.$$

Hence there is a unique $\alpha_* \in [-\delta_1, \delta_1]$ such that $\ell(\alpha_*) = 0$.

4° For each $|\mathbf{Y}| \leqslant \delta$ set

$$I(\mathbf{Y}) = \alpha_*;$$

Then, for all $|\mathbf{Y}| \leqslant \delta$,

$$0 = \ell(\alpha_*) = \Psi(\mathscr{P}\mathbf{Y}, I(\mathbf{Y})) - \Psi(Y_d\bar{\mathbf{P}}_0),$$

and (7.7) is satisfied.

5° Our last step will be to show that I is Lipschitz continuous. Let $|\mathbf{Y}|$, $|\mathbf{X}| \leqslant \delta$ be given. Then (7.8a) and the Lipschitz continuity of ψ yield

$$\begin{aligned}
\Phi(I(\mathbf{X}) + \rho; \mathbf{Y}) &= \Psi(\mathscr{P}\mathbf{Y}, I(\mathbf{X}) + \rho) - \Psi(Y_d\bar{\mathbf{P}}_0) \\
&\geqslant \Psi(\mathscr{P}\mathbf{X}, I(\mathbf{X}) + \rho) - \Psi(X_d\bar{\mathbf{P}}_0) - 2L\,|\mathbf{X} - \mathbf{Y}| \\
&= \Psi(\mathscr{P}\mathbf{X}, I(\mathbf{X}) + \rho) - \Psi(\mathscr{P}\mathbf{X}, I(\mathbf{X})) + \Psi(\mathscr{P}\mathbf{X}, I(\mathbf{X})) \\
&\quad - \Psi(X_d\bar{\mathbf{P}}_0) - 2L\,|\mathbf{X} - \mathbf{Y}| \\
&\geqslant \rho\,|\mathbf{P}_0|/2 + \Phi(I(\mathbf{X}); \mathbf{X}) - 2L\,|\mathbf{X} - \mathbf{Y}| \geqslant 0
\end{aligned}$$

provided

$$\rho \geqslant 4L\,|\mathbf{X} - \mathbf{Y}|/|\mathbf{P}_0|. \tag{7.10}$$

A similar argument shows that

$$\Phi(I(\mathbf{X}) - \rho; \mathbf{Y}) \leqslant 0$$

if ρ satisfies (7.10). Hence $I(\mathbf{Y}) \in [I(\mathbf{X}) - \rho, I(\mathbf{X}) + \rho]$. ∎

Remark 7.1. Note that the Lipschitz constant of I is $\leqslant 4L/|\mathbf{P}_0|$, with L the Lipschitz constant of Ψ on $\mathscr{B}(\delta_1)$.

The next result, the key technical contribution of the paper, will be used in an essential manner in the proof of Theorem 3.3. For $\mathbf{p} \neq \mathbf{0}$, let

$$\theta(\mathbf{p}) = \sin^{-1}(-\bar{p}_2), \qquad \bar{\mathbf{p}} = \mathbf{p}/|\mathbf{p}|.$$

PROPOSITION 7.1. *Let v be a semiconvex function on $[0, \infty) \times \mathbb{R}^2$ (so that $v(t, \mathbf{x}) + \kappa(t^2 + |\mathbf{x}|^2)$ is convex for some constant κ). Suppose that v is differentiable at $(t_0, \mathbf{x}_0)$ with*

$$\mathbf{p}_0 = \nabla v(t_0, \mathbf{x}_0) \neq 0.$$

Then there exist $(t_n, \mathbf{x}_n) \to (t_0, \mathbf{x}_0)$ and $(q_n, \mathbf{p}_n, \mathbf{A}_n) \in cD^+ v(t_n, \mathbf{x}_n)$ such that

$$\lim_{n \to \infty} (q_n, \mathbf{p}_n) = (v_t(t_0, \mathbf{x}_0), \mathbf{p}_0), \qquad |\mathbf{p}_n| \neq 0, \tag{7.11a}$$

$$\liminf_{n \to \infty} \min\{\theta(\mathbf{p}_n) - \theta(\mathbf{p}_0), T(\mathbf{p}_n, \mathbf{A}_n)\} \leqslant 0, \tag{7.11b}$$

where

$$T(\mathbf{p}, \mathbf{A}) = \mathrm{trace}[I - \bar{\mathbf{p}} \otimes \bar{\mathbf{p}}) \mathbf{A}].$$

To motivate the proof of this proposition, assume, for the moment, that v is smooth. For $r \in \mathbb{R}$, let

$$h(r) = \theta(\nabla v(t_0, \mathbf{x}_0 + r\mathbf{w})), \qquad \mathbf{w} = ((p_0)_2, -(p_0)_1).$$

Then

$$h'(0) = T(\mathbf{p}_0, \nabla^2 v(t_0, \mathbf{x}_0)).$$

Hence if $T(\mathbf{p}_0, \nabla^2 v(t_0, \mathbf{x}_0)) > 0$ then $h(r) \leqslant h(0)$ for $r > 0$.

This argument works only for smooth v. However, if v is semiconvex, then its second derivative is bounded from below. We will use this lower bound to prove (7.11b), with $\hat{H}$ playing the role of h (cf 7°).

The argument given above indicates the possible validity of the following assertion, which is dual to (7.11b):

$$\liminf_{n \to \infty} \min\{\theta(\mathbf{p}_0) - \theta(\mathbf{p}_n), T^*(\mathbf{p}_n, \mathbf{A}_n)\} \leqslant 0 \tag{7.11c}$$

Indeed, the proof of Proposition 7.1 with minor changes establishes the existence of a sequence satisfying (7.11a) and (7.11c). One might believe further that

$$\liminf_{n \to \infty} \min\{\theta(\mathbf{p}_n) - \theta(\mathbf{p}_0), -T(\mathbf{p}_n, \mathbf{A}_n)\} \leqslant 0, \tag{7.11d}$$

but (7.11d) is not valid, the reason being that, since v is assumed semiconvex, its second derivatives are necessarily bounded only from below, but the proof of (7.11d) requires an upper bound on the second derivatives; in fact, (7.11d) holds for semiconcave functions.

The proof of Proposition 7.1 will utilize the following result, which connects the subdifferentials of convex analysis to cD^- and cD^+. We omit the proof; similar results may be found in [FS, Sect. 2.8 and Chap. 5].

LEMMA 7.1. *Let v be a semiconvex function on $[0, \infty) \times \mathbb{R}^2$. Then*

$$(q, \mathbf{p}, \mathbf{A}) \in cD^+v(t, \mathbf{x}) \cup cD^-v(t, \mathbf{x})$$

only if

$$(q, \mathbf{p}) \in \hat{\partial} v(t, \mathbf{x}). \tag{7.12}$$

Conversely, (7.12) *implies that*

$$(q, \mathbf{p}, -2\kappa I) \in D^-v(t, \mathbf{x}),$$

where κ is the constant appearing in the definition of semiconvexity.

Proof of Proposition 7.1. 1° Let (y, z) denote a generic point of $\mathbb{R}^2$. Assume, without loss in generality, that v is defined on $\mathbb{R} \times \mathbb{R}^2$, that $(t_0, \mathbf{x}_0) = (0, 0, 0)$, and that

$$\mathbf{p}_0 = |\mathbf{p}_0| \, (0, 1).$$

If

$$\liminf_{\varepsilon \downarrow 0} \inf\{ T(\mathbf{p}, \mathbf{A}) \colon (q, \mathbf{p}, \mathbf{A}) \in cD^+v(t, y, z),\ |t| + |y| + |z| \leqslant \varepsilon\} \leqslant 0,$$

then (7.11b) follows directly; we therefore assume that there are $\gamma, \varepsilon_1 > 0$ such that

$$T(\mathbf{p}, \mathbf{A}) \geqslant \gamma, \qquad \forall (q, \mathbf{p}, \mathbf{A}) \in cD^+v(t, y, z),\ (t, y, z) \in \mathscr{B}(\varepsilon_1). \tag{7.13}$$

The semicontinuity of v yields the existence of a $\delta_1 > 0$ satisfying (7.6), and hence of a $\delta_1 > 0$ such that, for all $(q, \mathbf{p}) \in \hat{\partial} v(t, y, z)$, $(t, y, z) \in \mathscr{B}(2\delta_1)$:

$$|\mathbf{p} - \mathbf{p}_0| \leqslant |\mathbf{p}_0|/4, \tag{7.14a}$$

$$\mathbf{p}_2 \geqslant |\mathbf{p}_0|/4. \tag{7.14b}$$

2° Since v is semiconvex, v is locally Lipschitz continuous. Therefore, by Rademacher's Theorem, v is differentiable almost everywhere; we define $H(t, y, z)$ at the points of differentiability by

$$H(t, y, z) = \theta(\nabla v(t, y, z)).$$

3° By Theorem 7.1, there are a $\delta \in (0, \delta_1]$ and a function $I(t, y, z)$ satisfying

$$v(t, y, I(t, y, z)) = v(0, 0, z), \qquad \forall (t, y, z) \in \mathscr{B}(\delta). \tag{7.15}$$

Further, by Remark 7.1, $I(t, y, z)$ is Lipschitz continuous on $\mathscr{B}(\delta)$ with Lipschitz constant no more than $4L/|\mathbf{p}_0|$, where L is the Lipschitz constant of v on $\mathscr{B}(\delta_1)$.

