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Motion of a Set by the Curvature of Its Boundary

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The study of a crystal shrinking or growing in a melt gives rise to equations relating the normal velocity of the motion to the curvature of the crystal boundary. Often these equations are anisotropic, indicating the preferred directions of the crystal structure. In the isotropic case this equation is called the curve shortening or the mean curvature flow equation, and has been studied by differential geometric tools. In general, there are no classical solutions to these equations. In this paper we develop a weak theory for the generalized mean curvature equation using the newly developed theory of viscosity solutions. Our approach is closely related to that of Osher and Sethian, Chen, Giga and Goto, and Evans and Spruck, who view the boundary of the crystal as the level set of a solution to a nonlinear parabolic equation. Altough we use their results in an essential way, we give an *intrinsic* definition. Our main results are the existence of a solution, large time asymptotics of this solution, and its connection to the level set solution of Osher and Sethian, Chen, Giga and Goto, and Evans and Spruck. In general there is no uniqueness, even for classical solutions, but we prove a uniqueness result under restrictive assumptions. We also construct a class of explicit solutions which are dilations of Wulff crystals. © 1993 Academic Press, Inc.

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1. Introduction

In a series of papers, Gurtin studied the nonequilibrium thermomechanics of two-phase continua [26, 27]. Based on the assumption of a sharp interface with its own thermomechanical structure, he derived the thermomechanic restrictions on the constitutive relations and several free boundary problems that are consistent with these restrictions. In simple versions of these problems, the temperature distribution solves the heat equation in the interior of each phase, and certain boundary conditions are satisfied at the interface. Hence the interface motion is coupled with the evolution of the temperature, and this coupling renders the problem difficult.[1] For perfect conductors, however, the temperature is constant and the free boundary condition reduces to a single equation[2]

$$\beta(\theta)\mathbf{V} + \operatorname{trace}(G(\theta)\cdot\mathbf{R}) - v = 0, \tag{E}$$

where θ, $\mathbf{V}$, and $\mathbf{R}$ are the outward unit normal vector, the normal velocity, and the second fundamental form of the interface $\partial C(t)$, respectively. The kinetic coefficient $\beta(\theta)$ measures the drag opposing the motion in the θ direction, $G(\theta)$ is a linear function of the interfacial energy (or surface tension) and its second derivatives, and the constant v is the energy density difference between two phases.

Angenent and Gurtin [6] developed an extensive theory for the perfect conductors in two space dimensions. Further development of a general theory including the nonperfect conductors in space dimensions higher than two, however, requires an analytical study of the interface motion equation (E). In this paper we develop a weak theory for (E), and then obtain asymptotic results for large time.

We begin with the isotropic version of the interface equation.[3] In this case (E) takes a particularly simple form

$$\mathbf{V} = -\kappa, \tag{MCE}$$

where κ is the mean curvature. Following the standard terminology in differential geometry, we refer to (MCE) as the curve shortening or the mean curvature flow equation. Since (MCE) is the gradient flow for the area functional, the surface area of any classical solution of the isotropic equa-

[1] Recently Gurtin and Soner [28] studied these problems in one space dimension. Also a preliminary analysis of multi-dimensional problems is contained in [29].

[2] See Angenent and Gurtin [6].

[3] In a slightly different set-up, Mullins [37] derived the isotropic version of this equation as a simple model. For a more detailed discussion, we refer the reader to Sekerka [39]. Also see Allen and Cahn [2].

tion is nonincreasing in time. Owing to this property, the mean curvature flow has been studied extensively by differential geometric tools. The flow of an embedded plane curve under the mean curvature flow equation was analysed by Gage [19, 20], Gage and Hamilton [21], and Grayson [23]. They showed that a smooth embedded plane curve first becomes convex and then shrinks to a point in finite time. Also the limiting shape is a circle. Huisken [30] generalized this result showing that any convex set, in any space dimension, shrinks to a point smoothly. The flow of a smooth curve embedded in a smooth Riemannian surface was pursued by Grayson [24]. Recently, Angenent generalized some of the two dimensional results [3–5] to the nonlinear case.

All of the above results show the existence of a classical flow. However, this is not always the case if the space dimension is larger than two and the initial shape is not convex. Indeed, Abresch and Langer [1] and Epstein and Weinstein [14], in their study of nonembedded curves, proved that these curves develop singularities before they shrink to a point. In higher dimensions, even the smooth embedded hypersurfaces may develop singularities. Grayson [25] gives the example of a dumbell shape in R^3. This is a region obtained by connecting two spheres by a thin long pipe. Grayson argues that under the mean curvature flow, the boundary of this region will pinch off, leaving two bubbles. Also numerical studies of Sethian [41] support this observation. Hence a weaker formulation of this equation is necessary to define the subsequent evolution after the onset of singularities.

Brakke [9] was the first to study the mean curvature flow past the singularities. Using varifolds of geometric measure theory, he constructed global generalized solutions for a large class of initial conditions. An alternate approach, initially suggested for numerical calculations by Sethian [40] and Osher and Sethian [38] and for a flame propogation model by Barles [7],[4] represents the solution as the zero level set of the auxiliary continuous function Φ. This latter suggestion has been extensively developed by Evans and Spruck [15], [16] and Chen, Giga, and Goto [10].[5] To describe this approach in detail, let $\partial C(t)$ be the interface and Φ be an auxiliary function satisfying

$$\partial C(t) = \{x \in R^d : \Phi(x, t) = 0\} \tag{1.1}$$

with

$$C(t) = \{x \in R^d : \Phi(x, t) > 0\}.$$

[4] The equation studied in [7] is of first order.

[5] Chen, Giga, and Goto consider a more general class of equations than (E), which they call geometric. A discussion of this class of equations is given by Giga and Goto [22].

Then, Φ formally satisfies the parabolic equation

$$\beta\left(-\frac{D\Phi(x,t)}{|D\Phi(x,t)|}\right)\frac{(\partial/\partial t)\,\Phi(x,t)}{|D\Phi(x,t)|}$$
$$=\operatorname{trace}\left[G\left(-\frac{D\Phi(x,t)}{|D\Phi(x,t)|}\right)D\left(\frac{D\Phi(x,t)}{|D\Phi(x,t)|}\right)\right]+\nu, \qquad \forall t>0,\ \Phi(x,t)=0, \tag{1.2}$$

where D denotes the differentiation with respect to the spatial variable x. The above equation is nonlinear, degenerate, and undefined when $D\Phi(x,t)=0$. Evans and Spruck and Chen *et al.* circumvent these rather subtle technical problems using the newly developed theory of viscosity solutions of nonlinear partial differential equations as in Crandall and Lions [12], Crandall, Evans, and Lions [11], Jensen [33], Jensen, Lions, and Souganidis [34], Lions [35, 36], and Ishii [32].

In this paper, we define a notion of viscosity solutions of (E) which is closely related to the one given in [10, 15]. We observe that the "signed" distance function

$$d_C(x,t)=\begin{cases}\text{distance}\,(x,\partial C(t)) & \text{if } x\in C(t)\\ -\text{distance}\,(x,\partial C(t)) & \text{if } x\notin C(t)\end{cases} \tag{1.3}$$

of $C(\cdot)$ satisfies

$$\beta(-Dd_C(x,t))\frac{\partial}{\partial t}d_C(x,t)$$
$$=\operatorname{trace}[G(-Dd_C(x,t))\,D^2\,d_C(x,t)]+\nu \qquad \forall x\in\partial C(t), \tag{1.4}$$

as long as $\partial C(t)$ is smooth. Using the viscosity formulations of first and second derivatives of semi-continuous functions, we give an *intrinsic* definition of viscosity solutions of (E) by requiring that (1.4) should hold in the viscosity sense.[6] Following the techniques and the results of [10, 15], we obtain an existance result and prove that any supersolution $\{C(t)\}_{t\geqslant 0}$ of (E) includes any subsolution $\{\Gamma(t)\}_{t\geqslant 0}$ of (E) provided that the closure of $\Gamma(0)$ is a compact subset of $C(0)$. In general, there is more than one solution of the initial value problem. In fact, this is the case whenever the level set of the solution of Evans and Spruck and Chen *et al.* develops a nonempty interior (Section 8). However, the comparison result enables us to define two solutions. One of these solutions contains all the subsolutions of

[6] See Definition 5.1 and Section 4 for the precise formulation.

the initial value problem and the other is contained in all the supersolutions of the initial value problem. These solutions are given by

$$U(t) = \{x \in R^d : \Phi(x, t) \geqslant 0\},$$

and

$$L(t) = \{x \in R^d : \Phi(x, t) > 0\},$$

where Φ is a solution of (1.2) on the whole domain $R^d x(0, \infty)$ (Section 10). For a nonpositive v, we also prove the uniqueness of solutions to the initial value problem provided that the solution is initially "strictly" starshaped (Section 11).

The stationary version of (E) is formally related to a variational problem and its celebrated solution is known as the Wulff crystal [44] of the interfacial energy.[7] It is also known that any solution of (E) shrinks to an empty set in finite time if v is nonpositive, or if the solution is initially small.[8] But if v is strictly positive and initially the solution is large enough, then the solution grows for all time.[9] In Sections 12 and 13 we construct a class of explicit solutions which are dilations of the Wulff crystal of $(1/\beta)$ and then use them to obtain asymptotic results. In particular, we show that the solution asymptotically looks like the Wulff crystal of $(1/\beta)$, when it is growing.[10]

In a forthcoming paper with Evans and Souganidis [17], we use the signed distance function to prove asymptotic results for a reaction-diffusion equation with fast reaction and slow diffusion. We show that in the limit, the motion of the zero level curve of the solution is governed by the generalized mean curvature flow. This is an independent check on the reasonableness of the weak theory developed here and in [10, 15].

2. Preliminaries

In this section we make several definitions which will be used throughout the paper. We will use the notation cl A, int A, and A^c to denote the closure of A, the interior of A, and the complement of A, respectively. Let S^{d-1} be the set of all unit vectors in R^d. We assume that $G(\theta)$ is a $d x d$ symmetric matrix, and for all $\theta \in S^{d-1}$,

$$\beta \text{ and } G \text{ are continuous on } S^{d-1}, \text{ and } \beta(\theta) > 0, G(\theta) \geqslant 0, G(\theta)\theta = 0. \tag{A}$$

[7] Also see [13, 42, 43, 18] for its properties and definition of it, and Section 6.1 of Angenent and Gurtin [6] for the connection between the equation (E) and the Wulff crystal.

[8] This result is proved in [6, 15, 10], under different assumptions.

[9] In two space dimensions it is proved in [6].

[10] This result was conjectured by Angenent and Gurtin [6].

DEFINITION 2.1. (a) A subset of R^d is called *proper* if the interior of itself and its complement are nonempty.

(b) Let $\{C(t)\}_{t\in[0,T)}$ be a collection of proper subsets of R^d. The *signed distance function* of $\{C(t)\}_{t\in[0,T)}$ is

$$d_C(x,t)=\begin{cases}\operatorname{dist}(x,\partial C(t)) & \text{if } x\in C(t)\\ -\operatorname{dist}(x,\partial C(t)) & \text{if } x\notin C(t)\end{cases}\quad \forall t\in[0,T),$$

where $\operatorname{dist}(x, A)$ denotes the distance between the point x and the set A.

Finally we define the notion of a classical solution of (E). Basically we require that the signed distance function is smooth on a neighborhood of the boundary, and satisfies (1.4). Also an additional "causality" condition is needed (see Example 5.5), but we do not require the global continuity of the distance function.

DEFINITION 2.2. We say that an open collection of smooth subsets $\{C(t)\}_{t\geq 0}$ of R^d is a *classical solution* of (E) if:

(a) There is $0<T\leqslant\infty$ such that

$$C(t)=\varnothing \qquad \forall t\geqslant T,$$

or

$$C(t)=R^d \qquad \forall t\geqslant T,$$

and for $t<T$, $C(t)$ is proper, d_C is smooth in a (space-time) neighborhood of every boundary point $x\in\partial C(t)$, and d_C satisfies (1.4).

(b) For every $t<T$ and $x\in R^d$,

$$\limsup_{(z,s)\to(x,t)}\,[d_C(z,s)\wedge 0]=\limsup_{(z,s)\uparrow(x,t)}\,[d_C(z,s)\wedge 0], \tag{2.1}$$

$$\liminf_{(z,s)\to(x,t)}\,[d_C(z,s)\vee 0]=\liminf_{(z,s)\uparrow(x,t)}\,[d_C(z,s)\vee 0]. \tag{2.2}$$

3. SEMI-CONTINUOUS ENVELOPES

In this section, we define the upper and lower semi-continuous envelopes of a collection of subsets of R^d. We also prove several elementary properties of them which will be used in later sections.

For a given collection of proper subsets $\{C(t)\}_{t \in [0, T)}$ of R^d and for $t \in [0, T)$ define

$$C^*(t) = \bigcap_{\varepsilon > 0} \mathrm{cl}\left[\bigcup_{|t-s| \leq \varepsilon, s < T} C(s) \right], \tag{3.1}$$

$$C_*(t) = \bigcup_{\varepsilon > 0} \mathrm{int}\left[\bigcap_{|t-s| \leq \varepsilon, s < T} C(s) \right]. \tag{3.2}$$

Observe that for every $t \in [0, T)$, $C^*(t)$ is a closed set and $C_*(t)$ is an open set.

LEMMA 3.1. *Let $\{C(t)\}_{t \in [0, T)}$ be a collection of proper subsets of R^d.*

(a) $R^d \backslash C_*(t) = (R^d \backslash C(\cdot))^*(t)$, $\forall t \in [0, T)$,

(b) $R^d \backslash C^*(t) = (R^d \backslash C(\cdot))_*(t)$, $\forall t \in [0, T)$,

(c) *if $(x_n, t_n) \to (x, t)$ and $x_n \in C^*(t_n)$, then $x \in C^*(t)$,*

(d) *if $(x_n, t_n) \to (x, t)$ and $x_n \notin C_*(t_n)$, then $x \notin C_*(t)$,*

(e) *the map $(x, t) \to d_{C^*}(x, t)$ is upper semi-continuous,*

(f) *the map $(x, t) \to d_{C_*}(x, t)$ is lower semi-continuous.*

Proof. (a) and (b) These follow from De Morgan's rule.

(c) Since $x_n \in C^*(t_n)$ and $(x_n, t_n) \to (x, t)$, there is a sequence $(y_n, s_n) \to (x, t)$ as n tends to infinity, and satisfying $y_n \in C(s_n)$ for all n. Set

$$\varepsilon_k = \sup\{ |s_n - t| : n \geq k \}.$$

Observe that

$$y_n \in C(s_n) \subset \bigcup_{|t-s| \leq \varepsilon_k, s \geq 0} C(s), \qquad \text{if} \quad n \geq k.$$

Hence,

$$x \in \mathrm{cl}\left\{ \bigcup_{|t-s| \leq \varepsilon_k, s \geq 0} C(s) \right\}, \qquad \forall k.$$

(d) This follows from (a) and (c).

(e) Fix $(x, t) \in R^d \times [0, T)$. Set $\beta = d_{C^*}(x, t)$. Suppose that $\beta \geq 0$. Then, there is a sequence $y_n \notin C^*(t)$ satisfying

$$|x - y_n| \to \beta.$$

The definition of $C^*(t)$ yields that there are $\varepsilon_n > 0$ satisfying

$$y_n \notin \mathrm{cl}\left[\bigcup_{|t-s| \leqslant \varepsilon_n, s \geqslant 0} C(s)\right].$$

Hence $y_n \notin C^*(s)$ for all $s \in ((t-\varepsilon_n) \vee 0,\ t+\varepsilon_n)$ and consequently

$$d_{C^*}(z, s) \leqslant |z - y_n|, \qquad \forall z \in R^d, s \in ((t-\varepsilon_n) \vee 0, (t+\varepsilon_n) \wedge T).$$

Therefore,

$$\limsup_{(z,s) \to (x,t)} d_{C^*}(z, s) \leqslant \lim_{n \to \infty, z \to x} |z - y_n| = \beta.$$

Now suppose that $\beta < 0$. We first claim that

$$\alpha = \limsup_{(z,s) \to (x,t)} d_{C^*}(x, t) < 0.$$

Indeed if it is not the case, there exists a subsequence $(\omega_n, t_n) \to (x, t)$ such that $\omega_n \in C^*(t_n)$. Then, part (c) implies that $x \in C^*(t)$ which contradicts the assumption $\beta < 0$.

Choose $(z_n, t_n) \to (x, t)$ such that

$$\alpha = \lim d_{C^*}(z_n, t_n).$$

Choose another sequence $y_n \in C^*(t_n)$ satisfying

$$\alpha = \lim -|z_n - y_n|.$$

Since $|x - y_n| \leqslant |z_n - y_n| + |z_n - x|$, $|y_n|$ is uniformly bounded in n. Hence we may assume that y_n is convergent. Let $y = \lim y_n$. Then, $\alpha = -|x - y|$, and part (c) yields that $y \in C^*(t)$. So

$$\alpha = -|x - y| \leqslant d_{C^*}(x, t) = \beta.$$

(f) This follows from (b) and (e). ∎

We will use $(a \wedge b)$ and $(a \vee b)$ to denote $\min\{a, b\}$ and $\max\{a, b\}$, respectively.