4° Our next step is to show that the map $(t, y, z) \mapsto (t, y, I(t, y, z))$ with domain $\mathscr{B}(\delta)$ is one-to-one. Suppose $(t, y, I(t, y, z)) = (\bar{t}, \bar{y}, I(\bar{t}, \bar{y}, \bar{z}))$ Then $(t, y) = (\bar{t}, \bar{y})$ and

$$v(0, 0, z) = v(t, y, I(t, y, z)) = v(\bar{t}, \bar{y}, I(\bar{t}, \bar{y}, \bar{z})) = v(0, 0, \bar{z}).$$

Since $\mathbf{p}_0 = |\mathbf{p}_0| \, (0, 1)$, (7.14b) yields

$$|v(0, 0, \alpha) - v(0, 0, \bar{\alpha})| \geqslant |\alpha - \bar{\alpha}| \, |\mathbf{p}_0|/2 \tag{7.16}$$

for all $(0, 0, \alpha), (0, 0, \bar{\alpha}) \in \mathscr{B}(2\delta_1)$; hence $z = \bar{z}$.

5° The inverse of the map defined in 4° has the form $(t, y, J(t, y, z))$ with

$$v(t, y, z) = v(0, 0, J(t, y, z)),$$

and we may use (7.16) to show that J is Lipschitz continuous.

6° Thus the map $(t, y, z) \mapsto (t, y, I(t, y, z))$ with domain $\mathscr{B}(\delta)$ is one-to-one and Lipschitz, with Lipschitz inverse; hence it transforms null sets into null sets.

7° In view of 6°,

$$\hat{H}(t, y, z) = H(t, y, I(t, y, z))$$

is defined for almost every $(t, y, z) \in \mathscr{B}(\delta)$. We now define, for $0 < \varepsilon, \zeta \leqslant \delta/2$,

$$k(\varepsilon, \zeta) = \int_{\mathscr{B}(\varepsilon)} [\hat{H}(t, y + \zeta, z) - \hat{H}(t, y, z)] \, dt \, dy \, dz.$$

In 9°–15°, we shall show that, for sufficiently small $\varepsilon, \zeta > 0$,

$$k(\varepsilon, \zeta) \geqslant |\mathscr{B}_1| \, \gamma \zeta \varepsilon^3/2 \, |\mathbf{p}_0| \tag{7.17}$$

with γ as in (7.13), where $|\mathscr{B}_1|$ is the volume of the unit ball in $\mathbb{R}^3$. The above estimate provides a weak method of proving that $\hat{H}(t, y, z)$ is increasing in y. Indeed, if v were smooth, a direct calculation would yield

$$\hat{H}_y(0, 0, z) = |\mathbf{p}_0|^{-1} \, T(\mathbf{p}_0, \nabla^2 v(0, 0, z)) \geqslant \gamma/|\mathbf{p}_0|.$$

(The details are given in 9°).

We shall assume that (7.17) is valid and complete the proof of (7.11a, b), before proving (7.17).

8° Let $\mathcal{O}$ denote the set of points of differentiability of v. For all sufficiently small $\varepsilon, \zeta > 0$ there are $(\bar{t}, \bar{y}, \bar{z}) \in \mathcal{B}(\varepsilon)$ satisfying

$$(\bar{t}, \bar{y}, I(\bar{t}, \bar{y}, \bar{z})) \in \mathcal{O}, \qquad (\bar{t}, \bar{y}+\zeta, I(\bar{t}, \bar{y}+\zeta, \bar{z})) \in \mathcal{O},$$
$$\hat{H}(\bar{t}, \bar{y}+\zeta, \bar{z})) - \hat{H}(\bar{t}, \bar{y}, \bar{z}) \geqslant \gamma\zeta /2|\mathbf{p}_0|. \tag{7.18}$$

Moreover, by (7.5) and Lemma 7.1,

$$\lim_{\rho \downarrow 0} (\sup\{ |H(t, y, z) - \theta(\mathbf{p}_0)| : (t, y, z) \in \mathcal{B}(\rho) \cap \mathcal{O}\}) = 0.$$

Since $I(0, 0, 0) = 0$, by choosing $\varepsilon > 0$ small we can make $|H(\bar{t}, \bar{y}, \bar{z}) - \theta(\mathbf{p}_0)|$ smaller than $\gamma\zeta/2\,|\mathbf{p}_0|$. Therefore for every $\zeta = 1/n$ there are $\varepsilon_n \downarrow 0$ and $(\bar{t}_n, \bar{y}_n, \bar{z}_n) \in \mathcal{B}(\varepsilon_n)$ satisfying

$$(t_n, y_n, z_n) := (\bar{t}_n, \bar{y}_n + n^{-1}, I(\bar{t}_n, \bar{y}_n + n^{-1}, \bar{z}_n)) \in \mathcal{O},$$
$$H(t_n, y_n, z_n) > \theta(\mathbf{p}_0).$$

This completes the proof of (7.11a, b), granted (7.17).

We now turn to proof of (7.17).

9° We now assume that v is smooth, a restriction we will later remove using mollification. Since v is smooth, H is defined everywhere. Recall that

$$H(t, y, z) = \theta(\nabla v(t, y, z)), \qquad \hat{H}(t, y, z) = H(t, y, I(t, y, z)),$$
$$\theta(\mathbf{p}) = \sin^{-1}(-\bar{p}_2), \qquad \bar{\mathbf{p}} = \mathbf{p}/|\mathbf{p}|,$$
$$v(t, y, I(t, y, z)) = v(0, 0, z).$$

We claim that

$$\hat{H}_y(t, y, z) = T(\nabla v(\xi), \nabla^2 v(\xi))\, v_z(\xi)^{-1}, \qquad \xi = (t, y, I(t, y, z)). \tag{7.19}$$

This formula may be verified using a direct but tedious calculation; instead we give an indirect derivation, which also motivates our reason for computing $\hat{H}_y$.

For (t, z) fixed, the parametrized curve $\Gamma\colon y \mapsto (y, I(t, y, z))$, $|y| \leqslant \delta$ is a subset of the $v(0, 0, z)$ level curve of v. Hence the normal angle of Γ is $\theta = \hat{H}(t, y, z)$ and the curvature is given by

$$\theta_s = T(\nabla v, \nabla^2 v)/|\nabla v|$$

with s the arc length. Thus

$$\hat{H}_y = T(\nabla v, \nabla^2 v)\, s_y / |\nabla v|, \qquad s_y = (1 + I_y^2)^{1/2}.$$

By differentiating (7.15) with respect to y, we obtain

$$v_y + v_z I_y = 0;$$

hence, $s_y = |\nabla v| / v_z$, which, when substituted into the previous formula, yields (7.19)

10° We continue to assume that v is smooth. The definition of $k(\varepsilon, \zeta)$ yields

$$k(\varepsilon, \zeta) = \left[\int_{\mathscr{B}(\varepsilon)} \int_0^1 \hat{H}_y(t, y + r\zeta, z)\, dr\, dt\, dy\, dz \right] \zeta.$$

Let

$$K(t, y, z) = T(\nabla v(\xi), \nabla^2 v(\xi))\, v_z(\xi)^{-1}.$$

Then by (7.19),

$$k(\varepsilon, \zeta) = \left[\int_{\mathscr{B}(\varepsilon)} \int_0^1 K(t, y + r\zeta, z)\, dr\, dt\, dy\, dz \right] \zeta$$

for all $0 < \varepsilon, \zeta \leqslant \delta/2$, and (7.17) follows from (7.13) and (7.14). Thus we have established (7.17) for v smooth. We now remove this restriction; here the manner in which δ depends on v is important. The constant δ comes from Theorem 7.1 and hence depends only on $|Dv(0, 0, 0)|$, δ_1, and the Lipschitz constant of v on $\mathscr{B}(\delta_1)$; the constant δ_1 is chosen in 1° and satisfies both (7.6) and (7.14).

11° Let v_n be a molification of v. Then v_n converges to v uniformly on compact sets; $v_n(t, y, z) + \kappa(t^2 + y^2 + z^2)$ is convex, with κ as in the statement of the proposition; on compacts sets, the Lipschitz constant of v_n is $\leqslant$ the Lipschitz constant of v.

Let k_n and K_n be defined as in 10°, but with v replaced by v_n. Then 10° yields

$$k_n(\varepsilon, \zeta) = \left[\int_{\mathscr{B}(\varepsilon)} \int_0^1 K_n(t, y + r\zeta, z)\, dr\, dt\, dy\, dz \right] \zeta. \tag{7.20}$$

for all $0 < \varepsilon, \zeta \leqslant \delta_n/2$.

12° Consider a sequence $(t_n, y_n, z_n) \to (t, y, z)$. Since $Dv_n(t_n, y_n, z_n)$ (the derivative in $\mathbb{R}^3$) is uniformly bounded in n, it has a subsequence, also denoted by n, such that $Dv_n(t_n, y_n, z_n)$ is convergent with limit $(q, \mathbf{p})$. Then, by (7.5), $(q, \mathbf{p}) \in \partial v(t, y, z)$. Thus $Dv_n(t_n, y_n, z_n) \to Dv(t, y, z)$ for any sequence $(t_n, y_n, z_n) \to (t, y, z) \in \mathcal{O}$. In particular,

$$\lim_{n \to \infty} Dv_n(0, 0, 0) = Dv(0, 0, 0) \tag{7.21a}$$

$$\lim_{n \to \infty} k_n(\varepsilon, \zeta) = k(\varepsilon, \zeta) \tag{7.21b}$$

for all $0 < \varepsilon, \zeta \leqslant \delta_n/2$ for sufficiently large n.

13° Recall that $\delta_1 > 0$ satisfies (7.6) and (7.14). In view of the previous steps, we may choose $\delta_{1n} \to \delta_1$, as $n \to \infty$, satisfying (7.6) and (7.14) with v replaced by v_n. Since the Lipschitz constant of v_n is $\leqslant$ that of v on each compact set, 12° and the discussion just before 11° imply that $\delta_n \to \delta$.

14° Recall the definition of k_n given in 11°. We claim that there is a subsequence, also labeled by n, such that

$$\liminf_{n \to \infty} k_n(\varepsilon, \zeta) \geqslant |\mathscr{B}_1| \, \gamma\varepsilon^3/2 \, |\mathbf{p}_0| \tag{7.22}$$

for all sufficiently small $0 < \varepsilon, \zeta$. Indeed, since v is semiconvex;

$$D^2 v = M + \Lambda,$$

where M is an integrable matrix-valued function and Λ is a matrix-valued measure orthogonal to the Lebesgue measure (cf. [J, Proposition 3.3]). Moreover, $\Lambda \geqslant 0$ and

$$\mathbf{v} \cdot M(t, y, z)\, \mathbf{v} \geqslant -\kappa\, |\mathbf{v}|^2, \qquad \forall \mathbf{v} \in \mathbb{R}^3, \tag{7.23}$$

with κ as in the statement of the proposition. Since $v_n = v * m_n$ for some smooth mollifier m_n,

$$D^2 v_n = M_n + \Lambda_n, \qquad M_n = M * m_n, \qquad \Lambda_n = \Lambda * m_n.$$

The measure Λ_n has density with respect to Lebesgue measure. Moreover, $\Lambda_n \geqslant 0$ and M_n satisfies (7.23).

The monotonicity of $T(\mathbf{p}, \mathbf{A})$ in $\mathbf{A}$ and the positivity of Λ imply that

$$K_n(t, y, z) = T(\nabla v_n(\xi_n), \nabla^2 v_n(\xi_n))/(\partial/\partial z)\, v_n(\xi_n),$$
$$\xi_n = (t, y, I_n(t, y, z)).$$

Suppose $\xi = (t, y, I(t, y, z)) \in \mathcal{O}$. Then, by 12°,

$$\nabla v_n(\xi_n) \to \nabla v(\xi), \qquad (\partial/\partial z)\, v_n(\xi_n) \to (\partial/\partial z)\, v(\xi).$$

Further,

$$K_n(t, y, z) \geqslant T(\nabla v_n(\xi_n), M_n(\xi_n))/2\,|\mathbf{p}_0|$$

for all $(t, y, z) \in \mathscr{B}(\delta_n)$ and sufficiently large n. Also $M_n \to M$ in L^1. Recall that the map $(t, y, z) \to (t, y, l(t, y, z))$, on $\mathscr{B}(\delta)$, is one-to-one and Lipschitz, with Lipschitz inverse cf. 6°). Let

$$\hat{M}_n(t, y, z) = M_n(\xi_n), \qquad \hat{M}(t, y, z) = M(\xi).$$

Then $\hat{M}_n \to \hat{M}$ in $L^1(\mathscr{B}(\delta))$. Therefore, by passing to a subsequence, also labeled by n,

$$\lim_{n \to \infty} \hat{M}_n(t, y, z) = \hat{M}(t, y, z)$$

for almost every $(t, y, z) \in \mathscr{B}(\delta)$. Since

$$\mathbf{v} \cdot M_n(\xi_n)\, \mathbf{v} \geqslant -\kappa\, |\mathbf{v}|^2,$$

it follows that

$$T(\nabla v_n(\xi_n), M_n(\xi_n)) \geqslant -\kappa.$$

Fatou's Lemma then yields

$$\liminf_{n \to \infty} \int_{\mathscr{B}(\varepsilon)} T(\nabla v_n(\xi_n), M_n(\xi_n))\, dt\, dy\, dz$$
$$\geqslant \int_{\mathscr{B}(\varepsilon)} T(\nabla v(\xi), M(\xi))\, dt\, dy\, dz$$

for all $\varepsilon < \delta/2$. Hence, for $0 < \varepsilon, \zeta < \delta/2$,

$$\liminf_{n \to \infty} k_n(\varepsilon, \zeta) \geqslant \left[\int_{\mathscr{B}(\varepsilon)} \int_0^1 T(t, y + r\zeta, z)\, dr\, dt\, dy\, dz \right] \zeta \Big/ 2\,|\mathbf{p}_0|,$$

where

$$T(t, y, z) = T(\nabla v(\xi), M(\xi)).$$

Also for $\xi \in \mathcal{O}$, $\xi \notin \operatorname{supp} \Lambda$,

$$(Dv(\xi), M(\xi)) \in D^+ v(\xi).$$

Let $\hat{\mathcal{O}} = \mathcal{O} \cap \text{complement} (\operatorname{supp} \Lambda)$. Then $\hat{\mathcal{O}}$ has full measure and, by 6°, so also has $\{(t, y, z) \in \mathscr{B}(\delta) \colon \xi(t, y, z) \in \hat{\mathcal{O}}\}$. Moreover, by (7.13), $T(t, y, z) \geqslant \gamma$ for every $(t, y, z) \in \mathscr{B}(\varepsilon_1) \cap \hat{\mathcal{O}}$. We have therefore proved (7.22).

15° The desired result (7.17) follows from (7.20), (7.21a, b), and (7.22). ∎

C. Semiconvex and Semiconcave Approximations

For $\varepsilon > 0$ and $(t, \mathbf{x}) \in [0, \infty) \times \mathbb{R}^2$, we define

$$\varphi^\varepsilon(t, \mathbf{x}) = \sup\{\varphi^*(s, \mathbf{y}) - (4\varepsilon)^{-2}(|t-s|^4 + |\mathbf{x}-\mathbf{y}|^4) \colon (s, \mathbf{y}) \in [0, \infty) \times \mathbb{R}^2\},$$

$$\varphi_\varepsilon(t, \mathbf{x}) = \inf\{\varphi_*(s, \mathbf{y}) + (4\varepsilon)^{-2}(|t-s|^4 + |\mathbf{x}-\mathbf{y}|^4) \colon (s, \mathbf{y}) \in [0, \infty) \times \mathbb{R}^2\}.$$

These definitions are similar to the sup and inf convolutions of the theory of viscosity solutions [LL, FS, JLS, CIL], in which the second power rather than the fourth is used in the translations. Our reasons for using the fourth power are its simplification of our proof of comparison (cf. Lemma 7.4c).