LEMMA 3.2. *Let* $\{C(t)\}_{t \in [0,T)}$ *be a collection of proper subsets of* R^d. *Then,* $[d_{C^*}(x, t) \wedge 0]$ *and* $[d_{C_*}(x, t) \vee 0]$ *are upper and lower semi-*

continuous envelopes of the functions $[d_C(x,t)\wedge 0]$ *and* $[d_C(x,t)\vee 0]$, *respectively, i.e., for* $(x,t)\in R^d\times[0,T)$,

$$d_{C^*}(x,t)\wedge 0=\limsup_{(z,s)\to(x,t)}[d_C(z,s)\wedge 0], \tag{3.3}$$

$$d_{C_*}(x,t)\vee 0=\liminf_{(z,s)\to(x,t)}[d_C(z,s)\vee 0]. \tag{3.4}$$

Proof. Since $C(t)$ is included in $C^*(t)$, $d_C(x,t)\leqslant d_{C^*}(x,t)$ for all $(x,t)\in R^d\times[0,T)$. Then, the upper semi-continuity of $d_{C^*}(x,t)$ yields

$$d_{C^*}(x,t)\wedge 0\geqslant \limsup_{(z,s)\to(x,t)}[d_C(z,s)\wedge 0].$$

Fix $(x,t)\in R^d\times[0,T)$. Suppose that $x\in C^*(t)$. Then there are $(z_n,t_n)\to(x,t)$ such that $z_n\in C(t_n)$. Hence,

$$\begin{aligned}0&=d_{C^*}(x,t)\wedge 0\\ &=\lim[d_C(z_n,t_n)\wedge 0]\\ &\leqslant \limsup_{(z,s)\to(x,t)}[d_C(z,s)\wedge 0].\end{aligned}$$

Suppose $x\notin C^*(t)$. Then, there is $y\in C^*(t)$ such that

$$d_{C^*}(x,t)=-|x-y|.$$

Since $y\in C^*(t)$, there is a sequence $(\omega_n,t_n)\to(y,t)$ such that $\omega_n\in C(t_n)$. Then,

$$d_C(x,t_n)\geqslant -|x-\omega_n|,$$

and consequently

$$\begin{aligned}-|x-y|&=[d_{C^*}(x,t)\wedge 0]\\ &=d_{C^*}(x,t)\\ &\geqslant \limsup_{(z,s)\to(x,t)}[d_C(z,s)\wedge 0]\\ &\geqslant \lim d_C(x,t_n)\\ &\geqslant \lim -|x-\omega_n|\\ &=-|x-y|.\end{aligned}$$

This completes the proof of (3.3), and (3.4) is proved after observing that

$$\begin{aligned} d_{C_*}(x,t) \vee 0 &= -[d_{(R^d\setminus C(\cdot))^*}(x,t) \wedge 0] \\ &= -\limsup_{(z,s)\to(x,t)} [d_{(R^d\setminus C(\cdot))}(z,s) \wedge 0] \\ &= \liminf_{(z,s)\to(x,t)} [d_C(z,s) \vee 0]. \quad \blacksquare \end{aligned} \tag{3.5}$$

4. Sub- and Superdifferentials

We first recall the definition of sub- and superdifferentials of semi-continuous functions; see [12, 11]. We then define the sub- and super-differentials of a set-valued map.

Let $S(d)$ be the collection of all $d \times d$ symmetric matrices.

Definition 4.1. Let $T > 0$, Φ be a function on $R^d \times [0, T)$, Φ^* and Φ_* be the upper semi-continuous and lower semi-continuous envelope of Φ, respectively (see (3.3) and (3.4)).

(a) The set of *superdifferentials* of Φ at $(x, t) \in R^d \times (0, T)$ is given by

$$\mathbf{D}^+\Phi(x,t) = \left\{ (n,p) \in R^d \times R: \limsup_{(y,h)\to 0} \frac{\Phi^*(x+y,t+h) - \Phi^*(x,t) - ph - n\cdot y}{|y| + |h|} \leqslant 0 \right\}.$$

(b) The set of *second superdifferentials* of Φ at $(x, t) \in R^d \times (0, T)$ is given by

$$\mathbf{D}_x^{+2}{}_t^{+1}\Phi(x,t) = \left\{ (n,A,p) \in R^d \times S(d) \times R: \limsup_{(y,h)\to 0} \frac{\Phi^*(x+y,t+h) - \Phi^*(x,t) - ph - n\cdot y - (1/2)\,Ay\cdot y}{|y|^2 + |h|} \leqslant 0 \right\}.$$

(c) The set of *subdifferentials* of Φ at $(x, t) \in R^d \times (0, T)$ is given by

$$\mathbf{D}^-\Phi(x,t) = \left\{ (n,p) \in R^d \times R: \liminf_{(y,h)\to 0} \frac{\Phi_*(x+y,t+h) - \Phi_*(x,t) - ph - n\cdot y}{|y| + |h|} \geqslant 0 \right\}.$$

(d) The set of *second subdifferentials* of Φ at $(x, t) \in R^d \times (0, T)$ is given by

$$\mathbf{D}_x^{-2}{}_t^{-1}\Phi(x, t)$$

$$= \Big\{(n, A, p) \in R^d \times S(d) \times R :$$

$$\liminf_{(y,h)\to 0} \frac{\Phi_*(x+y, t+h) - \Phi_*(x, t) - ph - n \cdot y - (1/2)\, Ay \cdot y}{|y|^2 + |h|} \geqslant 0\Big\}.$$

See Appendix A for several well-known properties of these sets. We continue with the definition of sub- and superdifferentials of a set-valued map.

DEFINITION 4.2. For a given collection of subsets $\{C(t)\}_{t \geqslant 0}$ of R^d,

$$\mathbf{D}^+ C(t) = \bigcup_{x \in R^d} \mathbf{D}_x^{+2}{}_t^{+1}[d_{C^*} \wedge 0](x, t) \qquad \forall t < T(C(\cdot)),$$

$$\mathbf{D}^- C(t) = \bigcap_{x \in R^d} \mathbf{D}_x^{-2}{}_t^{-1}[d_{C_*} \vee 0](x, t) \qquad \forall t < T(C(\cdot)),$$

where $T(C(\cdot))$ is called the *extinction time* and is given by

$$T(C(\cdot)) = \begin{cases} \inf\{t \geqslant 0 : C^*(t) = R^d \text{ or } C_*(t) = \varnothing\} \\ \infty \qquad \text{if } C(t) \text{ is a proper subset of } R^d \text{ for all } t \geqslant 0. \end{cases}$$

Remark 4.3. (a) Observe that for any $(z, s) \in R^d \times [0, T)$,

$$\begin{aligned}[d_{C^*} \wedge 0](z, s) &= -\inf\{|z - w| : w \notin C^*(s)\} \\ &= \sup\{-|z - w| : w \notin C^*(s)\}.\end{aligned}$$

Hence for any $(z, s) \in R^d \times [0, T)$ and $(\tilde{z}, s) \in R^d \times [0, T)$,

$$|[d_{C^*} \wedge 0](z, s) - [d_{C^*} \wedge 0](\tilde{z}, s)| \leqslant |z - \tilde{z}|.$$

Fix $(x, t) \in R^d \times (0, T)$. Since $C^*(t)$ is closed, there is $z^* \in C^*(t)$ satisfying

$$\begin{aligned}[d_{C^*} \wedge 0](x, t) &= -|x - z^*| \\ &= -|x - z^*| + [d_{C^*} \wedge 0](z^*, t).\end{aligned}$$

Now use the previous identity with $z = x + y$, and $\tilde{z} = z^* + y$, and $s = t + h$, to obtain

$$[d_{C^*} \wedge 0](x + y, t + s) \geqslant -|x - z^*| + [d_{C^*} \wedge 0](z^* + y, t + s),$$

for all $y \in R^d$ and $s \in (-t, T-t)$. Therefore

$$[d_{C^*} \wedge 0](x+y, t+s) - [d_{C^*} \wedge 0](x, t)$$
$$\geq [d_{C^*} \wedge 0](z^*+y, t+s) - [d_{C^*} \wedge 0](z^*, t).$$

Hence, $\mathbf{D}_{x}^{+2}{}_{t}^{+1}[d_{C^*} \wedge 0](x, t)$ is included in $\mathbf{D}_{x}^{+2}{}_{t}^{+1}[d_{C^*} \wedge 0](z^*, t)$, and consequently

$$\mathbf{D}^+ C(t) = \bigcup_{x \in C^*(t)} \mathbf{D}_{x}^{+2}{}_{t}^{+1}[d_{C^*} \wedge 0](x, t). \tag{4.1}$$

(b) Similarly

$$\mathbf{D}^- C(t) = \bigcup_{x \notin C_*(t)} \mathbf{D}_{x}^{-2}{}_{t}^{-1}[d_{C_*} \vee 0](x, t). \tag{4.2}$$

Other properties of the above sets are gathered in Appendix A. Also see Example 5.5 for a discussion of the definition. In particular, in Example 5.5 we discuss why the set

$$\bigcup_{x \in \partial C^*(t)} \mathbf{D}_{x}^{+2}{}_{t}^{+1}[d_{C^*} \wedge 0](x, t)$$

is not large enough to yield a complete theory.

5. Viscosity Solutions; Definition

We start with rewriting Eq. (1.2) as

$$\frac{1}{|D\Phi(x, t)|} F(D\Phi(x, t), D^2\Phi(x, t), \frac{\partial}{\partial t}\Phi(x, t)) = 0, \tag{1.2}$$

where D and D^2 denote the gradient and the Hessian matrix with respect to the x variable alone and for $(n, A, p) \in [R^d \backslash \{0\}] \times S(d) \times R$,

$$F(n, A, p) = \beta\left(-\frac{n}{|n|}\right) p - \sum_{ij=1}^{d} \left[G_{ij}\left(-\frac{n}{|n|}\right) A_{ij}\right] - v\,|n|. \tag{5.1}$$

Let F^* and F_* be the upper and lower semi-continuous envelopes of F. Note that F^* and F_* are both defined on $R^d \times S(d) \times R$ and are given by

$$F^*(n, A, p) = \limsup_{\substack{(m, B, q) \to (n, A, p) \\ m \neq 0}} F(m, B, q),$$

$$F_*(n, A, p) = \liminf_{\substack{(m, B, q) \to (n, A, p) \\ m \neq 0}} F(m, B, q).$$

Since $\beta > 0$, and $G \geqslant 0$, the function F and its upper and lower semi-continuous envelopes satisfy

$$F(n, A, p) \leqslant F(n, A - B, p + q) \qquad \forall q > 0, B \geqslant 0.$$

Hence Eq. (1.4) is degenerate parabolic. Also $G(\theta)\theta = 0$ implies that

$$F(\lambda(n, A, p)) = \lambda F\left(n, \left(I - \mu \frac{n \otimes n}{|n|^2}\right) A, p\right) \qquad \forall |n| \neq 0, \lambda, \mu > 0. \quad (5.2)$$

DEFINITION 5.1. (a) A collection $\{C(t)\}_{t \geqslant 0}$ of subsets of R^d is a *viscosity subsolution* of (E) if

$$F_*(n, A, p) \leqslant 0, \qquad \forall (n, A, p) \in \mathbf{D}^+ C(t), t \in (0, T(C(\cdot))).$$

(b) A collection $\{C(t)\}_{t \geqslant 0}$ of subsets of R^d is a *viscosity supersolution* of (E) if

$$F^*(n, A, p) \geqslant 0, \qquad \forall (n, A, p) \in \mathbf{D}^- C(t), t \in (0, T(C(\cdot))).$$

(c) A collection $\{C(t)\}_{t \geqslant 0}$ of subsets of R^d is a *viscosity solution* of (E) if it is both a viscosity subsolution and a viscosity supersolution of (E).

See Appendix B for an equivalent definition. Also a stability theorem is stated in Appendix C.

Remark 5.2. (a) Since $[d_{C^*} \wedge 0](x, t)$ is the upper semi-continuous envelope of $[d_C \wedge 0](x, t)$, $\{C(t)\}_{t \geqslant 0}$ is a subsolution of (E) if and only if $[d_C \wedge 0](x, t)$ is a viscosity subsolution of (1.2) on $R^d \times (0, T(C(\cdot)))$. The viscosity solutions of equations like (1.2) are defined in [10, 15]. Similarly $\{C(t)\}_{t \geqslant 0}$ is a supersolution of (E) if and only if $[d_C \wedge 0](x, t)$ is a viscosity supersolution of (1.2) on $R^d \times (0, T(C(\cdot)))$.

(b) $\{C(t)_{t \geqslant 0}$ is a supersolution of (E) if and only if its complement is a subsolution of

$$\beta(-\theta)\mathbf{V} = \text{trace } G(-\theta)\, \mathbf{R} - v.$$

(c) Suppose that $\{C(t)\}_{t \geqslant 0}$ is a viscosity subsolution of (E). For $0 < T \leqslant T(C(\cdot))$, define

$$\Gamma(t) = \begin{cases} C(t) & \text{if } \ t \leqslant T \\ \varnothing & \text{if } \ t > T. \end{cases}$$

Then, $T(\Gamma(\cdot)) = T$ and $\{\Gamma(t)\}_{t \geqslant 0}$ is also a subsolution.

We make the following definition to distinguish between a "maximal" sub- or supersolution and others constructed like $\{\Gamma(t)\}_{t \geqslant 0}$.

DEFINITION 5.3. We say that a collection of subsets $\{C(t)\}_{t \geq 0}$ of R^d is *maximal* if whenever $T(C(\cdot))$ is finite either

$$C_*(t) = \varnothing \qquad \forall t \geq T(C(\cdot)),$$

or

$$C^*(t) = R^d \qquad \forall t \geq T(C(\cdot)).$$

We continue by showing that any classical solution $\{C(t)\}_{t \geq 0}$ of (E) is a maximal viscosity solution. For simplicity we make a simplifying assumption which rules out several pathological cases. We assume that there is $I = \{t_1, t_2, ..., t_N\}$ such that

$$d_C \text{ is continuous on } R^d \times ([0, T(C(\cdot)))\backslash I). \tag{5.3}$$

LEMMA 5.4. *Any classical subsolution (or supersolution) of* (E) *satisfying* (5.3) *is a maximal viscosity subsolution (or viscosity supersolution) of* (E).

Proof. The maximality of classical sub- or supersolutions follow from Definition 2.2. Let $\{C(t)\}_{t \geq 0}$ be a classical subsolution of (E), and $(n, A, p) \in \mathbf{D}^+ C(t)$ with $t < T(C(\cdot))$. Then, for some $x \in R^d$, we have

$$(n, A, p) \in \mathbf{D}_x^{+2,+1}{}_t[d_{C^*} \wedge 0](x, t).$$

First assume that $[d_{C^*} \wedge 0](x, t) = [d_C \wedge 0](x, t)$. Using (4.1) we may assume that $x \in \text{cl}\, C(t)$. If $x \in \text{int}\, C(t)$ the smoothness of $\{C(t)\}_{t \geq 0}$ yields that $(x, t) \in \text{int}\{(y, s) \in R^d \times [0, \infty) : y \in C(s)\}$. Hence,

$$(n, p) = (0, 0), \qquad A \geq 0,$$

and

$$F_*(n, A, p) \leq F_*(0, 0, 0) = 0.$$

So suppose that $x \in \partial C(t)$. Set $\eta = Dd_C(x, t)$. Then, $-\eta$ is the outward unit normal vector at $x \in \partial C(t)$. Let $\xi \in R^d$ be a vector satisfying $\xi \cdot \eta > 0$. Then, the smoothness of $\partial C(t)$ yields that

$$(x - \tau\xi) \notin C(t) \qquad \text{for sufficiently small } \tau > 0.$$

Using the definition of the subdifferential and Corollary 14.3, we obtain

$$\begin{aligned} 0 &\geq \lim_{\tau \downarrow 0} \frac{[d_C \wedge 0](x - \tau\xi, t) - [d_C \wedge 0](x, t) + \tau\xi \cdot n}{\tau\, |\xi|} \\ &\geq \lim_{\tau \downarrow 0} \frac{d_C(x - \tau\xi, t) - d_C(x, t) + \tau\xi \cdot n}{\tau\, |\xi|} \\ &= [-\eta \cdot \xi + n \cdot \xi]/|\xi|. \end{aligned}$$

Hence,

$$n \cdot \xi \leqslant \eta \cdot \xi \qquad \forall \xi \in R^d \text{ satisfying } \xi \cdot \eta \geqslant 0. \tag{5.4}$$

Also,

$$(x + \tau\eta) \in C(t) \qquad \text{for sufficiently small } \tau > 0,$$

and a similar argument based on $[d_C \wedge 0](x + \tau\eta, t) = 0$ yields

$$n \cdot \eta \geqslant 0. \tag{5.5}$$

Inequalities (5.4) and (5.5) imply that

$$n = \rho\eta \tag{5.6}$$

for some $\rho \in [0, 1]$. Set

$$V = \frac{\partial}{\partial t} d_C(x, t).$$

Then, for every $\varepsilon > 0$,

$$x + \tau(1 - \varepsilon) V\eta \notin C(t - \tau) \qquad \text{for sufficiently small } \tau > 0.$$

Since $(n, p) \in \mathbf{D}^+ [d_C \wedge 0](x, t)$,

$$\begin{aligned} 0 &\geqslant \lim_{\tau \downarrow 0} \frac{[d_C \wedge 0](x + \tau(1-\varepsilon) V\eta, t - \tau) - [d_C \wedge 0](x, t) - \tau(1-\varepsilon) V\eta \cdot n + p\tau}{\tau} \\ &= \lim_{\tau \downarrow 0} \frac{d_C(x + \tau(1-\varepsilon) V\eta, t - \tau) - d_C(x, t) - \tau(1-\varepsilon) V\eta \cdot n + p\tau}{\tau} \\ &= V(1-\varepsilon) - V - (1-\varepsilon) V\eta \cdot n + p \\ &= -\varepsilon V - (1-\varepsilon) V\rho + p. \end{aligned}$$

Hence,

$$p \leqslant \rho V.$$

A similar argument based on

$$x - \tau(1 + \varepsilon) V\eta \in C(t + \tau) \qquad \text{for sufficiently small } \tau > 0,$$

yields that $p \geqslant \rho V$ and therefore

$$p = \rho V. \tag{5.7}$$

Let $\zeta \in R^d$ be a unit vector orthogonal to η. Since the boundary of $C(t)$ is smooth, there is a sequence $z_m \in \partial C(t)$ converging to x and

$$\lim \frac{z_m - x}{|z_m - x|} = \zeta.$$

Set $\tau_m = |z_m - x|$ and $w_m = (z_m - x)/\tau_m$. Then, w_m converges to ζ and

$$\begin{aligned} 0 &= d_C(z_m, t) \\ &= d_C(x, t) + \int_0^1 Dd_C(x + \tau\tau_m w_m, t) \cdot \tau_m w_m \, d\tau \\ &= \int_0^1 \left[Dd_C(x, t) + \int_0^1 D^2 d_C(x + r\tau\tau_m w_m, t)\, \tau\tau_m w_m \, dr \right] \cdot \tau_m w_m \, d\tau. \end{aligned}$$

Divide both sides of the above equation by $(\tau_m)^2$ and then let m go to ∞ to obtain

$$\lim \left[\frac{\eta \cdot w_m}{\tau_m} \right] = \lim \left[\frac{Dd_C(x, t) \cdot w_m}{\tau_m} \right] = -(1/2)\, D^2 d_C(x, t)\, \zeta \cdot \zeta.$$

Also,

$$\begin{aligned} 0 &\geqslant \lim_{m \to \infty} \frac{[d_C \wedge 0](z_m, t) - [d_C \wedge 0](x, t) - \tau_m w_m \cdot n - (\tau_m)^2 (1/2)\, A w_m \cdot w_m}{(\tau_m)^2} \\ &= \lim_{m \to \infty} \frac{d_C(z_m, t) - d_C(x, t) - \tau_m w_m \cdot n - (\tau_m)^2 (1/2)\, A w_m \cdot w_m}{(\tau_m)^2} \\ &= \lim_{m \to \infty} \left[-\rho \frac{\eta \cdot w_m}{\tau_m} - (1/2)\, A w_m \cdot w_m \right] \\ &= \rho (1/2)\, D^2 d_C(x, t)\, \zeta \cdot \zeta - (1/2)\, A\zeta \cdot \zeta. \end{aligned}$$

Hence,

$$[A - \rho\, D^2 d_C(x, t)]\, \zeta \cdot \zeta \geqslant 0 \qquad \forall \xi \in R^d \text{ satisfying } \zeta \cdot \eta = 0,$$

or equivalently

$$\begin{aligned} (I - \eta \otimes \eta) A &\geqslant \rho (I - Dd_C(x, t) \otimes Dd_C(x, t))\, D^2 d_C(x, t) \\ &= \rho\, D^2 d_C(x, t). \end{aligned} \tag{5.8}$$

Suppose that $\rho > 0$. Then, (5.6), (5.7), and (5.8) imply that

$$\begin{aligned} F_*(n, A, p) &= F(n, A, p) \\ &= F(\rho(n, A, p))/\rho \\ &\leqslant F(Dd_C(x, t), D^2 d_C(x, t), \frac{\partial}{\partial t} d_C(x, t))/\rho \\ &= 0. \end{aligned}$$

If $\rho = 0$, then $(n, p) = (0, 0)$. Also, $(I - \eta \otimes \eta) A \geqslant 0$. Hence

$$F_*(n, A, p) = F_*(0, A, 0) \leqslant 0.$$

Recall that we have assumed that $[d_C \wedge 0](x, t) = [d_{C^*} \wedge 0](x, t)$. If $[d_C \wedge 0](x, t) \neq [d_{C^*} \wedge 0](x, t)$, then using (2.1) and (5.3), we can construct a sequence x_m, t_m, n_m, A_m, p_m such that $[d_C \wedge 0](x_m, t_m) = [d_{C^*} \wedge 0](x_m, t_m)$,

$$\lim(x_m, t_m, n_m, A_m, p_m) = (x, t, n, A, p),$$

and

$$(n_m, A_m, p_m + K_m) \in \mathbf{D}_x^{+2} {}_t^{+1} [d_C \wedge 0](x_m, t_m)$$

for some $K_m \geqslant 0$. Such a sequence is constructed by considering the local maxima of the map $[d_C \wedge 0](y, s) - \Psi(y, s) - [m(t-s)]^{-1}$ on the region $R^d \times (0, t)$, where Ψ is a smooth function as in Theorem 14.1(b) with $\Phi = [d_C \wedge 0]$. Using the previous argument we conclude that

$$F_*(n_m, A_m, p_m) \leqslant F_*(n_m, A_m, p_m + K_m) \leqslant 0.$$

Passing to the limit as m tends to infinity yields that $\{C(t)\}_{t \geqslant 0}$ is a viscosity subsolution of of (E). The assertion about the supersolutions is proved by using the proven result and Remark 5.2(b). ∎

We give a simple example to clarify the definition.