Let φ be bounded. Then φ^ε is semiconvex. To verify this, choose a maximizer $(s_0, \mathbf{y}_0)$ in the definition of $\varphi^\varepsilon(t_0, \mathbf{x}_0)$, and set

$$r = t_0 - s_0, \qquad \mathbf{w} = \mathbf{x}_0 - \mathbf{y}_0.$$

Then

$$\varphi^\varepsilon(t_0, \mathbf{x}_0) = \varphi^*(s_0, \mathbf{y}_0) - (4\varepsilon)^{-2}(r^4 + |\mathbf{w}|^4),$$

$$\varphi^\varepsilon(t, \mathbf{x}) \geqslant \varphi^*(s_0, \mathbf{y}_0) - (4\varepsilon)^{-2}(|t-s_0|^4 + |\mathbf{x}-\mathbf{y}_0|^4)$$

for all $(t, \mathbf{x})$. For $0 < h \leqslant t_0$ and $\mathbf{z} \in \mathbb{R}^2$, we use this inequality at $(t, \mathbf{x}) = (t_0 \pm h, \mathbf{x}_0 \pm \mathbf{z})$ to obtain

$$\begin{aligned} Q(t_0, \mathbf{x}_0; h, \mathbf{z}) &= \varphi^\varepsilon(t_0 + h, \mathbf{x}_0 + \mathbf{z}) + \varphi^\varepsilon(t_0 - h, \mathbf{x}_0 - \mathbf{z}) - 2\varphi^\varepsilon(t_0, \mathbf{x}_0) \\ &\geqslant -(4\varepsilon)^{-2}[(r+h)^4 + (r-h)^4 - 2r^4 + |\mathbf{w}+\mathbf{z}|^4 \\ &\quad + |\mathbf{w}-\mathbf{z}|^4 - |\mathbf{z}|^4]. \end{aligned}$$

Hence

$$\liminf_{(h, \mathbf{z}) \to 0} [h^2 + |\mathbf{z}|^2]^{-1} Q(t_0, \mathbf{x}_0; h, \mathbf{z}) \geqslant -3(4\varepsilon)^{-2}[r^2 + |\mathbf{w}|^2].$$

Also,

$$(4\varepsilon)^{-2}[r^4 + |\mathbf{w}|^4] = \varphi^*(s_0, \mathbf{y}_0) - \varphi^\varepsilon(t_0, \mathbf{x}_0) \leqslant 2\,\|\varphi\|,$$

with $\|\cdot\|$ the sup norm. Therefore

$$D^2\varphi^\varepsilon \geqslant -(\kappa/\varepsilon)\, I \qquad \text{in } \mathscr{D}'$$

for some constant κ depending only on $\|\varphi\|$; hence φ^ε is semiconvex. A similar argument shows that φ_ε is semiconcave (cf. [CIL, Sect. 3; FS, Sect. 5.4]). Also, as $\varepsilon \downarrow 0$,

$$\varphi^\varepsilon(t, \mathbf{x}) \downarrow \varphi^*(t, \mathbf{x}), \qquad \varphi_\varepsilon(t, \mathbf{x}) \uparrow \varphi_*(t, \mathbf{x})$$

for all $(t, \mathbf{x}) \in [0, \infty) \times \mathbb{R}^2$.

The next lemma is similar to [FS, Lemma 7.2, Sect. 5.7] (see also [CIL, Sect. 3]). Let $M^\varepsilon(t, \mathbf{x})$ denote the set of all maximizers in the definition of $\varphi^\varepsilon(t, \mathbf{x})$ and $m^\varepsilon(t, \mathbf{x})$, the set of all minimizers in the definition of $\varphi_\varepsilon(t, \mathbf{x})$.

LEMMA 7.2. *Fix $(t, \mathbf{x}) \in (0, \infty) \times \mathbb{R}^2$ and $\varepsilon > 0$. Then*

$$|t-s|^4 + |\mathbf{x}-\mathbf{y}|^4 \leqslant 8\ \|\varphi\|\ \varepsilon^2, \tag{7.24}$$

for every $(s, \mathbf{y}) \in M^\varepsilon(t, \mathbf{x}) \cap m^\varepsilon(t, \mathbf{x})$. Suppose

$$t > t_\varepsilon := (8\ \|\varphi\|\ \varepsilon^2)^{1/4}. \tag{7.25}$$

Then

$$D^+\varphi^\varepsilon(t, \mathbf{x}) \subset D^+\varphi(s, \mathbf{y})$$

for every $(s, \mathbf{y}) \in M^\varepsilon(t, \mathbf{x})$ and

$$D^-\varphi_\varepsilon(t, \mathbf{x}) \subset D^-\varphi(s, \mathbf{y})$$

for every $(s, \mathbf{y}) \in m^\varepsilon(t, \mathbf{x})$.

Suppose that φ is a viscosity solution of (3.4) in $(0, \infty) \times \mathbb{R}^2$. Then we may use Lemma 7.2 and (7.1) to conclude that φ^ε is a viscosity subsolution of (3.2) in $(t_\varepsilon, \infty) \times \mathbb{R}^2$ and that φ_ε is a viscosity sypersolution of (3.2) in $(t_\varepsilon, \infty) \times \mathbb{R}^2$.

LEMMA 7.3. *Suppose that*

$$(q, \mathbf{p}, \mathbf{A}) \in cD^+\varphi^\varepsilon(t_0, \mathbf{x}_0).$$

Then

$$\mathbf{A} \geqslant -3(|\mathbf{p}|/\varepsilon)^{2/3} I. \tag{7.26}$$

Proof. We assume, without loss in generality, that $(q, \mathbf{p}, \mathbf{A}) \in D^+\varphi^\varepsilon(t_0, \mathbf{x}_0)$. Then there is a function $w \in C^{1,2}$ such that

$$w_t(t_0, \mathbf{x}_0) = q, \qquad \nabla w(t_0, \mathbf{x}_0) = \mathbf{p}, \qquad \nabla^2 w(t_0, \mathbf{x}_0) = \mathbf{A}$$

and $(t_0, \mathbf{x}_0)$ is a maximizer of $\varphi^\varepsilon - w$. Let

$$\psi(t, \mathbf{x}; s, \mathbf{y}) = \varphi^*(s, \mathbf{y}) - (4\varepsilon)^{-2}(|t-s|^4 + |\mathbf{x}-\mathbf{y}|^4) - w(t, \mathbf{x}).$$

Choose $(s_0, \mathbf{y}_0) \in M^\varepsilon(t_0, \mathbf{x}_0)$. Then ψ has a maximum at $(t_0, \mathbf{x}_0; s_0, \mathbf{y}_0)$. Thus

$$\varepsilon^{-2}(\mathbf{y}_0 - \mathbf{x}_0)\,|\mathbf{y}_0 - \mathbf{x}_0|^2 = \nabla w(t_0, \mathbf{x}_0) = \mathbf{p},$$

$$-\varepsilon^{-2}[\,|\mathbf{y}_0 - \mathbf{x}_0|^2 I + 2(\mathbf{y}_0 - \mathbf{x}_0) \otimes (\mathbf{y}_0 - \mathbf{x}_0)] \leqslant \nabla^2 w(t_0, \mathbf{x}_0) = \mathbf{A},$$

and (7.25) follows. ∎

A similar argument yields

$$\mathbf{B} \leqslant 3(|\mathbf{p}|/\varepsilon)^{2/3} I. \tag{7.27}$$

for all $(q, \mathbf{p}, \mathbf{B}) \in cD^-\varphi^\varepsilon(t_0, \mathbf{x}_0)$.

LEMMA 7.4. *Let $\varepsilon, \beta > 0$ and bounded functions φ, ψ on $[0, \infty) \times \mathbb{R}^2$ be given. Suppose that $(t_0, \mathbf{x}_0) \in (t_\varepsilon, \infty) \times \mathbb{R}^2$ is a maximizer of t_ε (cf. (7.25))*

$$W^\varepsilon(t, \mathbf{x}) = \varphi^\varepsilon(t, \mathbf{x}) - \psi_\varepsilon(t, \mathbf{x}) - \beta t.$$

Then:

(a) *φ^ε and ψ_ε are differentiable at $(t_0, \mathbf{x}_0)$ with*

$$\nabla\varphi^\varepsilon(t_0, \mathbf{x}_0) = \nabla\psi_\varepsilon(t_0, \mathbf{x}_0) =: \mathbf{p}_\varepsilon, \tag{7.28a}$$

$$-\beta + (\varphi^\varepsilon)_t\,(t_0, \mathbf{x}_0) = (\psi_\varepsilon)_t\,(t_0, \mathbf{x}_0) =: q_\varepsilon; \tag{7.28b}$$

(b) *there are symmetric matrices $\mathbf{A}_\varepsilon \leqslant \mathbf{B}_\varepsilon$ such that*

$$(q_\varepsilon + \beta, \mathbf{p}_\varepsilon, \mathbf{A}_\varepsilon) \in cD^+\varphi^\varepsilon(t_0, \mathbf{x}_0), \qquad (q_\varepsilon, \mathbf{p}_\varepsilon, \mathbf{B}_\varepsilon) \in cD^-\psi_\varepsilon(t_0, \mathbf{x}_0);$$

(c) *if $\mathbf{p}_\varepsilon = 0$, then $\mathbf{A}_\varepsilon = \mathbf{B}_\varepsilon = \mathbf{O}$.*

Proof. (a) Recall that φ^ε and ψ_ε are semiconvex and semiconcave, respectively. Thus there is a κ_ε such that

$$\bar\varphi(t, \mathbf{x}) = \varphi^\varepsilon(t, \mathbf{x}) - \kappa_\varepsilon(t^2 + |\mathbf{x}|^2)$$

is convex and

$$\bar\psi(t, \mathbf{x}) = \psi_\varepsilon(t, \mathbf{x}) - \kappa_\varepsilon(t^2 + |\mathbf{x}|^2)$$

is concave.