EXAMPLE 5.5. For $t \geqslant 0$, define

$$C(t) = \begin{cases} \{(x, y) \in R^2 : 2(1-t) < |x|^2 + |y|^2 < 4 - 2t\} & \text{if } t < 1 \\ \{(x, y) \in R^2 : |x|^2 + |y|^2 < 4 - 2t\} & \text{if } 1 \leqslant t < 2 \\ \varnothing & \text{if } t \geqslant 2. \end{cases}$$

Then, $C(\cdot)$is a classical solution of (MCE) with $d=2$. In fact it is the unique viscosity solution of (MCE) with initial condition

$$C(0)=\{(x, y)\in R^2 : 2<|x|^2+|y|^2<4\}. \tag{5.9}$$

Also define

$$\Gamma(t)=\begin{cases} C(t) & \text{if } \ t<(1/2) \\ \{(x, y)\in R^2 : |x|^2+|y|^2<4-2t\} & \text{if } \ (1/2)\leqslant t<2 \\ \varnothing & \text{if } \ t\geqslant 2. \end{cases}$$

Then for any $(x, y)\in \partial\Gamma^*(t)$,

$$F(n, A, p)\leqslant 0 \qquad \forall (n, A, p)\in \mathbf{D}_x^{+2}{}_t^{+1}\, d_{\Gamma^*}((x, y), t).$$

Observe that $\{\Gamma(t)\}_{t\geqslant 0}$ does not satisfy (2.1) and therefore it is not a classical solution. Also

$$(0, 0, p)\in \mathbf{D}_x^{+2}{}_t^{+1} d_{\Gamma^*}((0, 0), 0.5),$$

for any $p\geqslant 0$. Hence $\{\Gamma(t)\}_{t\geqslant 0}$ is not a viscosity solution of (E), either. This example indicates why we need a subdifferential which is larger than the set

$$\bigcup_{x\in\partial C^*(t)} \mathbf{D}_x^{+2}{}_t^{+1}\, d_{C^*}(x, t).$$

However, if $\{C(t)\}_{t\geqslant 0}$ is "continuous" in the time variable, there may be an equivalent definition which only uses the above set.

6. Existence by Perron's Method

In this section we obtain an existence result by assuming the existence of a viscosity subsolution and a viscosity supersolution of (E). Existence of a solution satisfying a given initial data is discussed in Section 10.

We use Perron's method to obtain existence of viscosity solutions. Our proof is very closely related to the proof of Proposition 2.3 in [10]. We also refer to Ishii [31] for the first use of Perron's method in the context of viscosity solutions.

Lemma 6.1. (a) *Let* $\mathbf{C}$ *be a collection of viscosity subsolutions of* (E). *Define*

$$C(t)=\bigcup \{\Gamma(t) : \Gamma(\cdot)\in \mathbf{C} \text{ and } t<T(\Gamma(\cdot))\}\cup\varnothing.$$

Then $\{C(t)\}_{t\geqslant 0}$ *is a viscosity subsolution of* (E).

(b) *Let* **A** *be a collection of viscosity supersolutions of* (E). *Define*

$$C(t) = \bigcap \{\Gamma(t) : \Gamma(\cdot) \in \mathbf{A} \text{ and } t < T(\Gamma(\cdot))\} \cap R^d.$$

Then $\{C(t)\}_{t \geq 0}$ *is a viscosity supersolution of* (E).

Proof. (a) Remark 5.2(a) and Proposition 2.2 of [10] imply the result provided that for all $t \in [0, T(C(\cdot)))$ and x,

$$[d_C \wedge 0](x, t) = \sup\{[d_\Gamma \wedge 0](x, t) : \Gamma(\cdot) \in \mathbf{C} \text{ and } t < T(\Gamma(\cdot))\}. \tag{6.1}$$

Indeed if $x \in C(t)$ with $t < T(C(\cdot))$, then $x \in \Gamma(t)$ for some $\Gamma(\cdot) \in \mathbf{C}$ which is proper at time t, and (6.1) follows easily. Suppose $x \notin C(t)$ with $t < T(C(\cdot))$. Then, there are $y_n \in C(t)$ such that

$$d_C(x, t) = \lim -|x - y_n|.$$

The definition of $C(t)$ implies that for each n, $y_n \in \Gamma_n(t)$ for some $\Gamma_n \in \mathbf{C}$ which is proper at time t. Hence,

$$-|x - y_n| \leqslant d_{\Gamma_n}(x, t) = [d_{\Gamma_n} \wedge 0](x, t) \qquad \forall n,$$

and

$$[d_C \wedge 0](x, t) \leqslant \sup\{[d_\Gamma \wedge 0](x, t) : \Gamma(\cdot) \in \mathbf{C} \text{ and } t < T(\Gamma(\cdot))\}.$$

To prove the reverse inequality observe that for every $t < T(C(\cdot))$ and positive integer n, there is $\Gamma_n(\cdot) \in \mathbf{C}$ which is proper at time t and

$$[d_{\Gamma_n} \wedge 0](x, t) \geqslant \sup\{[d_\Gamma \wedge 0](x, t) : \Gamma(\cdot) \in \mathbf{C} \text{ and } t < T(\Gamma(\cdot))\} - (1/n).$$

Also choose $z_n \in \Gamma_n(t)$ satisfying

$$-|x - z_n| \geqslant [d_{\Gamma_n} \wedge 0](x, t) - (1/n).$$

Then by the definition of $C(t)$, $z_n \in C(t)$ and

$$[d_C \wedge 0](x, t) \geqslant -|x - z_n|.$$

Combining the above inequalities, we arrive at (6.1).

(b) This follows from part (a) and Remark 5.2(b). ∎

We need the following technical lemma in our main existence result.

LEMMA 6.2. *Suppose that there are $\delta > 0$, $y_0 \in R^d$, $s_0 > 0$, and smooth functions f and g satisfying*

$$F^*\left(Df(y_0, s_0), D^2f(y_0, s_0), \frac{\partial}{\partial t} f(y_0, s_0)\right) < 0, \tag{6.2}$$

$$f(y_0, s_0) = g(y_0, s_0) = 0, \tag{6.3}$$

and

$$f(x, t) > 0 \Rightarrow g(x, t) \geqslant 0, \qquad \forall |x - y_0| + |t - s_0| < \delta. \tag{6.4}$$

Then,

$$F_*\left(Dg(y_0, s_0), D^2g(y_0, s_0), \frac{\partial}{\partial t} g(y_0, s_0)\right) \leqslant 0. \tag{6.5}$$

Proof. We analyse three cases separately.

(1) $(Df(y_0, s_0), (\partial/\partial t) f(y_0, s_0)) \neq 0$. Set

$$\eta = \left(Df(y_0, s_0), \frac{\partial}{\partial t} f(y_0, s_0)\right).$$

Then, for any $v \in R^{d+1}$ satisfying $\eta \cdot v > 0$, there is $\rho(v) > 0$ such that

$$f((y_0, s_0) + \rho v) > 0 \qquad \forall \rho \in (0, \rho(v)].$$

Assumption (6.4) yields

$$g((y_0, s_0) + \rho v) \geqslant 0 \qquad \forall \rho \in [0, \rho(v)],\ v \in R^{d+1}, \text{ and } \eta \cdot v > 0,$$

and therefore there is $\alpha \geqslant 0$ such that

$$\begin{aligned}(Dg(y_0, s_0), \frac{\partial}{\partial t} g(y_0, s_0)) &= \alpha\eta \\ &= \alpha\left(Df(y_0, s_0), \frac{\partial}{\partial t} f(y_0, s_0)\right).\end{aligned} \tag{6.6}$$

Let $\zeta \in R^{d+1}$ be a unit vector orthogonal to η. By the implicit function theorem, there is a sequence (z_m, t_m) converging to (y_0, s_0) such that $f(z_m, t_m) = 0$ and

$$\lim \frac{z_m - x}{|z_m - x|} = \zeta.$$

Set $\tau_m = |(z_m - x, t_m - t)|$ and $w_m = (z_m - x, t_m - t)/\tau_m$. Then, w_m converges to ζ and

$$\begin{aligned}
0 &= f(z_m, t_m) \\
&= f(y_0, s_0) + \int_0^1 \left(Df((y_0, s_0)\tau\tau_m w_m), \frac{\partial}{\partial t} f((y_0, s_0) + \tau\tau_m w_m) \right) \cdot \tau_m w_m \, d\tau \\
&= \int_0^1 \left[\eta + \int_0^1 D_{x\,t}^{\,2} f((y_0, s_0) + r\tau\tau_m w_m)\, \tau\tau_m w_m \, dr \right] \cdot \tau_m w_m \, d\tau,
\end{aligned}$$

where $D_{x\,t}^{\,2} f$ is the Hessian matrix of f with respect to all of its variables. Divide both sides of the above equation by $(\tau_m)^2$ and then let m go to ∞ to obtain

$$\lim \left[\frac{\eta \cdot w_m}{\tau_m} \right] = -(1/2)\, D_{x\,t}^{\,2} f(y_0, s_0)\, \zeta \cdot \zeta.$$

Also, an approximation argument based on (6.4) and $(Df(y_0, s_0), (\partial/\partial t) f(y_0\, s_0)) \neq 0$ yields $0 \leqslant g(z_m, t_m)$. Hence,

$$\begin{aligned}
0 &\leqslant g(z_m, t_m) \\
&= g(y_0, s_0) + \int_0^1 (Dg((y_0, s_0) + \tau\tau_m w_m), \frac{\partial}{\partial t} g((y_0, s_0) + \tau\tau_m w_m)) \cdot \tau_m w_m \, d\tau \\
&= \int_0^1 \left[\left(Dg(y_0, s_0), \frac{\partial}{\partial t} g(y_0, s_0) \right) \right. \\
&\quad \left. + \int_0^1 D_{x\,t}^{\,2} g((y_0, s_0) + r\tau\tau_m w_m)\, \tau\tau_m w_m \, dr \right] \cdot \tau_m w_m \, d\tau.
\end{aligned}$$

Divide both sides of the above equation by $(\tau_m)^2$ and then let m go to ∞ to obtain

$$\begin{aligned}
0 &\leqslant \lim \left[\frac{(Dg(g_0, s_0), (\partial/\partial t)\, g(y_0, s_0)) \cdot w_m}{\tau_m} \right] + (1/2)\, D_{x\,t}^{\,2} g(y_0, s_0)\, \zeta \cdot \zeta \\
&= \lim \left[\frac{\alpha\eta \cdot w_m}{\tau_m} \right] + (1/2)\, D_{x\,t}^{\,2}(y_0, s_0)\, \zeta \cdot \zeta \\
&= -\alpha(1/2)\, D_{x\,t}^{\,2} f(y_0, s_0)\, \zeta \cdot \zeta + (1/2)\, D_{x\,t}^{\,2} g(y_0, s_0)\, \zeta \cdot \zeta.
\end{aligned}$$

Hence,

$$[\alpha\, D_{x\,t}^{\,2} f(y_0, s_0) - D_{x\,t}^{\,2} g(y_0, s_0)]\, \zeta \cdot \zeta \leqslant 0 \qquad \forall \zeta \cdot \eta = 0.$$

Combining (5.2), (6.2), (6.6), and the above inequality, we obtain (6.5).

(2) $(Dg(y_0, s_0), (\partial/\partial t)\, g(y_0, s_0)) \neq 0$. Using the negation of (6.4), proceed exactly as in the previous case to obtain

$$\begin{aligned} &[D_{x_i}^2 f(y_0, s_0) - \alpha\, D_{x_i}^2 g(y_0, s_0)]\, \zeta \cdot \zeta \leqslant 0 \\ &\text{for all } \zeta \text{ satisfying, } \zeta \cdot \left(Dg(y_0, s_0), \frac{\partial}{\partial t} g(x_0, s_0) \right) = 0, \end{aligned} \tag{6.7}$$

$$\left(Df(y_0, s_0), \frac{\partial}{\partial t} f(y_0, s_0) \right) = \alpha \left(Dg(y_0, s_0), \frac{\partial}{\partial t} g(y_0, s_0) \right), \tag{6.8}$$

for some $\alpha \geqslant 0$. If $\alpha > 0$, then (6.5) follows from (6.2), (6.7), and (6.8). Now suppose that $\alpha = 0$. Then, (6.2), (6.7), and the definition F^* imply that

$$\begin{aligned} 0 &> F^* \left(Df(y_0, s_0), D^2 f(y_0, s_0), \frac{\partial}{\partial t} f(y_0, s_0) \right) \\ &= F^*(0, D^2 f(y_0, s_0), 0) \\ &\geqslant \limsup_{\rho \downarrow 0} F(\rho\, Dg(y_0, s_0), D^2 f(y_0, s_0), 0). \end{aligned}$$

However, (6.7) with $\alpha = 0$ yields

$$F(\rho\, Dg(y_0, s_0), D^2 f(y_0, s_0), 0) \geqslant 0$$

for every $\rho > 0$. Clearly the above inequality contradicts (6.2), hence $\alpha \neq 0$.

(3) $(Df(y_0, s_0), (\partial/\partial t) f(y_0, s_0)) = (Dg(y_0, s_0), (\partial/\partial t)\, g(y_0, s_0)) = 0$. Set

$$\begin{aligned} A &= D^2 f(y_0, s_0), \\ B &= D^2 g(y_0, s_0). \end{aligned}$$

Then, (6.4) and the fact that $(Df(y_0, s_0), (\partial/\partial t) f(y_0, s_0)) = (Dg(y_0, s_0), (\partial/\partial t)\, g(y_0, s_0)) = 0$ yield

$$Ae \cdot e > 0 \Rightarrow Be \cdot e \geqslant 0 \qquad \forall e \in R^d. \tag{6.9}$$

Let $e, f \in R^d$ be given. We claim that

$$Ae \cdot e + Af \cdot f > 0 \Rightarrow Be \cdot e + Bf \cdot f \geqslant 0.$$

Indeed if $Ae \cdot e$ and $Af \cdot f$ are both strictly positive, then (6.9) yields the result. If they are both negative, then there is nothing to prove. So we may assume that

$$Ae \cdot e > 0 \geqslant Af \cdot f \qquad \text{and} \qquad Be \cdot e \geqslant 0 > Bf \cdot f.$$

Define two second order polynomials by

$$P_1(r) = A(f+re)\cdot(f+re) \qquad \text{and} \qquad P_2(r) = B(f+re)\cdot(f+re).$$

For large r, $P_1(r)>0$. Hence (6.9) implies that $P_2(r)$ is nonnegative. But this is possible only if $Be\cdot e>0$. Let $\lambda_1 \leqslant \lambda_2$ be the roots of P_1 and $\mu_1 \leqslant \mu_2$ be the roots of P_2. Observe that

$$\lambda_1\lambda_2 = Af\cdot f/Ae\cdot e, \qquad \mu_1\mu_2 = Bf\cdot f/Be\cdot e.$$

Hence to prove the claim it suffices to show that

$$\lambda_1\lambda_2 + 1 > 0 \Rightarrow \mu_1\mu_2 + 1 \geqslant 0.$$

Using (6.9) we conclude that whenever $P_1(r)>0$, then $P_2(r)\geqslant 0$. We also know that $Ae\cdot e>0\geqslant Af\cdot f$, and $Be\cdot e>0>Bf\cdot f$. Hence

$$\lambda_1 \leqslant \mu_1 < 0 < \mu_2 \leqslant \lambda_2.$$

Consequently $\lambda_1\lambda_2 \leqslant \mu_1\mu_2$ and the claim is proved.