Let $(q_1, \mathbf{p}_1) \in \partial\bar{\varphi}(t_0, \mathbf{x}_0)$, $(q_2, \mathbf{p}_2) \in -\partial(-\bar{\psi})(t_0, \mathbf{x}_0)$. Since $(t_0, \mathbf{x}_0)$ is a maximizer of $W^\varepsilon(t, \mathbf{x})$,

$$\begin{aligned}\bar{\varphi}(t, \mathbf{x}) - \bar{\psi}(t, \mathbf{x}) &= W^\varepsilon(t, \mathbf{x}) + \beta t + 2\kappa_\varepsilon(t^2 + |\mathbf{x}|^2)\\ &\leqslant W^\varepsilon(t_0, \mathbf{x}_0) + \beta t + 2\kappa_\varepsilon(t^2 + |\mathbf{x}|^2).\end{aligned}$$

Also, by the definition of the subdifferentials $\partial\bar{\varphi}$ and $-\partial(-\bar{\psi})$,

$$\begin{aligned}&\bar{\varphi}(t, \mathbf{x}) - \bar{\psi}(t, \mathbf{x})\\ &\quad\geqslant \bar{\varphi}(t_0, \mathbf{x}_0) - \bar{\psi}(t_0, \mathbf{x}_0) + (q_1 - q_2)(t - t_0) + (\mathbf{p}_1 - \mathbf{p}_2) \cdot (\mathbf{x} - \mathbf{x}_0)\\ &\quad= W^\varepsilon(t_0, \mathbf{x}_0) + \beta t_0 + 2\kappa_\varepsilon(t_0^2 + |\mathbf{x}_0|^2) + (q_1 - q_2)(t - t_0)\\ &\qquad+ (\mathbf{p}_1 - \mathbf{p}_2) \cdot (\mathbf{x} - \mathbf{x}_0).\end{aligned}$$

Thus

$$(q_1 - q_2 - \beta)(t - t_0) + (\mathbf{p}_1 - \mathbf{p}_2) \cdot (\mathbf{x} - \mathbf{x}_0) \leqslant 2\kappa_\varepsilon(t^2 + |\mathbf{x}|^2 - t_0^2 - |\mathbf{x}_0|^2)$$

for all $(t, \mathbf{x})$, so that

$$q_1 = q_2 + \beta, \qquad \mathbf{p}_1 = \mathbf{p}_2$$

for all $(q_1, \mathbf{p}_1) \in \partial\bar{\varphi}(t_0, \mathbf{x}_0)$, $(q_2, \mathbf{p}_2) \in -\partial(-\bar{\psi})(t_0, \mathbf{x}_0)$. Hence $\partial\bar{\varphi}(t_0, \mathbf{x}_0)$ and $-\partial(-\bar{\psi})(t_0, \mathbf{x}_0)$ are singletons, and $\bar{\psi}$ and $\bar{\varphi}$ are differentiable at $(t_0, \mathbf{x}_0)$. Assertion (a) then follows from the definitions of $\bar{\psi}$ and $\bar{\varphi}$.

(b) Since φ^ε is semiconvex and ψ_ε semiconcave, this assertion follows from (7.25), (7.26), and Jensen's maximum principle [Je; CIL, Sect. 3; FS, Theorem 5.1, Sect. 5.5].

(c) Using (7.26) and (7.27) , we obtain

$$-3(|\mathbf{p}_\varepsilon|/\varepsilon)^{2/3} I \leqslant \mathbf{A}_\varepsilon \leqslant \mathbf{B}_\varepsilon \leqslant 3(|\mathbf{p}_\varepsilon|/\varepsilon)^{2/3} I. \quad \blacksquare$$

D. *Proof of Theorem* 3.3.

We will prove Theorem 3.3 by contradiction. Suppose that conclusion (3.13) is invalid.

1° By hypothesis, $\varphi \in \hat{M}([0, T] \times \mathbb{R}^2)$ and $\psi \in \hat{M}([0, T] \times \mathbb{R}^2)$; thus there are constants α, $\hat{\alpha}$, R such that

$$\varphi(t, \mathbf{x}) = \alpha, \qquad \psi(t, \mathbf{x}) = \hat{\alpha} \qquad \text{for} \quad |\mathbf{x}| \geqslant R, \quad t \in [0, T].$$

Thus, by (7.24), for all sufficiently small ε,

$$\varphi^{\varepsilon}(t, \mathbf{x}) = \alpha, \qquad \psi_{\varepsilon}(t, \mathbf{x}) = \hat{\alpha} \qquad \text{for} \quad |\mathbf{x}| \geqslant R+1, \quad t \in [0, T].$$

2° Set

$$I = \sup_{\mathbf{x} \in \mathbb{R}^2} [\varphi^*(0, \mathbf{x}) - \psi_*(0, \mathbf{x})]$$

Then $I \geqslant \alpha - \hat{\alpha}$. Since (3.13) does not hold, there are $(s, \mathbf{y}) \in (0, T] \times \mathscr{B}(R)$ and $\gamma > 0$ such that

$$\varphi^*(s, \mathbf{y}) - \psi_*(s, \mathbf{y}) \geqslant I + \gamma. \tag{7.29}$$

3° For $\varepsilon, \beta > 0$ consider the function

$$H^{\varepsilon}(t, \mathbf{x}) = \varphi^{\varepsilon}(t, \mathbf{x}) - \psi_{\varepsilon}(t, \mathbf{x}) - \beta t$$

for $(t, \mathbf{x}) \in [0, T] \times \mathbb{R}^2$. Then, by the definitions of φ^{ε} and ψ_{ε},

$$H^{\varepsilon}(t, \mathbf{x}) \geqslant \varphi^*(t, \mathbf{x}) - \psi_*(t, \mathbf{x}) - \beta t$$

for all $(t, \mathbf{x}) \in [0, T] \times \mathbb{R}^2$. In particular,

$$H^{\varepsilon}(s, \mathbf{y}) \geqslant I + \gamma - \beta T.$$

Therefore, by 1° and the inequality $I \geqslant \alpha - \hat{\alpha}$, for $\beta < \gamma/T$, H^{ε} achieves its maximum at some $(t(\varepsilon), \mathbf{x}(\varepsilon)) \in (0, T] \times \mathscr{B}(R+1)$.

4° Suppose $t(\varepsilon) \leqslant t_{\varepsilon}$ for all sufficiently small $\varepsilon > 0$, where t_{ε} is defined in (7.25). Since $|\mathbf{x}(\varepsilon)| \leqslant R+1$, there is a subsequence, also labeled by ε, such that $(t(\varepsilon), \mathbf{x}(\varepsilon)) \to (0, \mathbf{z})$ as $\varepsilon \downarrow 0$, and

$$I \geqslant \varphi^*(0, \mathbf{z}) - \psi_*(0, \mathbf{z}). \tag{7.30}$$

Also,

$$\varphi^{\varepsilon}(t(\varepsilon), \mathbf{x}(\varepsilon)) - \psi_{\varepsilon}(t(\varepsilon), \mathbf{x}(\varepsilon)) - \beta t(\varepsilon) \geqslant I + \gamma - \beta T. \tag{7.31}$$

Choose $(s(\varepsilon), \mathbf{y}(\varepsilon)) \in M^{\varepsilon}(t(\varepsilon), \mathbf{x}(\varepsilon))$. (Recall that $M^{\varepsilon}(t, \mathbf{x})$ is the set of all maximizers in the definition of $\varphi^{\varepsilon}(t, \mathbf{x})$.) Then by (7.24), $(s(\varepsilon), \mathbf{y}(\varepsilon)) \to (0, \mathbf{z})$ as $\varepsilon \downarrow 0$. Moreover,

$$\varphi^{\varepsilon}(t(\varepsilon), \mathbf{x}(\varepsilon)) \leqslant \varphi_*(s(\varepsilon), \mathbf{y}(\varepsilon))$$

and therefore

$$\limsup_{\varepsilon \downarrow 0} \varphi^{\varepsilon}(t(\varepsilon), \mathbf{x}(\varepsilon)) \leqslant \varphi^*(0, \mathbf{z}).$$

Similarly,

$$\liminf_{\varepsilon \downarrow 0} \psi_\varepsilon(t(\varepsilon), \mathbf{x}(\varepsilon)) \geq \psi_*(0, \mathbf{z}).$$

Using (7.30), (7.31), and the inequalities above, we are led to the inequality $\beta \geq \gamma/T$.