The hypothesis (6.2) and the assumption (A) of Section 2 yield that trace $G(\theta^*)A>0$ for some $\theta^* \in S^{d-1}$. Using this inequality and the nonnegativity of the symmetric matrix $G(\theta^*)$ we may represent $G(\theta^*)$ as

$$G(\theta^*) = \sum_{i=1}^{2M} e_i \otimes e_i$$

for some eigenvectors e_i of $G(\theta^*)$ (not necessarily distinct) satisfying

$$Ae_{2k-1}\cdot e_{2k-1} + Ae_{2k}\cdot e_{2k} > 0 \qquad \forall k=1, ..., M.$$

Then we have

$$Be_{2k-1}\cdot e_{2k-1} + Be_{2k}\cdot e_{2k} \geqslant 0.$$

Sum this inequality over k to obtain that trace $G(\theta^*)B\geqslant 0$, and (6.5) follows. ∎

THEOREM 6.3. *Let* $\{L(t)\}_{t\geqslant 0}$ *be a viscosity subsolution of* (E) *and* $\{U(t)\}_{t\geqslant 0}$ *be a viscosity supersolution of* (E), *respectively. If*

$$L(t) \subset U(t), \qquad \forall t < T(L(\cdot)) \wedge T(U(\cdot), \tag{6.10}$$

then there exists a viscosity solution $\{C(t)\}_{t\geqslant 0}$ *of* (E) *satisfying*

$$L(t) \subset C(t) \subset U(t) \qquad \forall t < T(L(\cdot)) \wedge T(U(\cdot)). \tag{6.11}$$

Proof. Set $T_0 = T(L(\cdot)) \wedge T(U(\cdot))$, $S_0 = T(U(\cdot))$, and

$$\mathbf{C} = \left\{ \{\Gamma(t)\}_{t \geq 0} : \begin{array}{c} \{\Gamma(t)\}_{t\geq 0} \text{ is a subsolution of (E)} \\ \Gamma(t) \subset U(t)\ \forall t < S_0 \wedge T(\Gamma(\cdot)) \end{array} \right\}.$$

Then Lemma 6.1(a) yields that

$$C(t) = \begin{cases} \bigcup \{\Gamma(t) : \Gamma(\cdot) \in \mathbf{C} \text{ and } t < T(\Gamma(\cdot))\} \cup \varnothing & \text{if } t < S_0 \\ \varnothing & \text{if } t \geq S_0 \end{cases}$$

is a subsolution of (E). Also (6.10) implies that $\{L(t)\}_{t\geq 0} \in \mathbf{C}$. Hence $T_0 \leq T(C(\cdot)) \leq S_0$. Suppose that $\{C(t)\}_{t\geq 0}$ is not a supersolution of (E). Using Lemma 14.6 we conclude that there exists a smooth function $\tilde{\Phi}$ and $(x_0, t_0) \in R^d \times (0, T(C(\cdot)))$ such that

$$\begin{aligned} 0 &= [d_{C_*} \vee 0](x_0, t_0) - \tilde{\Phi}(x_0, t_0) \\ &= \min\{[d_{C_*} \vee 0](x, t) - \tilde{\Phi}(x, t) : (x, t) \in R^d \times [0, T(C(\cdot)))\} \end{aligned}$$

and

$$F^* \left(D\tilde{\Phi}(x_0, t_0), D^2\tilde{\Phi}(x_0, t_0), \frac{\partial}{\partial t} \tilde{\Phi}(x_0, t_0) \right) < 0.$$

In view of (4.2) we may assume that

$$x_0 \notin C_*(t_0). \tag{6.12}$$

Set

$$\Phi(x, t) = \tilde{\Phi}(x, t) - [|x - x_0|^2 + |t - t_0|^2]^2.$$

Then,

$$\begin{aligned} &\text{(a)} \quad [d_{C_*} \vee 0](x, t) - \Phi(x, t) \geq [|x - x_0|^2 + |t - t_0|^2]^2\ \forall (x, t) \\ &\text{(b)} \quad \Phi(x_0, t_0) = [d_{C_*} \vee 0](x_0, t_0) \\ &\text{(c)} \quad F^* \left(D\Phi(x_0, t_0), D^2\Phi(x_0, t_0), \frac{\partial}{\partial t} \Phi(x_0, t_0) \right) < 0. \end{aligned} \tag{6.13}$$

Since $C(t) \subset U(t)$ for all $t < T(C(\cdot))$ and $\{U(t)\}_{t\geq 0}$ is a supersolution of (E), (6.13)(c) implies that

$$[d_{C_*} \vee 0](x_0, t_0) = \Phi(x_0, t_0) < [d_{U_*} \vee 0](x_0, t_0).$$

Hence by the lower semicontinuity of $[d_{U_*} \vee 0]$ there is a δ_1 such that

$$B_{\delta_1}(x_0, t_0) = \{(x, t) \in R^d \times [0, \infty) : |x - x_0|^2 + |t - t_0|^2 < (\delta_1)^2\}$$

$$\subset \bigcup_{t \geqslant 0} [U(t) \times \{t\}]. \tag{6.14}$$

Also due to the smoothness of Φ and the upper semi-continuity of F^* there is a δ_2 satisfying

$$F^*\left(D\Phi(x, t), D^2\Phi(x, t), \frac{\partial}{\partial t}\Phi(x, t)\right) < 0 \qquad \forall (x, t) \in B_{\delta_2}(x_0, t_0). \tag{6.15}$$

For $t < T(C(\cdot))$, let

$$\Psi(x, t) = \max\{\Phi(x, t) + (\delta_0)^4/2, [d_C \vee 0](x, t)\},$$

where

$$\delta_0 = \min\{\delta_1, \delta_2\}.$$

For any $(x, t) \notin B_{\delta_0}(x_0)$, (6.13)(a) yields that

$$\begin{aligned} \Phi(x, t) + (\delta_0)^4/2 &< \Phi(x, t) + [|x - x_0|^2 + |t - t_0|^2]^2 \\ &\leqslant [d_{C_*} \vee 0](x, t) \\ &\leqslant [d_C \vee 0](x, t). \end{aligned}$$

Hence, for $t < T(C(\cdot))$,

$$\Psi(x, t) = [d_C \vee 0](x, t), \qquad \forall (x, t) \notin B_{\delta_0}(x_0, t_0). \tag{6.16}$$

Finally define

$$S(t) = \{(x, t) \in R^d \times [0, \infty) : \Psi(x, t) > 0\} \qquad \text{if} \quad t < T(C(\cdot)),$$

and $S(t)$ is defined to be the empty set of $t \geqslant T(C(\cdot))$. The definition of $S(t)$ and (6.16) imply that

$$\bigcup_{t \geqslant 0} [S(t) \times \{t\}] \subset \bigcup_{t \geqslant 0} [C(t) \times \{t\}] \cup B_{\delta_0}(x_0, t_0).$$

Since $C(t)$ is included in $U(t)$ for all $t < S_0 \wedge T(C(\cdot)) = S_0 \wedge T(S(\cdot))$ and $\delta_0 \geqslant \delta_1$, (6.14) and (6.16) yield that

$$S(t) \subset U(t) \qquad \forall t \in [0, S_0 \wedge T(S(\cdot))). \tag{6.17}$$

Using the smoothness of Φ and the inequalities

$$\Psi \geqslant \Phi + (\delta_0)^4/2,$$

and

$$\Phi(x_0, t_0) = [d_{C_*} \vee 0](x_0, t_0) \geqslant 0,$$

we conclude that

$$B_\delta(x_0, t_0) \subset \bigcup_{t \geqslant 0} [S(t) \times \{t\}] \tag{6.18}$$

for some $\delta > 0$. Suppose that

$$B_\delta(x_0, t_0) \subset \bigcup_{t \geqslant 0} [C(t) \times \{t\}] \tag{6.19}$$

for some $\delta > 0$. Then, $x_0 \in C_*(t_0)$. Recall that x_0 is chosen from the complement of $C_*(t_0)$ (see (6.12)). Hence, (6.19) does not hold for any positive δ and (6.18) yields

$$\bigcup_{t < T(C(\cdot))} [C(t) \times \{t\}] \neq \bigcup_{t < T(C(\cdot))} [S(t) \times \{t\}]. \tag{6.20}$$

Since $\{C(t)\}_{t \geqslant 0}$ is the largest subsolution of (E) included in $\{U(t)\}_{t \geqslant 0}$, (6.17), (6.20), and the fact that $C(t)$ is included in $S(t)$ imply that $\{S(t)\}_{t \geqslant 0}$ is not a subsolution of (E). Hence to complete the proof of this theorem it suffices to prove that $\{S(t)\}_{t \geqslant 0}$ is a subsolution of (E).

First note that $T(S(\cdot)) = T(C(\cdot))$. Suppose that a smooth function γ and $(y_0, s_0) \in R^d \times (0, T(S(\cdot)))$ satisfy

$$0 = [d_{S^*} \wedge 0](y_0, s_0) - \gamma(y_0, s_0) > [d_{S^*} \wedge 0](y, s) \qquad \forall (y, s) \neq (y_0, s_0).$$

We need to show that

$$F_*\left(D\gamma(y_0, s_0), D^2\gamma(y_0, s_0), \frac{\partial}{\partial t}\gamma(y_0, s_0)\right) \leqslant 0. \tag{6.21}$$

Since $[d_{C^*} \wedge 0] \leqslant [d_{S^*} \wedge 0]$, if

$$[d_{C^*} \wedge 0](y_0, s_0) = [d_{S^*} \wedge 0](y_0, s_0),$$

(6.21) follows from the fact that $\{C(t)\}_{t \geqslant 0}$ is a subsolution of (E). So we may assume that

$$[d_{C^*} \wedge 0](y_0, s_0) < [d_{S^*} \wedge 0](y_0, s_0). \tag{6.22}$$

We analyse four cases separately.

(1) $\Phi(y_0, s_0) + (\delta_0)^4/2 > 0$. Then, $[d_{S^*} \wedge 0]$ is equal to zero on a neighborhood of (y_0, s_0). Hence, $D\gamma(y_0, s_0) = 0$, $(\partial/\partial t)\gamma(y_0, s_0) = 0$, $D^2\gamma(y_0, s_0) \geqslant 0$, and (6.21) follows easily.

(2) $\Phi(y_0, s_0) + (\delta_0)^4/2 = [d_{S^*} \wedge 0](y_0, s_0) = 0$. We claim that the hypotheses of the previous lemma are satisfied with $f = \Phi + (\delta_0)^4/2$, and $g = \gamma$. Indeed we have

$$\gamma(y_0, s_0) = \Phi(y_0, s_0) + (\delta_0)^4/2 = 0.$$

Now suppose that $\Phi(y, s) + (\delta_0)^4/2 > 0$ for some (y, s). Since $\Psi \geqslant \Phi + (\delta_0)^4/2$, we have $[d_{S^*} \wedge 0](y, s) = 0$. We also know that $\gamma \geqslant [d_{S^*} \wedge 0]$. Hence,

$$\gamma(y, s) \geqslant 0 \qquad \text{whenever} \quad \Phi(y, s) + (\delta_0)^4/2 > 0.$$

Also $y_0 \in S^*(s_0)$ and consequently $\Psi > 0$ in a neighborhood of (y_0, s_0). Since, by (6.22), $[d_{C^*} \wedge 0](y_0, s_0) < 0$, we conclude that $\Psi > [d_{C^*} \wedge 0]$ in a neighborhood of (y_0, s_0). Therefore by (6.16), $(y_0, s_0) \in B_{\delta_0}(x_0, t_0)$ and (6.15) implies (6.2) with $f = \Phi + (\delta_0)^4/2$. Now use the previous lemma to obtain (6.5) which is equivalent to (6.21).

(3) $[d_{S^*} \wedge 0](y_0, s_0) < 0$. Hence $y_0 \notin S^*(s_0)$. Choose $z_0 \in \partial S^*(s_0)$ such that

$$[d_{S^*} \wedge 0](y_0, s_0) = -|y_0 - z_0|.$$

Define

$$\xi(y, s) = \gamma(y + y_0 - z_0, s) + |y_0 - z_0|.$$

Then,

$$\begin{aligned}[d_{S^*} \wedge 0](z_0, s_0) - \xi(z_0, s_0) &= [d_{S^*} \wedge 0](y_0, s_0) - \gamma(y_0, s_0)\\ &\geqslant [d_{S^*} \wedge 0](y, s) - \gamma(y, s) \qquad \forall y, s,\\ &\geqslant -|z - y| + [d_{S^*} \wedge 0](z, s) - \gamma(y, s) \quad \forall y, s, z.\end{aligned}$$

Let $y = z + y_0 - z_0$,

$$\begin{aligned}&[d_{S^*} \wedge 0](z_0, s_0) - \xi(z_0, s_0)\\ &\quad \geqslant -|y_0 - z_0| + [d_{S^*} \wedge 0](z, s) - \gamma(z + y_0 - z_0, s)\\ &\quad = [d_{S^*} \wedge 0](z, s) - \xi(z, s) \qquad \forall y, s.\end{aligned}$$

Since $z_0 \in \partial S^*(s_0)$, $\Psi(z_0, s_0) \geqslant 0$. Also (6.22) yields that $z_0 \notin C^*(s_0)$. Hence

$\Psi(z_0, s_0) = \Phi(z_0, s_0) + (\delta_0)^4/2$. Using the previous two cases at the point z_0, s_0 we obtain

$$F_*\left(D\xi(z_0, s_0), D^2\xi(z_0, s_0)), \frac{\partial}{\partial t}\xi(z_0, s_0)\right) \leqslant 0.$$

Now (6.21) follows after observing that

$$\left(D\xi(z_0, s_0), D^2\xi(z_0, s_0), \frac{\partial}{\partial t}\xi(z_0, s_0)\right)$$
$$= \left(D\gamma(y_0, s_0), D^2\gamma(y_0, s_0), \frac{\partial}{\partial t}\gamma(y_0, s_0)\right).$$

(4) $\Phi(y_0, s_0) + (\delta_0)^4/2 < 0$, $[d_{S^*} \wedge 0](y_0, s_0) = 0$. Using (6.22) we conclude that $\Psi = \Phi + (\delta_0)^4/2 < 0$ in a neighborhood of (y_0, s_0). This contradicts the fact $[d_{S^*} \wedge 0](y_0, s_0) = 0$. ∎

7. Comparison

THEOREM 7.1. *Let* $\{L(t)\}_{t \geqslant 0}$ *be a viscosity subsolution of* (E) *and let* $\{U(t)\}_{t \geqslant 0}$ *be a viscosity supersolution of* (E), *respectively. Assume that for each* $T > 0$, *there is a positive constant* $R(T)$ *satisfying*

$$L(t), U(t) \subset B_{R(T)} \qquad \forall t \leqslant T. \tag{7.1}$$

Also, assume that there is $\alpha > 0$, *such that*

$$[d_{L^*} \wedge 0](x, 0) \leqslant [d_{U_*} \vee 0](x, 0) - \alpha \qquad \forall x \in R^d. \tag{7.2}$$

Then,

$$L^*(t) \subset U_*(t) \qquad \forall 0 \leqslant t < T(L(\cdot)) \wedge T(U(\cdot)). \tag{7.3}$$

Proof. Set $T_0 = T(L(\cdot)) \wedge T(U(\cdot))$, for $(x, t) \in R^d \times [0, T_0)$ define

$$u(x, t) = [d_{L^*} \wedge 0](x, t),$$

and

$$v(x, t) = [d_{U_*} \vee 0](x, t) - \alpha.$$

Remark 5.2(a) yields that u and v are a viscosity subsolution and a viscosity supersolution of (1.2) on $R^d \times [0, T_0)$, respectively. Clearly, (7.2) yields that

$$u(x, 0) \leqslant v(x, 0) \qquad \forall x \in R^d.$$

Also, using (7.1) we obtain that for any $t < T \wedge T_0$ and $|x| = R(T) + \alpha$,

$$\begin{aligned} u(x, t) &= d_{L^*}(x, t) \\ &\leqslant -\text{distance}(x, \partial B_{R(T)}) \\ &= -\alpha \\ &\leqslant [d_{U_*} \vee 0](x, t) - \alpha \\ &= v(x, t). \end{aligned}$$

Hence, Theorem 4.1 of [10] implies that

$$u(x, t) \leqslant v(x, t) \qquad \forall t < T \wedge T_0 \text{ and } |x| \leqslant R(T) + \alpha. \tag{7.4}$$

Suppose $x \in L^*(t)$ for some $t < T \wedge T_0$, then $|x| \leqslant R(T)$ and

$$u(x, t) = 0 \leqslant v(x, t) \leqslant [d_{U_*} \vee 0](x, t) - \alpha.$$

Therefore $d_{U_*}(x, t) \geqslant \alpha$, in particular $x \in U_*(t)$. ∎

For $\delta > 0$, define

$$L^\delta(t) = \{(x, t) \in R^d \times [0, \infty) : d_{L^*}(x, t) > -\delta\},$$
$$U_\delta(t) = \{(x, t) \in R^d \times [0, \infty) : d_{U_*}(x, t) > \delta\}.$$

Remark 7.2. Condition (7.2) is equivalent to

$$L^*(0) \subset (U_*)_\alpha(0). \tag{7.5}$$

The following is a weak regularity result in the time variable and it will be used in Sections 9 and 10.

LEMMA 7.3. *Let* $\{L(t)\}_{t \geqslant 0}$ *be a viscosity subsolution of* (E) *and let* $\{U(t)\}_{t \geqslant 0}$ *be a viscosity supersolution of* (E), *respectively. Then,*

(a) $\limsup_{s \uparrow t} [d_{C^*} \wedge 0](x, s) = [d_{C^*} \wedge 0](x, t)$
$\qquad \forall (x, t) \in R^d \times (0, T(C(\cdot)))$,

(b) $\liminf_{s \uparrow t} [d_{U_*} \vee 0](x, s) = [d_{U_*} \vee 0](x, t)$
$\qquad \forall (x, t) \in R^d \times (0, T(U(\cdot)))$.

Proof. (a) Fix $(x, t) \in R^d \times (0, T(U(\cdot)))$. We analyse two cases separately.

(1) $x \in C^*(t)$. Suppose the contrary. Then

$$-\alpha = \limsup_{s \uparrow t} [d_{C^*} \wedge 0](x, s) < 0,$$

and consequently there is $\delta > 0$ such that

$$[d_{C^*} \wedge 0](y, s) \leqslant -(\alpha/2), \qquad \forall\, |x - y| \leqslant \delta,\ s \in [t - \delta, t).$$

Hence, for any positive p

$$\begin{aligned}
&[d_{C^*} \wedge 0](y, s) - [d_{C^*} \wedge 0](x, t) - (s - t)\, p \\
&\quad \leqslant -(\alpha/2) - 0 - (s - t)\, p, \qquad \forall\, |x - y| \leqslant \delta,\ s \in [t - \delta, t) \\
&\quad \leqslant 0 \qquad \forall\, |x - y| \leqslant \delta,\ s \in [t - ((\alpha/2)\, p \wedge \delta), t).
\end{aligned}$$

For $s \geqslant t$, and positive p, $(s - t)\, p \geqslant 0$. Hence the above inequality also holds, for $(y, s) \in R^d \times [t, \infty)$. Therefore, $(0, 0, p) \in \mathbf{D}^+ C(t)$. Since $\beta > 0$ (see assumption (A)),

$$F_*(0, 0, p) \geqslant \min\{\beta(-n/|n|)\, p : n \in R^d\} > 0 \qquad \forall p > 0.$$

The above inequality contradicts the subsolution property of $\{C(t)\}_{t \geqslant 0}$.

(2) $x \notin C^*(t)$. Choose $z \in \partial C^*(t)$ such that

$$d_{C^*}(x, t) = -|x - z|.$$

Then, the upper semi-continuity and the sublinearity of $[d_{C^*} \wedge 0]$ imply that

$$\begin{aligned}
d_{C^*}(x, t) &\geqslant \limsup_{s \uparrow t}\, [d_{C^*} \wedge 0](x, s) \\
&\geqslant \limsup_{s \uparrow t}\, [d_{C^*} \wedge 0](z, s) - |x - z| \\
&= \limsup_{s \uparrow t}\, [d_{C^*} \wedge 0](z, s) + d_{C^*}(x, t).
\end{aligned}$$

Since $z \in \partial C^*(t)$, we apply case (1) to obtain

$$\limsup_{s \uparrow t}\, [d_{C^*} \wedge 0](z, s) = 0.$$

(b) This follows from part (a) and Remark 5.2(b). ∎

8. Nonuniqueness; Examples

We give two planar isotropic examples to show that there is no general uniqueness result. Nonuniqueness of solutions is related to the development of an interior of the level sets of viscosity solutions to (1.2). Similar examples are also discussed in Section 8.2 of [15].

EXAMPLE 8.1. Let $h(z, t)$ be the solution of

$$\frac{\partial}{\partial t} h(z, t) = \frac{\partial^2}{\partial^2 z} h(z, t) \left[1 + \left(\frac{\partial}{\partial z} h(z, t) \right) \right]^{-1} \qquad \forall t > 0, z > 0, \quad (8.1)(a)$$

$$\frac{\partial}{\partial z} h(0, t) = 0 \qquad \forall t > 0, \qquad (8.1)(b)$$

$$\lim_{\xi \to \infty} \frac{\partial}{\partial z} h(\xi, t) = 1 \qquad \forall t > 0, \qquad (8.1)(c)$$

$$h(z, 0) = z \qquad \forall z \geqslant 0. \qquad (8.1)(d)$$

The existence of such a solution can be proved by an approximation argument. Define

$$D(t) = \{(x, y) \in R^2 : |x| > h(|y|, t)\},$$

and

$$C(t) = \{(x, y) \in R^2 : |y| < h(|x|, t)\}.$$

A straightforward calculation shows that both $\{C(t)\}_{t \geqslant 0}$ and $\{D(t)\}_{t \geqslant 0}$ are classical solutions of (MCE) with initial condition

$$C(0) = D(0) = \{(x, y) \in R^2 : |x| > |y|\}.$$

EXAMPLE 8.2. Let $(h(z, t), A(t))$ and T be a solution of

$$\frac{\partial}{\partial t} h(z, t) = \frac{\partial^2}{\partial^2 z} h(z, t) \left[1 + \left(\frac{\partial}{\partial z} h(z, t) \right)^2 \right]^{-1} \qquad \forall t \in [0, T), z \in (0, A(t)), \quad (8.2)(a)$$

$$\frac{\partial}{\partial z} h(0, t) = h(A(t), t) = 0 \qquad \forall t \in [0, T), \qquad (8.2)(b)$$

$$\lim_{\xi \uparrow A(t)} \frac{\partial}{\partial z} h(\xi, t) = -\infty \qquad \forall t \in [0, T), \qquad (8.2)(c)$$

$$h(z, 0) = z\sqrt{1 - z^2} \qquad \forall z \in (0, 1), \qquad (8.2)(d)$$

$$A(0) = 1. \qquad (8.2)(e)$$

For $t \leqslant T$ define

$$D(t) = \{(x, y) \in [-A(t), A(t)] \times R : |y| < h(|x|, t)\}.$$

Let $(p(z,t), b(t), B(t))$ and T be a solution of

$$\frac{\partial}{\partial t} p(z,t) = \frac{\partial^2}{\partial^2 z} p(z,t)\left[1 + \left(\frac{\partial}{\partial z} p(z,t)\right)^2\right]^{-1} \qquad \forall t \in [0,T),\ z \in (b(t), B(t)), \quad (8.3)(\mathrm{a})$$

$$p(b(t),t) = p(B(t),t) = 0 \qquad \forall t \in [0,T), \quad (8.3)(\mathrm{b})$$

$$\lim_{\xi \downarrow b(t)} \frac{\partial}{\partial z} p(\xi,t) = -\lim_{\xi \uparrow B(t)} \frac{\partial}{\partial z} p(\xi,t) = \infty \qquad \forall t \in [0,T), \quad (8.3)(\mathrm{c})$$

$$p(z,0) = z\sqrt{1-z^2} \qquad \forall z \in (0,1), \quad (8.3)(\mathrm{d})$$

$$b(0) = 0, \qquad B(0) = 1. \quad (8.3)(\mathrm{e})$$

For $t \leqslant T$ define

$$C(t) = \{(x,y) \in ([-B(t), -b(t)] \cup [b(t), B(t)]) \times R : |y| < p(|x|, t)\}.$$

If there are solutions to (8.2) and (8.3), it is easy to show that both $\{C(t)\}_{t \geqslant 0}$ and $\{D(t)\}_{t \geqslant 0}$ are classical solutions of (E) with initial condition

$$C(0) = D(0) = \Gamma = \{(x,y) \in [-1,1] \times [-1,1] : |y| < |x|\sqrt{1-x^2}\}.$$

See Fig. 1.

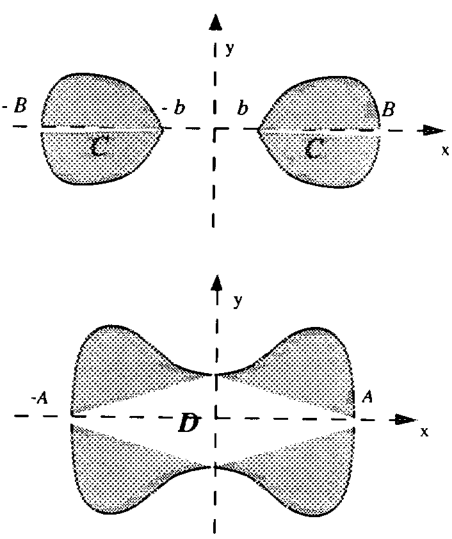

FIG. 1. Two solutions with initial data $C(0)$.

9. Uniqueness for Nonpositive v

In this section we prove a uniqueness result for $v \leqslant 0$ and a class of initial conditions. Along with other properties, the boundaries of these initial conditions do not have self-intersections. We start with the description of the condition we impose on the initial conditions.

For $\rho \in (0, 1)$, $\Gamma \subset R^d$, and $x \in R^d$ define

$$\begin{aligned}\Gamma(x, \rho) &= \rho(\Gamma \oplus \{-x\}) \oplus \{x\}\\ &= \{\rho y + (1-\rho)x : y \in \Gamma\}.\end{aligned}$$

Definition 9.1. We say that a bounded open subset Γ of R^d is *strictly starshaped* around a point $x \in R^d$ if there is $\rho_0 \in (0, 1)$ such that

$$\min\{\operatorname{dist}(z, \Gamma^c) : z \in \operatorname{cl} \Gamma(x, \rho)\} > 0, \qquad \forall \rho \in [\rho_0, 1).$$

The precompactness of Γ implies the following.

Lemma 9.2. *A bounded open set Γ is strictly starshaped around x if and only if there is ρ_0 such that*

$$\sup\{-\operatorname{dist}(z, \Gamma(x, \rho)) - \operatorname{dist}(z, \Gamma^c) : z \in R^d\} = -\alpha(\rho) < 0 \qquad \forall \rho \in [\rho_0, 1). \tag{9.1}$$

Theorem 9.3. *Let $\{L(t)\}_{t \geqslant 0}$ be a viscosity subsolution of* (E) *and let $\{U(t)\}_{t \geqslant 0}$ be a viscosity supersolution of* (E), *respectively. Assume that $v \leqslant 0$ and there is a bounded open set Γ satisfying*

$$L^*(0) \subset \operatorname{cl} \Gamma \qquad \textit{and} \qquad \Gamma \subset U_*(0). \tag{9.2}$$

Futher assume that Γ is strictly starshaped around a point x and for each $T > 0$, there is $R(T)$ such that

$$L(t),\ U(t) \subset B_{R(T)} \qquad \forall t \leqslant T.$$

Then,

$$L^*(t) \subset \operatorname{cl} U_*(t) \subset U^*(t),$$

and

$$L_*(t) \subset U_*(t) \qquad \forall 0 \leqslant t < T(L(\cdot)) \wedge T(U(\cdot)).$$

Proof. Since the equation (E) is invariant under translation, without loss of generality we may assume that Γ is strictly starshaped around the origin. Fix $\rho < 1$, and define

$$C(t) = \rho L\left(\frac{t}{\rho^2}\right), \qquad \forall t \geq 0.$$

We claim that $\{C(t)\}_{t\geq 0}$ is a subsolution of (E). Indeed suppose

$$(n, A, p) \in \mathbf{D}_x^{+2}{}_t^{+1}[d_{C^*} \wedge 0](x, t)$$

at some $(x, t) \in R^d \times [0, T(L(\cdot)))$. Since $C^*(s) = \rho L^*(s/\rho^2)$ for every $s \geq 0$, we have

$$d_{C^*}(y, s) = \rho\, d_{L^*}\left(\frac{y}{\rho}, \frac{s}{\rho^2}\right), \qquad \forall (y, s).$$

Then the definition of the set of second superdifferentials yields

$$\begin{aligned}
0 &\geq \limsup_{(y,h)\to 0} \left\{\frac{[d_{C^*} \wedge 0](x+y, t+h) - [d_{C^*} \wedge 0](x, t) - n \cdot y - ph - (1/2)\, Ay \cdot y}{|h| + |y|^2}\right\} \\
&= \limsup_{(y,h)\to 0} \left\{\frac{\rho[d_{L^*} \wedge 0]((x+y)/\rho, (t+h)/\rho^2) - \rho[d_{L^*} \wedge 0](x/\rho, t/\rho^2)}{|h| + |y|^2}\right. \\
&\quad \left. + \frac{-n \cdot y - ph - (1/2)\, Ay \cdot y}{|h| + |y|^2}\right\} \\
&= \frac{1}{\rho}\left\{\limsup_{(z,s)\to 0} \frac{[d_{L^*} \wedge 0]((x/\rho) + z, (t/\rho^2) + s) - [d_{L^*} \wedge 0](x/\rho, t/\rho^2)}{|s| + |z|^2}\right. \\
&\quad \left. + \frac{-n \cdot z - \rho p s - (1/2)\, \rho A z \cdot z}{|s| + |z|^2}\right\}.
\end{aligned}$$

Hence,

$$(n, \rho A, \rho p) \in \mathbf{D}_x^{+2}{}_t^{+1}[d_{L^*} \wedge 0](x/\rho, t/\rho^2).$$

Since $\{L(t)\}_{t\geq 0}$ is a subsolution of (E),

$$\begin{aligned}
0 &\geq F_*(n, \rho A, \rho p) \\
&= \rho F_*(n, A, p) - v\, |n| (1 - \rho) \\
&\geq \rho F_*(n, A, p).
\end{aligned}$$

Combine (9.1) and (9.2) to obtain

$$[d_{C^*} \wedge 0](x, 0) \leq [d_{U^*} \vee 0](x, 0) - \alpha(\rho). \tag{9.3}$$

Hence the hypotheses of Theorem 7.1 are satisfied by $\{C(t)\}_{t\geq 0}$ and $\{U(t)\}_{t\geq 0}$. Thus

$$\rho L^*\left(\frac{t}{\rho^2}\right)=C^*(t)\subset U_*(t),\qquad \forall t\in[0,T(L(\cdot))\wedge T(U(\cdot))). \tag{9.4}$$

Suppose $x\in L^*(t)$ with $t<T(L(\cdot))\wedge T(U(\cdot))$, using Lemma 7.3(a) we obtain a sequence $(x_n,t_n)\to(x,t)$ satisfying

$$t_n<t \qquad \text{and} \qquad x_n\in L^*(t_n)\ \forall n.$$

Set $\rho_n=\sqrt{t/t_n}$. Then

$$\rho_n x_n\in\rho_n L^*(t_n)=C^*(t). \tag{9.5}$$

Since $\rho_n<1$ and $\rho_n\to 1$, (9.4) and (9.5) yield that for sufficiently large n

$$\rho_n x_n\in U_*(t).$$

Now let n tend to infinity to conclude that $x\in \operatorname{cl} U_*(t)$. Suppose that $x\in L_*(t)$. Then by the definition of $L_*(t)$,

$$\rho x\in\rho L\left(\frac{t}{\rho^2}\right)$$

for all ρ sufficiently close to one. Since $L(s)$ is included in $L^*(s)$ for all s, (9.4) yields that $x\in U_*(t)$. ∎

10. Existence; Initial Value Problem

In this section we construct a maximal viscosity solution to (E) with a given initial data. Our construction is closely related to Section 6 of [10]. In view of Section 6, to obtain an existence result it suffices to construct a viscosity subsolution and a viscosity supersolution satisfying the given initial data

$$C^*(0)\subset \operatorname{cl}\Gamma, \tag*{(10.1)(a)}$$

$$C_*(0)\supset \operatorname{int}\Gamma, \tag*{(10.1)(b)}$$

where Γ is a given bounded subset of R^d.

For $r\geq 0$, define $w(r)$ by

$$w(r)=\frac{rR-\ln(rR+1)}{KR^2},$$

where

$$K = \max\{\text{trace } G(n)/\beta(n) : n \in R^d \text{ and } |n| = 1\} \vee 1,$$
$$R = (\max\{|v|/\beta(n) : n \in R^d \text{ and } |n| = 1\}/K) \vee 1.$$

For a given $x_0 \in R^d$ and $\rho > 0$, define

$$L(x_0, \rho)(t) = \{x \in R^d : \rho - t - w(|x - x_0|) > 0\},$$

and

$$U(x_0, \rho)(t) = \{x \in R^d : -\rho + t + w(|x - x_0|) > 0\}.$$

Then,

$$L(x_0, \rho)(t) = B_{\rho(t)}(x_0),$$

and

$$U(x_0, \rho)(t) = R^d \backslash B_{\rho(t)}(x_0),$$

where $B_\rho(x)$ denotes the d-dimensional sphere with radius ρ and center x, and $\rho(t)$ is the unique solution of

$$\rho = t + w(\rho(t)).$$

Hence,

$$\begin{aligned} \frac{d}{dt}\rho(t) &= -\left[\frac{d}{dr} w(\rho(t))\right]^{-1} \\ &= -\frac{K}{\rho(t)} - KR \\ &\leqslant -\frac{1}{\rho(t)} \frac{\text{trace } G(n)}{\beta(n)} - \frac{v}{\beta(n)}, \end{aligned}$$

for all n. A straightforward calculation, using the above estimate shows that $\{L(x_0, \rho)(t)\}_{t \geqslant 0}$ is a classical subsolution of (E), and $\{U(x_0, \rho)(t)\}_{t \geqslant 0}$ is a classical supersolution of (E). Finally, define

$$L(t) = \bigcup \{L(x_0, \rho)(t) : L(x_0, \rho)(0) \subset \Gamma\}, \tag{10.2}$$

and

$$U(t) = \bigcap \{U(x_0, \rho)(t) : \Gamma \subset U(x_0, \rho)(0)\}. \tag{10.3}$$

Then, Lemma 6.1 implies that $\{L(t)\}_{t\geqslant 0}$ and $\{U(t)\}_{t\geqslant 0}$ are a viscosity subsolution and a viscosity supersolution of (E), respectively. Observe that

$$L(t)\subset U(t) \qquad \forall t<T(L(\cdot))\wedge T(U(\cdot)). \tag{10.4}$$

THEOREM 10.1. *For any given proper subset Γ of R^d there is a viscosity solution $\{C(t)\}_{t\geqslant 0}$ of* (E) *satisfying* (10.1). *Moreover, the extinction time $T(C(\cdot))$ is strictly positive.*

Proof. The existence of a viscosity solution $\{C(t)\}_{t\geqslant 0}$ satisfying

$$L(t)\subset C(t)\subset U(t), \qquad \forall t<T(L(\cdot))\wedge T(U(\cdot)),$$

follows from the preceding calculations and Theorem 6.3. To show that (10.1) is satisfied by $\{C(t)\}_{t\geqslant 0}$, suppose that $x_0\notin \mathrm{cl}\,\Gamma$. Then using the definitions of $U(t)$ and $U(x_0,\rho)(t)$, we conclude that there exists $\rho>0$ satisfying

$$C(t)\subset U(t)\subset U(x_0,2\rho)(t),$$

for all $t\geqslant 0$. Since

$$U(x_0,2\rho)(t)\subset R^d\backslash B_\rho(x_0)$$

for all small t, $C(t)$ does not intersect $B_\rho(x_0)$ for all small t. Hence $x_0\notin C^*(0)$, and consequently (10.1)(a) holds. A similar analysis also yields (10.1)(b).

We continue by showing that the extinction time of $\{C(t)\}_{t\geqslant 0}$ is positive. Since Γ is a proper subset of R^d, there are $R_1>0$, $R_2>0$ and $x_0\in R^d$, $y_0\in R^d$ such that

$$L(x_0,R_1)(0)\subset\Gamma\subset U(y_0,R_2)(0).$$

Then, the definitions of $L(x_0,R_1)(t)$ and $U(y_0,R_2)(t)$ imply that

$$\varnothing\neq L(x_0,R_1)(t)\subset L(t)\subset C(t) \qquad \forall t<R_1,$$

and

$$C(t)\subset U(t)\subset U(y_0,R_2)(t)\neq R^d \qquad \forall t<R_2.$$

Therefore,

$$T(C(\cdot))\geqslant R_1\wedge R_2. \quad \blacksquare$$

Remark 10.2. Let $\{V(t)\}_{t\geqslant 0}$ be a viscosity supersolution of (E) satisfying (10.1)(a) and (10.4). Then,

$$Y(t)=U(t)\cap V(t)$$

is again a viscosity supersolution of (E) with initial condition (10.1) and satisfies (10.4). Hence, by using $\{Y(t)\}_{t\geqslant 0}$ instead of $\{U(t)\}_{t\geqslant 0}$, in the proof of the above theorem we obtain a viscosity solution included by the given viscosity supersolution $\{V(t)\}_{t\geqslant 0}$.