5° We now fix

$$\beta = \gamma/2T.$$

Then in view of the previous step, $t(\varepsilon) \leq t_\varepsilon$ for some small $\varepsilon > 0$. Let $(t_0, \mathbf{x}_0) = (t(\varepsilon), \mathbf{x}(\varepsilon))$. Then, by Lemma 7.4, φ^ε and ψ_ε are differentiable at $(t_0, \mathbf{x}_0)$ and (7.28) are satisfied. Moreover, there are $\mathbf{A}_\varepsilon \leq \mathbf{B}_\varepsilon$ satisfying Lemma 7.4b. We now write $q_0, \mathbf{p}_0, \mathbf{A}_0, \mathbf{B}_0$ for $q_\varepsilon, \mathbf{p}_\varepsilon, \mathbf{A}_\varepsilon, \mathbf{B}_\varepsilon$ to emphasize the fact that ε is now fixed. Since φ^ε and ψ_ε are, respectively, a viscosity subsolution and supersolution of (3.4), Lemma 7.4, (7.1), and (7.2) imply that

$$q_0 + \beta \leq \mathscr{F}^*(\mathbf{p}_0, \mathbf{A}_0), \tag{7.32}$$

$$q_0 \geq \mathscr{F}_*(\mathbf{p}_0, \mathbf{B}_0). \tag{7.33}$$

6° Suppose that $\mathbf{p}_0 = \mathbf{0}$. Then, by Lemma 7.4c, $\mathbf{A}_0 = \mathbf{B}_0 = 0$ and (7.32) and (7.33) yield

$$q_0 + \beta \leq \mathscr{F}^*(\mathbf{0}, \mathbf{0}) = 0 \leq q_0,$$

which contradicts the positivity of β. Hence $\mathbf{p}_0 \neq \mathbf{0}$.

7° Suppose that G is continuous at

$$\theta_0 = \theta(\mathbf{p}_0).$$

Then for any symmetric matrix $\mathbf{A}$,

$$\mathscr{F}^*(\mathbf{p}_0, \mathbf{A}) = \mathscr{F}_*(\mathbf{p}_0, \mathbf{A}) = \mathscr{F}(\mathbf{p}_0, \mathbf{A}).$$

Since $\mathbf{A}_0 \leq \mathbf{B}_0$, the ellipticity property (3.7), (7.32), and (7.33) yield

$$q_0 + \beta \leq \mathscr{F}^*(\mathbf{p}_0, \mathbf{A}_0) = \mathscr{F}(\mathbf{p}_0, \mathbf{A}_0) \leq \mathscr{F}(\mathbf{p}_0, \mathbf{B}_0) = \mathscr{F}_*(\mathbf{p}_0, \mathbf{B}_0) \leq q_0,$$

which again contradicts the positivity of β.

8° Suppose that G is discontinuous at θ_0. Then, by (1.2), there is a $\gamma > 0$ such that $G(\theta) = 0$ either for all $\theta \in [\theta_0, \theta_0 + \gamma]$ or for all $\theta \in [\theta_0 - \gamma, \theta_0]$. We will consider only the case in which

$$G(\theta) = 0, \qquad \forall \theta \in [\theta_0 - \gamma, \theta_0];$$

the other case is treated similarly. Let

$$\rho = B(\theta_0)^{-1}, \qquad G_0 = \rho \lim_{\theta \downarrow \theta_0} G(\theta).$$

Then

$$\mathscr{F}^*(\mathbf{p}_0, \mathbf{A}_0) = -\rho U |\mathbf{p}_0| + G_0(T(\mathbf{p}_0, \mathbf{A}_0))^+,$$
$$\mathscr{F}_*(\mathbf{p}_0, \mathbf{B}_0) = -\rho U |\mathbf{p}_0| - G_0(T(\mathbf{p}_0, \mathbf{B}_0))^-,$$

where $(\alpha)^+ = \max(\alpha, 0)$ and $(\alpha)^- = (-\alpha)^+$.

9° We will analyze three cases separately.

Case $\mathbf{A}_0 \geqslant 0$. Since $\mathbf{B}_0 \geqslant \mathbf{A}_0$, it follows that $\mathbf{B}_0 \geqslant 0$ and $T(\mathbf{p}_0, \mathbf{B}_0) \geqslant 0$. Then, by (7.33),

$$q_0 \geqslant \mathscr{F}_*(\mathbf{p}_0, \mathbf{B}_0) = -\rho U |\mathbf{p}_0|. \tag{7.34}$$

We now use Proposition 7.1 to construct

$$\begin{gathered}(q_n, \mathbf{p}_n, \mathbf{A}_n) \in cD^+\varphi^\varepsilon(t_n, \mathbf{x}_n), \\ (q_n, \mathbf{p}_n) \to (q_0 + \beta, \mathbf{p}_0), \qquad (t_n, \mathbf{x}_n) \to (t_0, \mathbf{x}_0)\end{gathered} \tag{7.35}$$

satisfying (7.11b). By (7.35)

$$q_n \leqslant \mathscr{F}^*(\mathbf{p}_n, \mathbf{A}_n)$$

and therefore

$$q_0 + \beta \leqslant \liminf_{n \to \infty} \mathscr{F}^*(\mathbf{p}_n, \mathbf{A}_n). \tag{7.36}$$

On the other hand, (7.11b) implies that

$$\liminf_{n \to \infty} \mathscr{F}^*(\mathbf{p}_n, \mathbf{A}_n) \leqslant -\rho U |\mathbf{p}_0|. \tag{7.37}$$

Indeed, (7.11b) yields either $\theta(\mathbf{p}_n) < \theta_0$ or

$$\liminf_{n \to \infty} T(\mathbf{p}_n, \mathbf{A}_n) \leqslant 0. \tag{7.38}$$

In the first case $G(\theta(\mathbf{p}_n)) = 0$ and

$$\mathscr{F}^*(\mathbf{p}_n, \mathbf{A}_n) = -\rho U |\mathbf{p}_n|,$$

and hence (7.37) follows from the convergence of $\mathbf{p}_n$ to $\mathbf{p}_0$. On the other hand, (7.38) and 8° yield

$$\mathscr{F}^*(\mathbf{p}_n, \mathbf{A}_n) \leqslant -\rho U |\mathbf{p}_n| + \max_\theta[G(\theta)\, B(\theta)^{-1}](T(\mathbf{p}_n, \mathbf{A}_n))^+,$$

which implies (7.37). Now combine (7.36) and (7.37) to obtain

$$\varepsilon+\beta \leqslant \rho U\,|\mathbf{p}_0|,$$

which, with (3.2) contradicts the positivity of ρ.

Case $\mathbf{B}_0 \leqslant 0$. Then $\mathbf{A}_0 \leqslant 0$ and

$$\mathscr{F}^*(\mathbf{p}_0, \mathbf{A}_0) = -\rho U\,|\mathbf{p}_0|,$$

and we may use Proposition 7.1 with $-\psi_\varepsilon$ and argue exactly as in the previous case to obtain a contradiction.

Case $\mathbf{A}_0 < 0 < \mathbf{B}_0$. Then

$$q_0+\beta \leqslant \mathscr{F}^*(\mathbf{p}_0, \mathbf{A}_0) = -\rho U\,|\mathbf{p}_0| = \mathscr{F}_*(\mathbf{p}_0, \mathbf{B}_0) = \mathscr{F}_*(\mathbf{p}_0, \mathbf{B}_0) \leqslant q_0,$$

which once again contradicts the positivity of β. ∎

8. Proof of Theorems 3.1 and 3.2

Proof of Theorem 3.2. The uniqueness of a level-set solution of (1.1) corresponding to an auxiliary function Φ_0 follows from Theorem 3.3.

Let $\{\theta_1, \theta_2, ..., \theta_M\}$ be the set of points of discontinuity of G. For n a sufficiently large positive integer, let G_n be the continuous 2π-periodic function with $G_n(\theta) = G(\theta)$ for $|\theta-\theta_k| \geqslant 1/n$, $k = 1, 2, ..., M$, and $G_n(\theta)$ linear otherwise. Further, let $\mathscr{F}_n$ denote the function defined by (3.3) with G replaced by G_n. Then $\mathscr{F}_n$ approximates $\mathscr{F}$ in the sense of the following lemma, whose proof we omit.