We need a technical lemma to prove that the viscosity solution constructed in the proofs of Theorem 6.3 and Theorem 10.1 is indeed a maximal one. Suppose that $\{C(t)\}_{t\geqslant 0}$ and $\{\Gamma(t)\}_{t\geqslant 0}$ are viscosity subsolutions of (E) satisfying,

$$[d_{\Gamma^*}\wedge 0](x,0)\leqslant [d_{C^*}\wedge 0](x,t_0),\qquad \forall x, \tag{10.5}$$

at some point t_0. Set

$$S(t)=\begin{cases} C(t) & \text{if } t\leqslant t_0\\ \Gamma(t-t_0) & \text{if } t>t_0.\end{cases}$$

LEMMA 10.3. *$\{S(t)\}_{t\geqslant 0}$ is a viscosity subsolution of* (E).

Proof. Let $(n,A,p)\in \mathbf{D}_x^{+2}{}_t^{+1}[d_{S^*}\wedge 0](x_0,s_0)$. We need to show that

$$F_*(n,A,p)\leqslant 0. \tag{10.6}$$

If $s_0\neq t_0$, then (10.6) follows easily from the subsolution properties of $\{C(t)\}_{t\geqslant 0}$ and $\{\Gamma(t)\}_{t\geqslant 0}$. So assume that $s_0=t_0$. Using (10.5) and Lemma 7.3(a), we obtain

$$[d_{S^*}\wedge 0](x,t)=\begin{cases}[d_{C^*}\wedge 0](x,t) & \text{if } t\leqslant t_0\\ [d_{\Gamma^*}\wedge 0](x,t-t_0) & \text{if } t>t_0.\end{cases}$$

Let Ψ be as in Theorem 14.1(b) with $\Phi=[d_{S^*}\wedge 0]$ and $(x,t)=(x_0,t_0)$. For $\varepsilon>0$, define

$$\Psi_\varepsilon(x,t)=\Psi(x,t)+[|x-x_0|^4+|t-t_0|^2]+\varepsilon\frac{1}{t_0-t}.$$

Choose $(x_\varepsilon,t_\varepsilon)$ such that $t_\varepsilon<t_0$, and

$$\begin{aligned}&[d_{C^*}\wedge 0](x_\varepsilon,t_\varepsilon)-\Psi_\varepsilon(x_\varepsilon,t_\varepsilon)\\ &\qquad\geqslant [d_{C^*}\wedge 0](x,t)-\Psi_\varepsilon(x,t)\qquad \forall x\in R^d,\ t\in[0,t_0).\end{aligned}$$

It is easy to show that $(x_\varepsilon, t_\varepsilon) \to (x_0, t_0)$ as ε tends to zero. Also the subsolution property of $\{C(t)\}_{t \geqslant 0}$ implies that

$$F_*\left(D\Psi_\varepsilon(x_\varepsilon, t_\varepsilon), D^2\Psi_\varepsilon(x_\varepsilon, t_\varepsilon), \frac{\partial}{\partial t}\Psi_\varepsilon(x_\varepsilon, t_\varepsilon)\right) \leqslant 0.$$

Since F_* is nondecreasing in the p-variable and

$$\liminf_{\varepsilon \downarrow 0} \frac{\partial}{\partial t}\Psi_\varepsilon(x_\varepsilon, t_\varepsilon) \geqslant \frac{\partial}{\partial t}\Psi(x_0, t_0),$$

we obtain (10.6) by letting ε go to zero. ∎

THEOREM 10.4. *For any given nonempty bounded open subset Γ of R^d there is a maximal viscosity solution $\{C(t)\}_{t \geqslant 0}$ of* (E) *satisfying* (10.1).

Proof. Since Γ is bounded, the extinction time of the viscosity supersolution $\{U(t)\}_{t \geqslant 0}$ given by (10.3) is ∞. Let $\{C(t)\}_{t \geqslant 0}$ be the viscosity solution of (E) constructed in the proofs of Theorems 6.3 and 10.1. Recall that for $t < T(U(\cdot))$,

$$C(t) = \bigcup \left\{\Gamma(t) : \begin{matrix} \{\Gamma(t)\}_{t \geqslant 0} \text{ is a subsolution of (E)} \\ \Gamma(t) \subset U(t)\ \forall t < T(U(\cdot)) \wedge T(\Gamma(\cdot)) \end{matrix}\right\}.$$

Since

$$C(t) \subset U(t),$$

and $T(U(\cdot)) = \infty$, we need to show that

$$\operatorname{int} C_*(t) = \varnothing \qquad \forall t \geqslant T(C(\cdot)).$$

Suppose to the contrary. Then, Theorem 10.1 and the Remark 10.2 imply that there exists a viscosity solution $\{\Gamma(t)\}_{t \geqslant 0}$ of (E) and initial condition $\Gamma = C^*(T(C(\cdot)))$. Moreover, it has a positive extinction time and

$$\Gamma(t) \subset U(t - T(C(\cdot))) \qquad \forall t \geqslant T(C(\cdot)).$$

Define

$$S(t) = \begin{cases} C(t) & \text{if } \ t \leqslant T(C(\cdot)) \\ \Gamma(t - T(C(\cdot))) & \text{if } \ t > T(C(\cdot)). \end{cases}$$

Since $\{\Gamma(t)\}_{t \geqslant 0}$ satisfies (10.1) with $\Gamma = C^*(T(C(\cdot)))$, the previous lemma yields that $\{S(t)\}_{t\geqslant 0}$ is a viscosity subsolution of (E). Then by the definition of $\{C(t)\}_{t\geqslant 0}$

$$T(C(\cdot))\geqslant T(S(\cdot))\geqslant T(C(\cdot))+T(\Gamma(\cdot)).$$

This contradicts the positivity of $T(\Gamma(\cdot))$. ∎

11. Connection between (E) and (1.2)

Let u be the unique viscosity solution of

$$F\left(Du(x,t), D^2u(x,t), \frac{\partial}{\partial t}u(x,t)\right)=0, \qquad \forall x\in R^d, t>0, \tag{1.2}$$

with initial condition

$$u(x,0)=\begin{cases}\operatorname{dist}(x,\partial\Gamma)\wedge 1 & \text{if } x\in\Gamma\\ -[\operatorname{dist}(x,\partial\Gamma)\wedge 1] & \text{if } x\notin\Gamma,\end{cases}$$

where Γ is a given bounded subset of R^d. The existence, the uniqueness, and the continuity of u are proved in Theorem 6.8 of [10]. Set

$$L(t)=\{x\in R^d; u(x,t)>0\}, \tag{11.1}$$

$$U(t)=\{x\in R^d; u(x,t)\geqslant 0\}. \tag{11.2}$$

THEOREM 11.1. (a) *For any bounded nonempty open subset Γ of R^d, $\{L(t)\}_{t\geqslant 0}$ and $\{U(t)\}_{t\geqslant 0}$ are maximal viscosity solutions of* (E) *satisfying* (10.1).

(b) *Let $\{\Gamma(t)\}_{t\geqslant 0}$ be a viscosity solution of* (E) *satisfying* (10.1). *Then,*

$$L_*(t)\subset\Gamma(t)\subset U^*(t) \qquad \forall t\in[0, T(L(\cdot))\wedge T(U(\cdot))). \tag{11.3}$$

Proof. Let $\mathbf{A}$ be the collection of all viscosity solutions of (E) with initial conditions which are compact in int Γ. Set

$$J(t)=\bigcup\{C(t):\{C(t)\}_{t\geqslant 0}\in\mathbf{A}, t<T(C(\cdot))\}\cup\varnothing.$$

Since Γ is bounded, if $\{C(t)\}_{t\geqslant 0}\in\mathbf{A}$ then there is $\alpha>0$ such that

$$[d_{C^*}\wedge 0](x,0)\leqslant u(x,0)-\alpha. \tag{11.4}$$

Also $\{C(t)\}_{t\geqslant 0}$ is included in the viscosity supersolution defined by (10.3).

In particular $\{C(t)\}_{t\geq 0}$ is bounded. Therefore (11.4) and Theorem 4.1 of [10] yield that

$$[d_{C^*} \wedge 0](x, t) \leq u(x, t) - \alpha.$$

Suppose that $x \in C(t)$ with $t \leq T(C(\cdot))$. Then, $u(x, t) \geq \alpha > 0$. Hence, $J(t)$ is included in $L(t)$ for all $t \geq 0$.

Suppose that $x_0 \in L(t_0)$ with $t_0 < T(L(\cdot))$, then $u(x_0, t_0) > \gamma$ for some $\gamma > 0$. By Theorem 10.4 there is a maximal viscosity solution $\{C(t)\}_{t\geq 0} \in \mathbf{A}$ such that

$$[d_{C_*} \vee 0](x, 0) \geq u(x, 0) - \gamma. \tag{11.5}$$

Then, Theorem 4.1 of [10] implies that $[d_{C_*} \vee 0](x, t) \geq u(x, t) - \gamma$ for all $(x, t) \in R^d \times [0, T(C(\cdot))]$. Since $\{C(t)\}_{t\geq 0}$ is maximal and Γ is bounded, if $T(C(\cdot))$ is finite,

$$C_*(T(C(\cdot))) = \varnothing.$$

Therefore the continuity of u and (11.5) yield that

$$T(C(\cdot)) \geq \inf\{t \geq 0 : \text{there is } x \text{ such that } u(x, t) > \gamma\}.$$

Hence $x_0 \in C(t_0)$, $t_0 < T(C(\cdot))$, and consequently $L(t) = J(t)$ for all $t \geq 0$. The stability theory, Appendix C, yields that $\{L(t)\}_{t\geq 0} = \{J(t)\}_{t\geq 0}$ is a viscosity solution.

Now let $\mathbf{B}$ be the collection of all viscosity solutions of (E) with initial data which compactly includes cl Γ. A similar argument yields

$$U(t) = \bigcap \{C(t) : \{C(t)\}_{t\geq 0} \in \mathbf{B}, t < T(C(\cdot))\} \cap R^d.$$

Let $\{C(t)\}_{t\geq 0}$ be a viscosity solution of (E) and (10.1). Theorem 7.1 implies that

$$K^*(t) \subset C(t) \subset V_*(t),$$

for all $\{K(t)\}_{t\geq 0} \in \mathbf{A}$, $\{V(t)\}_{t\geq 0} \in \mathbf{B}$, and $t \in [0, T(L(\cdot)) \wedge T(U(\cdot)))$. ∎

COROLLARY 11.2. *If the level set*

$$\Gamma(t) = \{x \in R^d ; u(x, t) = 0\}$$

has a nonempty interior then there is more than one viscosity solution to (E). *When there is a unique solution* $\{C(t)\}_{t\geq 0}$ *to* (E),

$$\Gamma(t) = \partial C(t) \qquad \forall t < T(C(\cdot)).$$

12. A Class of Explicit Solutions

In this section we construct a class of explicit solutions of (E) which are related to Wulff crystals [44] (also see Dinghast [13], Taylor [42, 43], and Fonseca [18]). We will use these solutions in the asymptotic analysis of (E). Let

$$B(\theta)=\frac{1}{\beta(\theta/|\theta|)}\,|\theta|$$

and we assume

$$\sum_{j=1}^{d}\sum_{i=1}^{d}\frac{\partial^2}{\partial\theta_i\,\partial\theta_j}B(\theta)\,\xi_i\,\xi_j>0 \qquad \forall\theta\cdot\xi=0. \tag{12.1}$$

For $x\in R^d$ and $x\neq 0$, set $\hat{x}=x/|x|$. Then define

$$R(x)=\min\left\{\frac{B(\theta)}{\theta\cdot\hat{x}}:\theta\in R^d \text{ and } \theta\cdot\hat{x}>0\right\} \qquad \forall x\neq 0, \tag{12.2}$$

and

$$\Theta(x)=\left\{\theta\in S^{d-1}:\theta\cdot\hat{x}>0 \text{ and } \frac{B(\theta)}{\theta\cdot\hat{x}}=R(x)\right\} \qquad \forall x\neq 0, \tag{12.3}$$

where $S^{d-1}=\{\theta\in R^d:|\theta|=1\}$. We gather several elementary properties of the above functions into a lemma.

LEMMA 12.1. (a) *There is a continuously differentiable function* $\theta(x)$ *such that*

$$\Theta(x)=\{\theta(x)\} \qquad \forall x\neq 0.$$

In particular,

$$DB(\theta(x))[\theta(x)\cdot\hat{x}]-B(\theta(x))\hat{x}=0. \tag{12.4}$$

(b) *For all* $x\neq 0$,

$$B(\theta(x))=R(x)[\theta(x)\cdot\hat{x}]=\max\{R(y)[\theta(x)\cdot\hat{y}]:y\neq 0\}.$$

In particular,

$$DR(x)[\theta(x)\cdot\hat{x}]+R(x)\frac{\theta(x)}{|x|}-R(x)[\theta(x)\cdot\hat{x}]\frac{x}{|x|^2}=0. \tag{12.5}$$

Proof. (a) For $i=1, \ldots, d$, set

$$H_i(\theta, x) = [\theta \cdot \hat{x}] \frac{\partial}{\partial \theta_i} B(\theta) - B(\theta)\, \hat{x}_i. \tag{12.6}$$

Then,

$$H(\theta, x) = (H_1(\theta, x), \ldots, H_d(\theta, x)) = 0 \qquad \forall \theta \in \Theta(x). \tag{12.7}$$

We directly calculate that

$$\frac{\partial}{\partial \theta_j} H_i(\theta, x) = [\theta \cdot \hat{x}] \frac{\partial^2}{\partial \theta_i \, \partial \theta_j} B(\theta) + \hat{x}_j \frac{\partial}{\partial \theta_i} B(\theta) - \hat{x}_i \frac{\partial}{\partial \theta_j} B(\theta).$$

Let ξ be a vector orthogonal to θ. Using (12.1) we conclude that

$$\begin{aligned} &\sum_{j=1}^{d} \sum_{i=1}^{d} \left[\frac{\partial}{\partial \theta_j} H_i(\theta, x) \right] \xi_i \xi_j \\ &\quad = [\theta \cdot \hat{x}] \sum_{j=1}^{d} \sum_{i=1}^{d} \left[\frac{\partial^2}{\partial \theta_i \, \partial \theta_j} B(\theta) \right] \xi_i \xi_j > 0, \qquad \forall \theta \cdot \xi = 0,\ \theta \cdot \hat{x} > 0. \end{aligned}$$

Hence for every x, there is a unique solution $\theta(x) \in S^{d-1}$ of Eq. (12.7). Using the implicit function theorem we conclude that $\theta(x)$ is continuously differentiable.

(b) This follows from straightforward calculations. ∎

Let h be a real-valued, continuously differentiable, strictly decreasing function on $[0, \infty)$. For $x \neq 0$, define

$$u(x) = h(|x|/R(x)).$$

Using the previous lemma, we directly calculate that

$$\begin{aligned} Du(x) &= h'(|x|/R(x)) \left[\frac{x}{|x|\, R(x)} - \frac{|x|}{(R(x))^2} DR(x) \right] \\ &= h'(x|x|/R(x)) \left[\frac{x}{|x|\, R(x)} - \frac{|x|}{(R(x))^2} \left\{ R(x) \frac{x}{|x|^2} - R(x) \frac{\theta(x)}{|x|\, [\theta(x) \cdot \hat{x}]} \right\} \right] \\ &= h'(|x|/R(x)) \frac{\theta(x)}{R(x)[\theta(x) \cdot \hat{x}]} \\ &= h'(|x|/R(x)) \frac{\theta(x)}{B(\theta(x))}. \end{aligned} \tag{12.8}$$

Since h is decreasing and $\theta(x) \in S^{d-1}$,

$$D\left(\frac{Du(x)}{|Du(x)|}\right) = -D(\theta(x)). \tag{12.9}$$

Set

$$g(\theta) = D^2(B(\theta))/(d-1). \tag{12.10}$$

LEMMA 12.2. *For any nonzero* $x \in R^d$,

$$\operatorname{trace}[g(\theta(x))(I - \theta(x) \otimes \theta(x))\, D(\theta(x))] = \frac{R(x)}{|x|}. \tag{12.11}$$

Proof. Recall that $H_i(\theta(x), x) = 0$, where H_i is as in (12.6). Differentiate this equation with respect to x_j and then use the same equation to obtain

$$[(\theta(x) \cdot x)\, D^2(B(\theta(x)))]\, D(\theta(x)) = B(\theta(x)) \left[I - \frac{1}{\theta(x) \cdot x} x \otimes \theta(x) \right]. \tag{12.12}$$

Since $B(\theta)$ is homogenous of degree one, $D^2(B(\theta))\theta = 0$ for every θ. Hence,

$$(d-1)\, g(\theta(x))(I - \theta(x) \otimes \theta(x)) = D^2(B(\theta(x))).$$

Using (12.12) we obtain

$$\begin{aligned} &g(\theta(x))(I - \theta(x) \otimes \theta(x))\, D(\theta(x)) \\ &= \frac{B(\theta(x))}{(d-1)(\theta(x) \cdot x)} \left(I - \frac{1}{\theta(x) \cdot x} x \otimes \theta(x) \right) \\ &= \frac{R(x)}{(d-1)\,|x|} \left(I - \frac{1}{\theta(x) \cdot x} x \otimes \theta(x) \right). \end{aligned}$$

We prove (12.11) after observing that $\operatorname{trace}(I - (1/\theta(x) \cdot x)\, x \otimes \theta(x)) = (d-1)$. ∎

For any $\alpha > 0$, and a real number λ define

$$C(t) = \{x \in R^d \backslash \{0\} : h(|x|/R(x)) > \alpha e^{\lambda t}\} \cup \{0\}.$$

Recall that for $\theta \in S^{d-1}$, $\beta(\theta) = (B(\theta))^{-1}$ and h is decreasing. Identities (12.8), (12.9) and (12.11) imply that at $x \in \partial C(t)$,

$$\text{outward normal velocity} = \mathbf{V} = \frac{\lambda \alpha e^{\lambda t}}{h'(|x|/R(x))\,\beta(\theta(x))}$$
$$= \frac{\lambda h(|x|/R(x))}{h'(|x|/R(x))\,\beta(\theta(x))},$$
$$\text{outward unit normal} = \theta = \theta(x),$$

and

$$\operatorname{trace} g(\theta(x))(I - \theta(x) \otimes \theta(x))\, D\left(\frac{Du(x)}{|Du(x)|}\right) = -\frac{R(x)}{|x|}.$$

Hence, $\{C(t)\}_{t \geqslant 0}$ is a classical solution of (E) with

$$G(\theta) = Cg(\theta)$$

provided that

$$\frac{\lambda h(\rho)}{h'(\rho)} = -\frac{C}{\rho} + v \qquad \forall \rho = |x|/R(x). \tag{12.13}$$

EXAMPLE 12.3. Suppose $v > 0$. Then a solution to (12.13) with $\lambda = -1$ is

$$h(\rho) = \exp\left[-\frac{\rho}{v} - \frac{C}{v^2} \ln\left(\rho - \frac{C}{v}\right)\right] \qquad \forall \rho > \frac{C}{v}.$$

Hence, for any $\alpha > 0$,

$$C_1(t) = \{x \in R^d \setminus \{0\} : h(|x|/R(x)) > \alpha e^{-t}\} \cup \{0\}$$

is a classical solution of

$$\beta(\theta)\mathbf{V} = -\operatorname{trace} Cg(\theta)\mathbf{R} + v.$$

The solution $\{C_1(t)\}_{t \geqslant 0}$ is increasing in time with infinite extinction time.