Lemma 8.1. *Let* $(\mathbf{p}_n, \mathbf{A}_n) \to (\mathbf{p}, \mathbf{A}) \in \mathbb{R}^2 \times S$ *as* $n \to \infty$. *Then*

$$\limsup_{n\to\infty}\, (\mathscr{F}_n)^*\,(\mathbf{p}_n, \mathbf{A}_n) \leqslant \mathscr{F}^*(\mathbf{p}, \mathbf{A}),$$

$$\liminf_{n\to\infty}\, (\mathscr{F}_n)_*\,(\mathbf{p}_n, \mathbf{A}_n) \geqslant \mathscr{F}_*(\mathbf{p}, \mathbf{A}).$$

Since G_n is continuous, we may use [CGG, Theorem 6.8] to conclude that there is a unique, continuous viscosity solution $\Phi_n \in \hat{M}([0, \infty) \times \mathbb{R}^2)$ —of (3.2) with $\mathscr{F}$ replaced by $\mathscr{F}_n$—satisfying $\Phi_n(\mathbf{x}, 0) = \Phi_0(\mathbf{x})$, and we define Φ^+ and Φ^- as in the proof of Theorem 5.2. Moreover, Lemma 8.1 together with classical stability results for viscosity solutions [FS, Sect 2.6 and 7.4] imply that Φ^+ and Φ^- are, respectively, a viscosity subsolution and a viscosity supersolution of (3.2) on $(0, \infty) \times \mathbb{R}^2$. We now follow the steps $4°$–$8°$ in the proof of Theorem 5.2 to conclude that $\Phi^+ = \Phi^- = \Phi$. Hence Φ is a level-set solution of (1.1) corresponding to Φ_0.

We complete the proof by establishing (3.12). Let Φ be a level-set solution of (1.1) corresponding to an auxiliary function Φ_0.

For $\delta > 0$, let $\eta_\delta: \mathbb{R} \to \mathbb{R}$ be smooth and satisfy: (i) $\eta'_\delta \geqslant 0$; (ii) $\eta_\delta(r) = 0$ for $r \leqslant c$; (iii) $\eta_\delta(r) = 1$ for $r \geqslant c + \delta$. Then the geometric property (3.6) implies that

$$\Phi_\delta(t, \mathbf{x}) = \eta_\delta(\Phi(t, \mathbf{x}))$$

is a level-set solution of (1.1).[24]

Next,

$$u^+(t, \mathbf{x}) = \limsup_{\substack{\delta \downarrow 0 \\ (s, \mathbf{y}) \to (t, \mathbf{x})}} \Phi_\delta(s, \mathbf{y})$$

is a viscosity subsolution of (3.2) (cf. [FS, Sect. 2.6 and 7.4]). If we let $\hat{u}(t, \mathbf{x})$ be the characteristic function of

$$\hat{\mathscr{U}} = \{\mathbf{x}: \Phi(t, \mathbf{x}) \geqslant c\},$$

then the continuity of Φ and the properties on η_δ yield

$$u^+(t, \mathbf{x}) = \hat{u}^*(t, \mathbf{x}) = \hat{u}(t, \mathbf{x}),$$

so that $\hat{\mathscr{U}}(t)$ is a χ-subsolution of (1.1). In fact, since

$$\operatorname{cl} \Omega_0 = \{\mathbf{x}: \Phi_0(\mathbf{x}) \geqslant c\} = \{\mathbf{x}: \limsup_{s \downarrow 0, \mathbf{y} \to \mathbf{x}} \hat{u}(s, \mathbf{y}) = 1\},$$

$\hat{\mathscr{U}}(t)$ is a χ-subsolution of (1.1) compatible with Ω_0.

Similarly,

$$u^-(t, \mathbf{x}) = \liminf_{\substack{\delta \downarrow 0 \\ (s, \mathbf{y}) \to (t, \mathbf{x})}} \Phi_\delta(s, \mathbf{y})$$

is a viscosity supersolution of (3.2), and, further, $u^- = \hat{u}_*$; hence $\hat{\mathscr{U}}(t)$ is also χ-supersolution of (1.1) compatible with Ω_0. Thus $\hat{\mathscr{U}}(t)$ is a χ-solution of (1.1) compatible with Ω_0.

Next, in view of the definition $\mathscr{U}(t)$, $\hat{\mathscr{U}}(t) \subseteq \mathscr{U}(t)$. In fact, they are equal. To verify this, let $\Omega(t)$ be a χ-subsolution compatible with Ω_0, and let $u(t, \mathbf{x})$ be the characteristic function of $\Omega(t)$. For any $d < c$, let $u(t, \mathbf{x}; d)$ be the characteristic function of

$$\mathscr{L}(t; d) = \{\mathbf{x}: \Phi(t, \mathbf{x}) > d\}.$$

[24] Cf. [CGG, Theorem 5.6.] for the proof of this fact when G is continuous.

Then $u^*(0, \mathbf{x}) \leqslant u_*(0, \mathbf{x}; d) = u(0, \mathbf{x}; d)$, since $\Omega(t)$ is compatible with Ω_0 and $\Phi_0(\mathbf{x}) = \Phi(0, \mathbf{x})$ is an auxiliary function for Ω_0. Then, by Corollary 3.1, $\Omega(t) \subset \mathscr{L}(t; d)$ for any χ-subsolution $\Omega(t)$ of (1.1) compatible with Ω_0. Hence $\mathscr{U}(t) \subseteq \mathscr{L}(t; d)$ for all $d < c$, and, since the intersection over all such d of $\mathscr{L}(t; d)$ is $\hat{\mathscr{U}}(t)$, $\mathscr{U}(t) \subseteq \hat{\mathscr{U}}(t)$. Thus $\mathscr{U}(t) = \hat{\mathscr{U}}(t)$.

The analogous assertion for $\mathscr{L}(t)$ is proved in the same manner. ∎

Proof of Theorem 3.1. Parts (a) and (b) follow from Theorem 3.2. To prove (c) note first that, since $\partial\Omega_0$ is C^3, there is a C^3 parametrization $\alpha \mapsto \mathbf{Y}(\alpha)$ ($\mathbb{R} \to \mathbb{R}^2$), periodic with period 1, such that

$$\partial\Omega_0 = \{\mathbf{Y}(\alpha) : \alpha \in [0, 1]\}$$

and

$$\mathbf{Y}'(\alpha)/|\mathbf{Y}'(\alpha)| = \mathbf{T}(\theta_0(\alpha)),$$

where $\theta_0(\alpha)$ the normal-angle at $\mathbf{Y}(\alpha)$ and $\mathbf{T}(\theta)$ is defined in (1.10).

Proceeding formally, let $\Omega(t)$ be a solution of (1.1) such that

$$\mathbf{X}(t, \alpha) = \mathbf{Y}(\alpha) + h(t, \alpha)\, \mathbf{N}(\theta_0(\alpha))$$

is a parametrization[25] of $\partial\Omega(t)$ for some real-valued function $h(t, \alpha)$. Then

$$h(t, \alpha) \text{ is periodic in } \alpha \text{ with period 1.} \tag{8.1}$$

Assume that h is C^2. Let $K_0(\alpha)$ denote the curvature of $\partial\Omega_0$ at $\mathbf{Y}(\alpha)$, and let $\theta(t, \alpha)$, $V(t, \alpha)$, and $K(t, \alpha)$ denote the normal-angle, normal-velocity, and curvature of $\partial\Omega(t)$ at $\mathbf{X}(t, \alpha)$. Then

$$\mathbf{T}(\theta(t, \alpha)) = \mathbf{X}_\alpha(t, \alpha)/|\mathbf{X}(t, \alpha)|$$

(where the subscript denotes differentiation with respect to that variable), and defining

$$F_1(\alpha, h, h_\alpha) = [F_2(\alpha, h)^2 + h_\alpha^2]^{1/2},$$
$$F_2(\alpha, h) = |\mathbf{Y}'(\alpha)| - hK_0(\alpha),$$

a tedious computation yields

$$\theta(t, \alpha) = \theta(\alpha, h(t, \alpha), h_\alpha(t, \alpha)),$$

with $\theta(\alpha, h, h_\alpha)$ the solution of

$$\mathbf{T}(\theta(\alpha, h, h_\alpha)) = [F_2(\alpha, h)\, \mathbf{T}(\theta_0(\alpha)) + h_\alpha \mathbf{N}(\theta_0(\alpha))]\, F_1(\alpha, h, h_\alpha)^{-1} \tag{8.2}$$

[25] A similar parametrization was used by Chen and Reitich [CR] in their proof of local existence for a modified Stefan problem.