EXAMPLE 12.4. Suppose $v > 0$. Then a solution to (12.13) with $\lambda = 1$ is

$$h(\rho) = \exp\left[\frac{\rho}{v} + \frac{C}{v^2} \ln\left(-\rho + \frac{C}{v}\right)\right] \qquad \forall \rho \in \left[0, \frac{C}{v}\right].$$

Hence, for any $\alpha > 0$,

$$C_2(t) = \{x \in R^d \setminus \{0\} : h(|x|/R(x)) > \alpha e^{t}\} \cup \{0\}$$

is a classical solution of

$$\beta(\theta)\mathbf{V} = -\text{trace } Cg(\theta)\mathbf{R} + v.$$

In this case the solution is decreasing in time with extinction time

$$T(C_2(\cdot)) = \frac{C}{v^2}\ln\left(\frac{C}{v}\right) - \ln\alpha.$$

EXAMPLE 12.5. Let $v = 0$. Then with $\lambda = 1$,

$$h(\rho) = \exp\left(-\frac{\rho^2}{2C}\right)$$

is a solution of (12.13). Hence, for any $\alpha > 0$,

$$\begin{aligned} C_3(t) &= \{x \in R^d \backslash \{0\} : h(|x|/R(x)) > \alpha e^t\} \cup \{0\} \\ &= \{x \in R^d \backslash \{0\} : [|x|/R(x)] < \sqrt{2C[-t - \ln\alpha]}\} \cup \{0\} \end{aligned}$$

is a classical solution of

$$\beta(\theta)\mathbf{V} = -\text{trace } Cg(\theta)\mathbf{R}.$$

In this case the solution is decreasing in time with extinction time

$$T(C_3(\cdot)) = -\ln\alpha.$$

DEFINITION 12.6. The open set

$$W(1/\beta) = \{x \in R^d \backslash \{0\} : |x| < R(x)\} \cup \{0\}$$

is called the *Wulff crystal* of the surface energy $(1/\beta)$. However, we should note that in the equation (E), the coefficient $(1/\beta)$ is the kinetic coefficient, *not* the surface energy.

All the solutions constructed in the above examples are dilations of the Wulff crystal $W(1/\beta)$. We collect the previous examples into the following lemma.

LEMMA 12.7. *For $i = 1, 2, 3$,*

$$C_i(t) = \alpha_i(t)\, W(1/\beta),$$

where $h(\alpha_i(t)) = \alpha e^{\lambda t}$. In particular, $\alpha_i(t)$ is the unique solution of

$$\frac{d}{dt}\alpha_i(t) = \left[-\frac{C}{\alpha_i(t)} + v\right] \qquad \forall t > 0, \tag{12.14}$$

with initial condition

$$\alpha_i(0) = h^{-1}(\alpha). \tag{12.15}$$

Remark 12.8. If β satisfies (12.1), then the Wulff crystal $\nu W(1/\beta)$ is a solution of the stationary problem,

$$0 = -\operatorname{trace} g(\theta)\mathbf{R} + \nu.$$

This fact was proved by Angenent and Gurtin in two dimensions (see Section 6.1 of [6]).

13. Large Time Asymptotics

In this section we show that any solution of (E), with bounded initial condition, has finite extinction time if $\nu \leqslant 0$ or if it is initially small. We also show that if $\nu > 0$, then solutions of (E) with large enough but bounded initial conditions have infinite extinction time. These results were already proved by Angenent and Gurtin [6] for classical solutions in two dimensions. Also Chen, Giga, and Goto [10] proved the finite extinction when $\nu = 0$. We use the comparison result Theorem 7.1 together with the explicit solutions constructed in the previous section. Our techniques also show that when the solution is growing, asymptotically it has the shape of the Wulff region $W(1/\beta)$.

We employ the notation $A \subset\subset B$, if A is a compact subset of B. When B is bounded, and $A \subset\subset B$, then A and B satisfy (7.5), i.e., $A \subset B_\alpha$ for a suitable α.

Lemma 13.1. *Suppose that* $\{C(t)\}_{t \geqslant 0}$ *is a viscosity subsolution of* (E) *and there is K such that*

$$C^*(0) \subset\subset B_K(0).$$

Further assume that

$$\nu \leqslant 0 \qquad \text{and} \qquad \frac{\operatorname{trace} G(\theta)}{\beta(\theta)} \geqslant g_0 > 0, \forall \theta.$$

Then,

$$T(C(\cdot)) \leqslant \frac{K^2}{2g_0}.$$

Proof. Let

$$\Gamma(t) = \alpha(t)\, W(1),$$

where α is the solution of (12.14) with $C = g_0$ and initial condition $\alpha(0) = K$. Then, Lemma 12.5 yields that $\{\Gamma(t)\}_{t \geq 0}$ is a classical solution of

$$\mathbf{V} = -g_0 \kappa.$$

Since the mean curvature of $\partial\Gamma(t)$ is always positive, $\{\Gamma(t)\}_{t \geq 0}$ is a supersolution of (E). Moreover, $\Gamma(0) = B_K(0)$. Hence Theorem 7.1 yields that

$$C^*(t) \subset \Gamma(t) \qquad \forall t < T(C(\cdot)) \wedge T(\Gamma(\cdot)).$$

We complete the proof after observing that $T(\Gamma(\cdot)) = K^2/2g_0$. ∎

Let $g(\theta)$ be as in the previous section and (12.1) is satisfied.

LEMMA 13.2. *Let $\{C(t)\}_{t \geq 0}$ be a viscosity subsolution of* (E). *Assume that $v > 0$ and that there is $g_1 > 0$ satisfying*

$$G(\theta) \geq g(\theta)\, g_1,$$

for all θ. Further assume that there is $K_1 < g_1/v$ such that

$$C^*(0) \subset \subset K_1 W(1/\beta).$$

Then, $T(C(\cdot))$ is finite.

Proof. Let

$$\Gamma_1(t) = \alpha_1(t)\, W(1/\beta),$$

where α_1 is the solution of (12.14) with $C = g_1$ and initial condition $\alpha(0) = K_1$. Then, Lemma 12.5 yields that $\{\Gamma_1(t)\}_{t \geq 0}$ is a classical solution of

$$\beta(\theta)\mathbf{V} = -\operatorname{trace} g(\theta)\, g_1 \mathbf{R} + v.$$

Since the second fundamental form of $\partial T_1(t)$ is always positive definite and $G(\theta) \geq g_1\, g(\theta)$, $\{\Gamma_1(t)\}_{t \geq 0}$ is a supersolution of (E). Therefore Theorem 7.1 yields

$$C^*(t) \subset \Gamma_1(t) \qquad \forall t < T(C(\cdot)) \wedge T(\Gamma_1(\cdot)).$$

Using the assumption $K_1 < g_1/v$, we obtain

$$\Gamma_1(t) = \left\{ x \in R^d : |x| < R(x)\frac{g_1}{v} \text{ and } h(|x|/R(x)) > h(K_1)\, e^{-t} \right\},$$

where

$$h(\rho)=\exp\left[\frac{\rho}{v}+\frac{g_1}{v^2}\ln\left(-\rho+\frac{g_1}{v}\right)\right] \qquad \forall\rho\in\left[0,\frac{g_1}{v}\right].$$

Recall that the above function is the one computed in Example 12.4 with $C=g_1$. Hence, $T(\Gamma_1(\cdot))$ is finite and so is $T(C(\cdot))$. ∎

LEMMA 13.3. *Let $\{C(t)\}_{t\geqslant 0}$ be a maximal viscosity supersolution of* (E). *Assume that $v>0$ and there is $g_2>0$ satisfying*

$$G(\theta)\leqslant g(\theta)\, g_2,$$

for all θ. Further assume that there is $K_2<(g_2/v)$ such that

$$K_2\, W(1/\beta)\subset\subset C^*(0),$$

and $C^*(0)$ is bounded. Then, $T(C(\cdot))$ is infinite.

Proof. Let

$$\Gamma_2(t)=\alpha_2(t)\, W(1/\beta),$$

where α_2 is the solution of (12.14) with $C=g_2$ and initial condition $\alpha(0)=K_2$. Then, Lemma 12.5 yields that $\{\Gamma_2(t)\}_{t\geqslant 0}$ is a classical solution of

$$\beta(\theta)\mathbf{V}=-\operatorname{trace} g(\theta)\, g_2\mathbf{R}+v.$$

Since the second fundamental form of $\partial\Gamma_2(t)$ is always positive definite, $\{\Gamma_2(t)\}_{t\geqslant 0}$ is a subsolution of (E). Therefore Theorem 7.1 yields

$$\Gamma_2(t)\subset C_*(t) \qquad \forall t<T(C(\cdot))\wedge T(\Gamma_2(\cdot)).$$

Using the assumption $K_2>g_2/v$, we proceed as in Example 12.3 to obtain

$$\Gamma_2(t)=\left\{x\in R^d:|x|\leqslant R(x)\frac{g_2}{v}\right\}$$
$$\cup\left\{x\in R^d:|x|>R(x)\frac{g_2}{v}\text{ and } h(|x|/R(x))>h(K_2)\,e^{-t}\right\},$$

where

$$h(\rho)=\exp\left[-\frac{\rho}{v}-\frac{g_2}{v^2}\ln\left(\rho-\frac{g_2}{v}\right)\right] \qquad \forall\rho>\frac{g_2}{v}.$$

We complete the proof after observing that $T(\Gamma_2(\cdot))$ is infinite, $\{C(t)\}_{t\geq 0}$ is maximal, and $C(t)$ is bounded for each t. ▮

PROPOSITION 13.4. *Let $\{C(t)\}_{t\geq 0}$ be a maximal viscosity solution of* (E) *and g_1 and g_2 be positive constants satisfying*

$$g(\theta)\, g_1 < G(\theta) \leq g(\theta)\, g_2,$$

for all θ. Further assume that there are K_1 and K_2 such that

$$K_1 < \frac{g_1}{v}, \qquad K_2 > \frac{g_2}{v}$$

$$K_1\, W(1/\beta) \subset\subset C^*(0),$$

and

$$C_*(0) \subset\subset K_2\, W(1/\beta).$$

Then,

$$\alpha_1(t)\, W(1/\beta) \subset C(t) \subset \alpha_2(t)\, W(1/\beta) \qquad \forall t > 0, \tag{13.1}$$

where $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ are solutions of (12.14) *with $C = g_i$ and initial conditions $\alpha_i(0) = K_i$ for $i = 1, 2$.*

Proof. Lemma 12.5 implies that $\alpha_i(t)\, W(1/\beta)$ is a solution of

$$\beta(\theta)\mathbf{V} = -\text{trace}\ g_i\, g(\theta)\mathbf{K} + v, \qquad \text{for} \quad i = 1, 2.$$

The positivity of the second fundamental form of $W(1/\beta)$ and the comparison principle yield the result. ▮

Remark 13.5. Since for $i = = 1, 2$ $\alpha_i(t)$ tends to infinity, from (12.14) we obtain that

$$\lim_{t\to\infty} \frac{\alpha_i(t)}{t} = v.$$

Hence,

$$\lim_{t\to\infty} \frac{1}{t}\, C(t) = v W(1/\beta).$$

The above asymptotic result is conjectured by Angenent and Gurtin (see [6, p. 354]).

14. APPENDIXES

A. *Properties of Sub- and Superdifferentials*

In this section we gather some properties of the set of sub- and superdifferentials. The proof of the following lemma is similar to the proof of Proposition 1.1 in [11] and Lemma I.4 in [12] (also see Lemma 2.15 in [33]).

For an open set O, $C^{2,1}(O)$ denotes the set of functions which are twice continuously differentiable in x and once continuously differentiable in t.

LEMMA 14.1. (a) $(n, p) \in \mathbf{D}^+\Phi(x, t)$ *if and only if there is a continuously differentiable function* Ψ *such that*

$$D\Psi(x, t) = n, \tag{14.1)(a}$$

$$\frac{\partial}{\partial t}\Psi(x, t) = p, \tag{14.1)(b}$$

$$\Phi^*(x, t) - \Psi(x, t) > \Phi^*(z, s) - \Psi(z, s), \qquad \forall (z, s) \neq (x, t). \tag{14.1)(c}$$

(b) $(n, A, p) \in \mathbf{D}_x^{+2}{}_t^{+1}\Phi(x, t)$ *if and only if there is* $\Psi \in C^{2,1}(R^d \times R)$ *such that*

$$D\Psi(x, t) = n, \tag{14.2)(a}$$

$$\frac{\partial^2}{\partial x_i\, x_j}\Psi(x, t) = A_{ij}, \qquad i, j = 1, ..., d, \tag{14.2)(b}$$

$$\frac{\partial}{\partial t}\Psi(x, t) = p, \tag{14.2)(c}$$

$$\Phi^*(x, t) - \Psi(x, t) > \Phi^*(z, s) - \Psi(z, s), \qquad \forall (z, s) \neq (x, t). \tag{14.2)(d}$$

(c) $(n, p) \in \mathbf{D}^-\Phi(x, t)$ *if and only if there is a continuously differentiable function* Ψ *such that*

$$D\Psi(x, t) = n, \tag{14.3)(a}$$

$$\frac{\partial}{\partial t}\Psi(x, t) = p, \tag{14.3)(b}$$

$$\Phi_*(x, t) - \Psi(x, t) < \Phi_*(z, s) - \Psi(z, s), \qquad \forall (z, s) \neq (x, t). \tag{14.3)(c}$$

(d) *$(n, A, p) \in \mathbf{D}_x^{-2}{}_t^{-1}\Phi(x, t)$ if and only if there is $\Psi \in C^{2,1}(R^d \times R)$ such that*

$$D\Psi(x, t) = n, \tag{14.4)(a}$$

$$\frac{\partial^2}{\partial x_i x_j} \Psi(x, t) = A_{ij}, \qquad i, j = 1, ..., d, \tag{14.4)(b}$$

$$\frac{\partial}{\partial t} \Psi(x, t) = p, \tag{14.4)(c}$$

$$\Phi_*(x, t) - \Psi(x, t) < \Phi_*(z, s) - \Psi(z, s), \qquad \forall (z, s) \neq (x, t). \tag{14.4)(d}$$

Proof. (a) and (c) These are proved in Proposition 1.1 of [11].

(b) Without loss of generality we may assume that $A = 0$, $n = 0$, $p = 0$, and $(x, t) = 0$. For $\rho > 0$, let

$$h(\rho) = \sup\left\{\frac{(\Phi^*(z, s)) \vee 0}{\sqrt{|z|^4 + |s|^2}} : \sqrt{|z|^4 + |s|^2} \leqslant \rho\right\}.$$

Then h is continuous on $[0, \infty)$, with $h(0) = 0$. Let

$$\Psi(z, s) = F(\sqrt{|z|^4 + |s|^2}),$$

where

$$F(r) = \frac{1}{r} \int_r^{2r} \int_\xi^{2\xi} h(\rho)\, d\rho\, d\xi.$$

Straightforward calculations show that Ψ satisfies (14.2).

(d) This is similar to part (b). ∎

Remark 14.2. In the above lemma, Ψ can not be smoother than $C^{2,1}$ in parts (b), (d) and C^1 in parts (a) and (c). However, by a simple approximation argument we obtain an equivalent definition of viscosity solutions which uses only smooth functions (see Appendix B). Also the global smoothness of these tests functions is not necessary.

COROLLARY 14.3. (a) *If $(n, A, p) \in \mathbf{D}_x^{+2}{}_t^{+1}\Phi(x, t)$, then $(n, p) \in \mathbf{D}^+\Phi(x, t)$.*

(b) *If $(n, A, p) \in \mathbf{D}_x^{-2}{}_t^{-1}\Phi(x, t)$, then $(n, p) \in \mathbf{D}^-\Phi(x, t)$.*

LEMMA 14.4. *Suppose $0 < |n| \leqslant 1$, and $An = 0$. Then, the following are equivalent:*

(a) $(n, A, p) \in \mathbf{D}^+C(t_0)$.

(b) *There are* $x_0 \in \partial C^*(t_0)$ *and a collection of open sets* $\{S(t)\}_{t \geq 0}$ *satisfying (see Fig. 2)*

$$d_S \textit{ is } C^{2,1}(N) \textit{ on a neighborhood } N \textit{ of } (x_0, t_0), \tag{14.5)(a}$$

$$Dd_S(x_0, t_0) = \frac{n}{|n|}, \tag{14.5)(b}$$

$$\frac{\partial^2}{\partial x_i x_j} d_S(x_0, t_0) = \frac{A_{ij}}{|n|}, \qquad i, j = 1, ..., d, \tag{14.5)(c}$$

$$\frac{\partial}{\partial t} d_S(x_0, t_0) = \frac{p}{|n|}, \tag{14.5)(d}$$

$$\bigcup_{s \geq 0} [[C^*(s) \cap (S(s))^c] \times \{s\}] = \{(x_0, t_0)\}. \tag{14.5)(e}$$

(c) $(1/|n|)(n, A, p) \in \mathbf{D}_x^{+2}{}_t^{+1}\, d_{C^*}(x_0, t_0)$ *for some* $x_0 \in \partial C^*(t_0)$.

Proof. (a) $\Rightarrow$ (b). The definition of $\mathbf{D}^+ C(t_0)$ implies that

$$(n, A, p) \in \mathbf{D}_x^{+2}{}_t^{+1}[d_{C^*} \wedge 0](x_0, t_0)$$

at some $x_0 \in R^d$. Since $|n| \neq 0$, $x_0 \notin \operatorname{int} C^*(t_0)$. We analyse the remaining two cases separately.

(1) $x_0 \in \partial C^*(t_0)$. Let Ψ be as in Lemma 14.1(b). Set

$$S(t) = \{(z, t) \in R^d \times [0, \infty): \Psi(z, t) > 0\}.$$

Then, (14.5)(a)–(d) is satisfied by $\{S(t)\}_{t \geq 0}$ due to the smoothness of Ψ and the positivity of $|n|$. Since $\Psi(x_0, t_0) = 0$,

$$x_0 \in [C^*(t_0) \cap (S(t_0))^c].$$

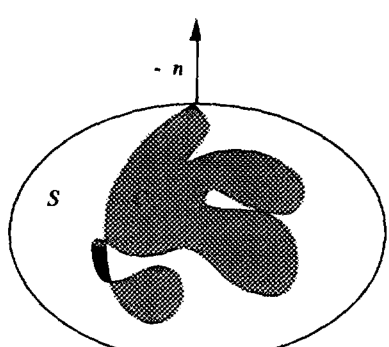

FIGURE 2

Suppose that

$$z \in [C^*(s) \cap (S(s))^c].$$

Then, $d_{C^*}(z, s) \geqslant 0 \geqslant \Psi(z, s)$, and (14.2)(d) yields that $(z, s) = (x_0, t_0)$. So, (14.5)(e) is satisfied by $\{S(t)\}_{t \geqslant 0}$.