and

$$V(t, \alpha) = \mathbf{X}_t(t, \alpha) \cdot \mathbf{N}(\theta(t, \alpha)) = h_t F_2(\alpha, h)/F_1(\alpha, h, h_\alpha), \tag{8.3a}$$

$$\begin{aligned} K(t, \alpha) &= \theta_\alpha(t, \alpha)/|\mathbf{X}_\alpha(t, \alpha)| = F_3(\alpha, h, h_\alpha, h_{\alpha\alpha}) \\ &= \{F_2(\alpha, h)[h_{\alpha\alpha} + hK_0(\alpha)^2] + K_0(\alpha)\, |\mathbf{Y}'(\alpha)|^3 \\ &\quad + h_\alpha[2K_0(\alpha)\, h_\alpha + K_0'(\alpha)\, h] - \mathbf{T}(\theta_0(\alpha)) \cdot \mathbf{Y}''(\alpha)\, h_\alpha \\ &\quad - \mathbf{N}(\theta_0(\alpha)) \cdot \mathbf{Y}''(\alpha)\, hK_0(\alpha)\}\; F_1(\alpha, h, h_\alpha)^{-3}, \end{aligned} \tag{8.3b}$$

(Note that $\theta(t, \alpha)$ is well defined provided the right side of (8.2) is nonzero.) Thus, since

$$B(\theta(t, \alpha))\; V(t, \alpha) = G(\theta(t, \alpha))\; K(t, \alpha) - U,$$

$h(t, \alpha)$ satisfies

$$\tilde{B}(\alpha, h, h_\alpha)\, h_t = \tilde{G}(\alpha, h, h_\alpha)\, h_{\alpha\alpha} - \tilde{F}(\alpha, h, h_\alpha), \tag{8.4}$$

with

$$\begin{aligned} \tilde{B}(\alpha, h, h_\alpha) &= B(\theta(\alpha, h, h_\alpha)), \\ \tilde{G}(\alpha, h, h_\alpha) &= G(\theta(\alpha, h, h_\alpha))\; F_1(\alpha, h, h_\alpha)^{-2}, \\ \tilde{F}(\alpha, h, h_\alpha) &= \tilde{G}(\alpha, h, h_\alpha)\{hK_0(\alpha)^2 + F_2(\alpha, h)^{-1}\, (K_0(\alpha)\, |\mathbf{Y}'(\alpha)|^3 \\ &\quad + h_\alpha[2K_0(\alpha)\, h_\alpha + K_0'(\alpha)\, h] - \mathbf{T}(\theta_0(\alpha)) \cdot \mathbf{Y}''(\alpha)\, h_\alpha \\ &\quad - \mathbf{N}(\theta_0(\alpha)) \cdot \mathbf{Y}''(\alpha)\, hK_0(\alpha))\} - UF_1(\alpha, h, h_\alpha)\, F_2(\alpha, h)^{-1}. \end{aligned}$$

We will complete the proof by solving (8.4) subject to $h(\alpha, 0) = 0$. Let

$$\mathcal{A} = \{(\alpha, h)\colon |h| \leqslant |\mathbf{Y}'(\alpha)|/2\, |K_0(\alpha)|\}.$$

Then

$$|\mathbf{Y}'(\alpha)| - hK_0(\alpha) \geqslant |\mathbf{Y}'(\alpha)|/2 > 0.$$

Hence the right side of (8.2) is nonzero and $\theta(\alpha, h, h_\alpha)$ is continuous on $\mathcal{A} \times \mathbb{R}$. Moreover, $\tilde{F}\colon \mathcal{A} \times \mathbb{R} \to \mathbb{R}$ is continuous; $\tilde{B}\colon \mathcal{A} \times \mathbb{R} \to [0, \infty)$ is continuous and strictly positive; $\tilde{G}\colon \mathcal{A} \times \mathbb{R} \to [0, \infty)$ is continuous except at finitely two-dimensional manifolds in $\mathcal{A} \times \mathbb{R}$, and suffers at most jump discontinuities across such manifolds.

Although $\tilde{G}$ has discontinuities, one can prove a comparison result for viscosity sub- and supersolutions of (8.4) using a modification of the analysis given in Section 7. Indeed, requisite modifications of all arguments except Proposition 7.1 are either straightforward or minor, and Proposition 7.1 should be replaced by

PROPOSITION 8.1. *Let $v(t, \alpha)$ be semiconvex on $[0, \infty) \times \mathbb{R}$ and differentiable at (t_0, α_0) with $v_\alpha(t_0, \alpha_0) \neq 0$. Then there are $(t_n, \alpha_n) \to (t_0, \alpha_0)$ and $(q_n, \mathbf{p}_n, a_n,) \in cD^+v(t_n, \alpha_n) \subset \mathbb{R}^3$ such that*

$$\lim_{n \to \infty} (q_n, \mathbf{p}_n) = (v_t, v_\alpha)(t_0, \alpha_0) = (q_0, p_0),$$

$$\liminf_{n \to \infty} \min\{\theta(\alpha_n, h_n, \mathbf{p}_n) - \theta(\alpha_0, h_0, \mathbf{p}_0), F_3(\alpha_n, h_n, \mathbf{p}_n, a_n)\} \leqslant 0,$$

where $h_n = v(t_n, \alpha_n)$, $h_0 = v(t_0, \alpha_0)$, and F_3 is defined in (8.3).

Once a comparison result has been obtained, the existence of a unique viscosity solution h of (8.4), satisfying (8.1) and an initial condition for $h(t, 0)$, can be established utilizing an approximation argument of the type used in the proof of Theorem 3.1. This solution is defined on $[0, \hat{T}_{\max}] \times \mathbb{R}$, where $\hat{T}_{\max}$ is the largest time satisfying $(\alpha, h(t, \alpha)) \in \mathcal{O}$ for all $(t, \alpha) \in [0, \hat{T}_{\max}] \times \mathbb{R}$.

Let

$$\varepsilon_0 = \inf_{\alpha \in \mathbb{R}} \{|\mathbf{Y}'(\alpha)|/4\, |K_0(\alpha)|\},$$

and for $|\varepsilon| \leqslant \varepsilon_0$, let $h(t, \alpha; \varepsilon)$, $(t, \alpha) \in [0, \hat{T}_{\max}(\varepsilon)] \times \mathbb{R}$, be the unique viscosity solution of (8.4) satisfying (8.1) and $h(0, \alpha; \varepsilon) = \varepsilon$. The uniqueness associated with such solutions ensures that $h(t, \alpha; \varepsilon)$ depends continuously on ε. Our next step will be to show that

$$T_* := \inf\{\hat{T}_{\max}(\varepsilon) \colon |\varepsilon| \leqslant \varepsilon_0\}$$

satisfies

$$0 < T_* \leqslant T_{\text{uniq}}.$$

To verify this assertion, define, for $(t, \mathbf{x}) \in [0, T_*] \times \mathbb{R}^2$,

$$\varphi(t, \mathbf{x}) = \begin{cases} \varepsilon & \text{if } \ \mathbf{x} \in \partial\Omega(t; \varepsilon), \quad |\varepsilon| \leqslant \varepsilon_0, \\ \varepsilon_0 & \text{if } \ \mathbf{x} \in \partial\Omega(t; \varepsilon_0), \\ -\varepsilon_0 & \text{if } \ \mathbf{x} \notin \partial\Omega(t; -\varepsilon_0), \end{cases}$$

where $\Omega(t; \varepsilon)$ is the closed region enclosed by

$$\{\mathbf{Y}(\alpha) + h(t, \alpha; \varepsilon)\, \mathbf{N}(\theta_0(\alpha)) \colon \alpha \in [0, 1]\}. \tag{8.5}$$

(Since $t \leqslant T_*$, $(\alpha, h(t, \alpha; \varepsilon)) \in \mathcal{A}$ and the curve (8.5) enclose a region.) Further, a tedious calculation shows that φ is a level-set solution of (1.1) corresponding to an auxiliary function compatible with Ω_0. By Theorem 3.2 (cf. (3.12)),

$$\mathscr{U}(t) = \{\mathbf{x}: \Phi(t, \mathbf{x}) \geqslant 0\}, \qquad \mathscr{L}(t) = \{\mathbf{x}: \Phi(t, \mathbf{x}) > 0\}.$$

Since h depends continuously on ε,

$$T_{\mathrm{uniq}} \geqslant T^*.$$

To establish the positivity of T^*, observe that, by the maximum principle (or comparison result for (8.4)),

$$|h(t, \alpha; \varepsilon) - \varepsilon| \leqslant \kappa t$$

for all $|\varepsilon| \leqslant \varepsilon_0$, $t \in [0, \hat{T}_{\max}(\varepsilon))$, where κ is a suitable constant depending on the C^3 norm of $\partial\Omega_0$. Hence

$$|h(t, \alpha; \varepsilon)| \leqslant 2\varepsilon_0$$

for all $t \leqslant \varepsilon_0/\kappa$. Finally, the definitions of $\mathcal{A}$ and ε_0 imply that $T^* \geqslant \varepsilon_0/\kappa$. ▮

Acknowledgment

This work was supported by the Army Research Office and the National Science Foundation.

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