(2) $x_0 \notin C^*(t_0)$. Let Ψ be as in the previous case. Choose $y_0 \in \partial C^*(t_0)$ such that

$$d_{C^*}(x_0, t_0) = -|x_0 - y_0|.$$

Define

$$\Phi(z, s) = \Psi(z + x_0 - y_0, s) + |x_0 - y_0|.$$

Then,

$$\begin{aligned}[d_{C^*} \wedge 0](y_0, t_0) - \Phi(y_0, t_0) &= [d_{C^*} \wedge 0](x_0, t_0) - \Psi(x_0, t_0)\\ &\geqslant [d_{C^*} \wedge 0](z, s) - \Psi(z, s) \qquad \forall z, s.\end{aligned}$$

Using the above inequality and

$$[d_{C^*} \wedge 0](z, s) \geqslant -|z - w| + [d_{C^*} \wedge 0](w, s) \qquad \forall w, s,$$

we obtain

$$\begin{aligned}&[d_{C^*} \wedge 0](y_0, t_0) - \Phi(y_0, t_0)\\ &\qquad \geqslant -|z - w| + [d_{C^*} \wedge 0](w, s) - \Psi(z, s) \qquad \forall z, w, s.\end{aligned}$$

Let $z = w + x_0 - y_0$ in the above inequality and then use the definition of Φ to obtain

$$\begin{aligned}&[d_{C^*} \wedge 0](y_0, t_0) - \Phi(y_0, t_0)\\ &\qquad \geqslant -|x_0 - y_0| + [d_{C^*} \wedge 0](w, s) - \Psi(w + x_0 - y_0, s)\\ &\qquad = [d_{C^*} \wedge 0](w, s) - \Phi(w, s) \qquad \forall w, s.\end{aligned}$$

By Lemma 14.1(b) we conclude that

$$\left(D\Phi, D^2\Phi, \frac{\partial}{\partial t}\Phi\right)(y_0, t_0) \in \mathbf{D}_x^{+2}{}_t^{+1}[d_{C^*} \wedge 0](y_0, t_0).$$

Also,

$$\begin{aligned}\left(D\Phi, D^2\Phi, \frac{\partial}{\partial t}\Phi\right)(y_0, t_0) &= \left(D\Psi, D^2\Psi, \frac{\partial}{\partial t}\Psi\right)(x_0, t_0)\\ &= (n, A, p).\end{aligned}$$

Therefore $(n, A, p) \in \mathbf{D}_x^{+2,+1}_t[d_{C^*} \wedge 0](y_0, t_0)$, and part (b) yield (14.5) at the point y_0.

(b) $\Rightarrow$ (c). We claim that

$$d_{C^*}(x_0, t_0) - d_S(x_0, t_0) \geqslant d_{C^*}(x, t) - d_S(x, t) \qquad \forall (x, t) \in R^d \times [0, \infty). \tag{14.6}$$

We analyse three cases separately.

(1) $x \in [C^*(t) \cap (S(t))^c]$. Then, (14.5)(e) yields that $(x, t) = (x_0, t_0)$ and (14.6) holds trivially.

(2) $x \in \mathrm{cl}\, S(t)$. Choose $z_0 \in \partial S(t)$ such that

$$|x - z_0| = d_S(x, t).$$

Since $d_{C^*}(x_0, t_0) = d_S(x_0, t_0) = 0$ and we are trying to prove (14.6), we may assume that $d_{C^*}(x, t) \geqslant d_S(x, t) = |x - z_0|$. Therefore $z_0 \in C^*(t)$, and consequently

$$d_{C^*}(x, t) \leqslant |x - z_0| + d_{C^*}(z_0, t).$$

Hence,

$$\begin{aligned} d_{C^*}(x, t) - d_S(x, t) &\leqslant |x - z_0| + d_{C^*}(z_0, t) - |x - z_0| \\ &= d_{C^*}(z_0, t) \\ &= d_{C^*}(z_0, t) - d_S(z_0, t). \end{aligned}$$

So

$$d_{C^*}(z_0, t) - d_S(z_0, t) \geqslant d_{C^*}(x, t) - d_S(x, t) \geqslant 0. \tag{14.7}$$

Since $d_S(z_0, t) = 0$, $d_{C^*}(z_0, t) \geqslant 0$. Therefore

$$z_0 \in [\mathrm{cl}\, C^*(t) \cap (S(t))^c].$$

We now apply case (1) to conclude that $(z_0, t) = (x_0, t_0)$ and (14.7) yields (14.6).

(3) $x \notin C^*(t)$. Choose $z_0 \in \partial C^*(t)$ such that

$$-|x - z_0| = d_{C^*}(x, t).$$

Since $d_{C^*}(x_0, t_0) = d_S(x_0, t_0) = 0$, we may assume that $-|x - z_0| = d_{C^*}(x, t) \geqslant d_S(x, t)$. Therefore $z_0 \notin \mathrm{cl}\, S(t)$, and consequently

$$d_S(x, t) \geqslant -|x - z_0| + d_S(z_0, t).$$

Proceed as in the previous case to obtain

$$d_{C^*}(z_0, t) - d_S(z_0, t) \geq d_{C^*}(x, t) - d_S(x, t) \geq 0. \tag{14.8}$$

Since $d_{C^*}(z_0, t) = 0$, $d_S(z_0, t) \leq 0$. Therefore

$$z_0 \in [C^*(t) \cap (S(t))^c].$$

We now apply case (1) to conclude that $(z_0, t) = (x_0, t_0)$ and (14.8) yields (14.6).

This completes the proof of (14.6). Since d_S is smooth near (x_0, t_0), Remark 14.2 and (14.6) yield

$$\left(Dd_S, D^2 d_S, \frac{\partial}{\partial t} d_S\right)(x_0, t_0) \in \mathbf{D}_x^{+2}{}_t^{+1} d_{C^*}(x_0, t_0).$$

(c) $\Rightarrow$ (a). The fact that

$$[d_{C^*} \wedge 0] \leq d_{C^*}$$

implies that $\mathbf{D}_x^{+2}{}_t^{+1} d_{C^*}(x, t)$ is included in $\mathbf{D}_x^{+2}{}_t^{+1}[d_{C^*} \wedge 0](x, t)$ at every $t \geq 0$, and $x \in \partial C^*(t)$. Therefore

$$\frac{1}{|n|}(n, A, p) \in \mathbf{D}_x^{+2}{}_t^{+1}[d_{C^*} \wedge 0](x_0, t_0).$$

Also, $(0, 0, 0) \in \mathbf{D}_x^{+2}{}_t^{+1}[d_{C^*} \wedge 0](x_0, t_0)$. Hence the convexity of $\mathbf{D}_x^{+2}{}_t^{+1}[d_{C^*} \wedge 0](x_0, t_0)$ yields (a). ∎

Observe that

$$\begin{aligned} \mathbf{D}^- C(t) &= \bigcup_{x \in R^d} \mathbf{D}_x^{-2}{}_t^{-1}(-d_{(R^d \setminus C(\cdot))^*} \vee 0)(x, t) \\ &= \bigcup_{x \in R^d} -\mathbf{D}_x^{+2}{}_t^{+1}(d_{(R^d \setminus C(\cdot))^*} \wedge 0)(x, t) \\ &= -\mathbf{D}^+(R^d \setminus C(\cdot))(t). \end{aligned}$$

Hence we have the following analogue of the previous lemma for the superdifferentials.

LEMMA 14.5. *Suppose* $0 < |n| \leq 1$, *and* $An = 0$. *Then, the following are equivalent*:

(a) $(n, A, p) \in \mathbf{D}^- C(t_0)$.

(b) *There are* $x_0 \in \partial C_*(t_0)$ *and a collection of open sets* $\{S(t)\}_{t \geq 0}$ *satisfying* (*see Fig.* 3)

$$d_S \text{ is } C^{2,1}(N) \text{ on a neighborhood } N \text{ of } (x_0, t_0), \tag{14.9)(a}$$

$$Dd_S(x_0, t_0) = \frac{n}{|n|}, \tag{14.9)(b}$$

$$\frac{\partial^2}{\partial x_i x_j} d_S(x_0, t_0) = \frac{A_{ij}}{|n|}, \qquad i, j = 1, ..., d, \tag{14.9)(c}$$

$$\frac{\partial}{\partial t} d_S(x_0, t_0) = \frac{p}{|n|}, \tag{14.9)(d}$$

$$\bigcup_{s \geq 0} [[(C_*(s))^c \cap S(s)] \times \{s\}] = \{(x_0, t_0)\}. \tag{14.9)(e}$$

(c) $(1/|n|)(n, A, p) \in \mathbf{D}_{x}^{2}{}_{t}^{1}\, d_{C_*}(x_0, t_0)$ *for some* $x_0 \in \partial C_*(t_0)$.

B. *An Equivalent Definition*

Following the proof of Theorem 1.1 of [11], we obtain an equivalent definition (also see Lemma 2.5 in [33]).

LEMMA 14.6. (a) $\{C(t)\}_{t \geq 0}$ *is a viscosity subsolution of* (E) *if and only if for all smooth* Ψ,

$$F_*(D\Psi(x, t), D^2\Psi(x, t), \frac{\partial}{\partial t} \Psi(x, t)) \leq 0,$$

whenever $[d_{C^*} \wedge 0] - \Psi$ *attains its maximum at* (x, t).

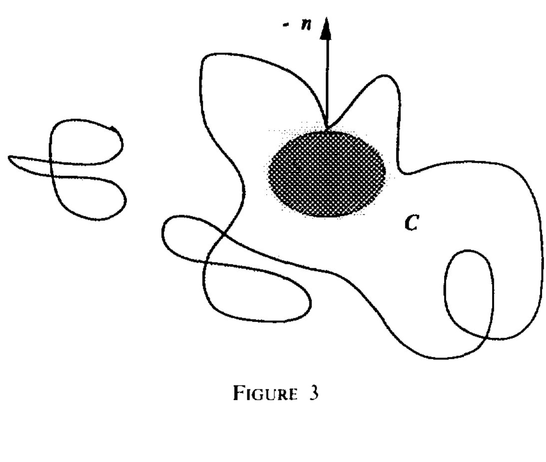

FIGURE 3

(b) $\{C(t)\}_{t\geq 0}$ *is a viscosity supersolution of* (E) *if and only if for all smooth* Ψ,

$$F^*(D\Psi(x,t), D^2\Psi(x,t), \frac{\partial}{\partial t}\Psi(x,t)) \geq 0,$$

whenever $[d_{C_*} \wedge 0] - \Psi$ *attains its minimum at* (x, t).

C. *Stability*

The following is an analogue of the stability theorem of Barles and Perthame [8] for the first order Hamilton–Jacobi equations. We state it only for subsolutions. Also, an analogue result holds for supersolutions. Let $\{C_n(t)\}_{t\geq 0}$ be a sequence of viscosity subsolutions of (E). Set $C_n(t) = \varnothing$ for $t \geq T(C_n(\cdot))$ and then define

$$\begin{aligned} C(t) &= \limsup_{s\to t\; n\to\infty} C_n(s) \\ &= \bigcap_{\substack{m=1,2,\dots\\ \varepsilon>0}} \left\{ \mathrm{cl}\left[\bigcup_{\substack{n=m,m+1,\dots\\ |t-s|\leq\varepsilon}} C_n(s) \right] \right\}. \end{aligned}$$

LEMMA 14.7. $\{C(t)\}_{t\geq 0}$ *is a viscosity subsolution of* (E).

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REFERENCES

1. U. ABRESCH AND J. LANGER, The normalized curve shortening flow and homothetic solutions, *J. Differential Geom.* **23** (1986), 175–196.
2. S. M. ALLEN AND J. W. CAHN, A macroscopic theory for antiphase boundary motion and its application to antiphase domain coarsing, *Acta Metall.* **27** (1979), 1085–1095.
3. S. ANGENENT, Parabolic equations for curves on surfaces. (I). Curves with p-integrable curvature, *Ann. of Math.* **132**, No. 3 (1990), 451–485.
4. S. ANGENENT, Parabolic equations for curves on surfaces. (II). Intersections, blow-up and generalized solutions, *Ann. of Math.* **133**, No. 1 (1991), 171–217.
5. S. ANGENENT, On the formation of singularities in the curve shortening flow, preprint.
6. S. ANGENENT AND M. E. GURTIN, Multiphase thermomechanics with interfacial structure. 2. Evolution of an isothermal interface, *Arch. Rational Mech. Anal.* **108** (1989), 323–391.
7. G. BARLES, "Remark on a Flame Propagation Model," Rapport INRIA, No. 464, December 1985.

8. G. Barles and B. Perthame, Exit time problems in optimal control and vanishing viscosity method, *SIAM J. Control. Optim.* **26**, No. 5 (1988), 1133–1148.
9. K. A. Brakke, "The Motion of a Surface by Its Mean Curvature," Princeton Univ. Press, Princeton, 1978.
10. Y.-G. Chen, Y. Giga, and S. Gogo, Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations, *J. Differential Geom.* **33** (1991), 749–786; Announcement, *Proc. Japan Acad. Ser. A* **65** (1989), 207–210.
11. M. G. Crandall, L. C. Evans, and P.-L. Lions, Some properties of viscosity solutions of Hamilton–Jacobi equations, *Trans. Amer. Math. Soc.* **282** (1984), 487–502.
12. M. G. Crandall and P.-L. Lions, Viscosity solutions of Hamiltonian–Jacobi equations, *Trans. Amer. Math. Soc.* **277** (1983), 1–42.
13. A. Dinghas, Uber einen geometrischen Satz von Wulff für die Gleichgewichtsform von Kristallen, *Z. Krist.* **105** (1944), 304–314.
14. C. L. Epstein and M. I. Weinstein, A stable manifold theorem for the curve shortening equation, *Comm. Pure Appl. Math.* **40** (1987), 119–139.
15. L. C. Evans and J. Spruck, Motion of level sets by mean curvature, I, *J. Differential Geom.* **33** (1991), 635–681.
16. L. C. Evans and J. Spruck, Motion of level sets by mean curvature, II, *Trans. Amer. Math. Soc.*, in press.
17. L. C. Evans, H. M. Soner, and P. E. Souganidis, The Allen–Cahn equation and generalized motion by mean curvature, *Comm. Pure Appl. Math.*, in press.
18. I. Fonseca, The Wulff's Theorem revisited, *Proc. Royal Soc. London A* **432** (1991), 125–145.
19. M. Gage, An isotropic inequality with applications to curve shortening, *Duke Math. J.* **50** (1983), 1225–1229.
20. M. Gage, Curve shorthening makes convex curves circular, *Invent. Math.* **76** (1984), 357–368.
21. M. Gage and R. S. Hamilton, The heat equation shrinking convex plane curves, *J. Differential Geom.* **23** (1986), 69–96.
22. Y. Giga and S. Goto, Motion of hypersurfaces and geometric equations, preprint.
23. M. Grayson, The heat equation shrinks embedded plane curves to round points, *J. Differential Geom.* **26** (1987), 285–314.
24. M. Grayson, Shorthening embedded curves, *Ann. of Math.* **129** (1989), 71–111.
25. M. Grayson, A short note on the evolution of surfaces via mean curvature, *Duke Math. J.* **58** (1989), 555–558.
26. M. E. Gurtin, Multiphase thermomechanics with interfacial structure. 1. Heat conduction and the capillary balance law, *Arch. Rational Mech. Anal.* **104** (1988), 195–222.
27. M. E. Gurtin, Multiphase thermomechanics with interfacial structure: Towards a nonequilibrium thermomechanics of two phase materials, *Arch. Rational Mech. Anal.* **104** (1988), 275–312.
28. M. E. Gurtin and H. M. Soner, The one dimensional superthermal Stefan problem, forthcoming.
29. M. E. Gurtin and H. M. Soner, Some remarks on the Stefan problem with surface structure, *Quart. Appl. Math.* **L/2** (1992), 291–303.
30. G. Huisken, Flow by mean curvature of convex surfaces into spheres, *J. Differential Geom.* **20** (1984), 237–266.
31. H. Ishii, Perron's method for Hamilton–Jacobi equations, *Duke Math. J.* **55** (1987), 369–384.
32. H. Ishii, On uniqueness and existence of viscosity solutions of fully nonlinear second order elliptic PDE's, *Comm. Pure Appl. Math.* **42** (1989), 15–45.
33. R. Jensen, The maximum principle for viscosity solutions of second order fully nonlinear partial differential equations, *Arch. Rational Mech. Anal.* **101** (1988), 1–27.

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34. R. JENSEN, P.-L. LIONS, AND P. E. SOUGANIDIS, A uniqueness result for viscosity solutions of second order fully nonlinear partial differential equations, *Proc. Amer. Math. Soc.* **102** (1988), 975–978.
35. P.-L. LIONS, Optimal control of diffusion processes and Hamilton–Jacobi–Bellman equations, I, *Comm. Partial Differential Equations* **8** (1983), 1101–1134.
36. P.-L. LIONS, Optimal control of diffusion processes and Hamilton–Jacobi–Bellman equations. Part II. Viscosity solutions and uniqueness, *Comm. Partial Differential Equations* **8** (1983), 1229–1276.
37. W. W. MULLINS, Grain boundary grooving by volume diffusion, *Trans. Metall. Soc. AIME* **218** (1960), 354–361.
38. S. OSHER AND J. A. SETHIAN, Fronts propagating with curvature dependent speed: Algorithms based on Hamilton–Jacobi formulations, *J. Comput. Phys.* **79** (1988), 12–49.
39. R. F. SEKERKA, "Morphological Stability, Crystal Growth: An Introduction," North-Holland, Amsterdam, 1973.
40. J. A. SETHIAN, Curvature and evolution of fronts, *Comm. Math. Phys.* **101** (1985), 487–499.
41. J. A. SETHIAN, Recent numerical algorithms for hypersurfaces moving with curvature dependent speed: Hamilton–Jacobi equations and conservation laws, *J. Differential Geom.* **31**, No. 1 (1990), 131–162.
42. J. TAYLOR, Existence and structure of solutions to a class of nonelliptic variational problems, *Sympos. Math.* **14** (1974), 499–508.
43. J. TAYLOR, Unique structure of solutions to a class of nonelliptic variational problems, *Proc. Sympos. Pure Math.* **27** (1975), 419–427.
44. G. WULFF, Frage der Geschwindigkeit des Wachstums und der Auflösung der Krystallflachen, *Z. Krystall. Min.* **34** (1901), 449–530.