## SINGULAR PERTURBATIONS IN MANUFACTURING\*

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Abstract. An asymptotic analysis for a large class of stochastic optimization problems arising in manufacturing is presented. A typical example of the problems considered in this paper is a production planning problem with random capacity and demand. In this example, it is assumed that the capacity of the system fluctuates faster than the other quantities. The general model considered here also has a fast controlled Markov process in its state description. By using the difference in the time scales of different quantities, the problem is simplified by "averaging" out the fast process. Then asymptotically optimal strategies are constructed from the optimal solutions of the limiting problems. The proofs of these results use the theory of viscosity solutions to dynamic programming equations. However, the formal construction of the asymptotically optimal strategies does not require knowledge of this theory.

Key words. dynamic programming, viscosity solutions, production planning, manufacturing, singular perturbations

## AMS(MOS) subject classifications. 90B30, 93E20, 35R35

1. Introduction. Most modern manufacturing systems are complex and large in scale, including several subsystems, a wide variety of equipment, and a number of different products. Moreover, operating policies of these systems must respond to discrete events that are quite different from one another, for example, machine setups, failure and repairs, demand changes, purchasing and building new facilities, etc. Because of the size of the systems, it is impossible to achieve optimal operating policies. The only practical strategies are the suboptimal ones, derived using the structure of a given system. Generally, these techniques amount to reduction of the complexity by decomposing the original system into simpler subsystems. We limit ourselves to systems that have hierarchical decomposition. Based on this structure we "average out" certain parameters, thus simplifying the optimization problems. Then suboptimal policies are obtained as solutions to these simplified problems. For further information on control of manufacturing systems, we refer the reader to Gershwin et al. [9]; on hierarchical production planning, see Gershwin [8] and Bitran and Tirupati [3].

Recently Lehoczky et al. [11] carried out the above procedure for a specific stochastic production planning problem. However, the scope of the mathematical tools used in [11] is not limited to the production planning problem. In this paper, we demonstrate the versatility of these techniques by introducing a general framework for asymptotic analysis of optimal stochastic control problems. This framework, in particular, includes the problem studied in [11] and its generalizations.

Typical of the problems we consider is a production planning problem subject to random changes in capacity and demand. We consider the case in which the capacity fluctuates faster than the other quantities, when the system is working. In other words, when there is production, the rate at which the capacity changes occur is much larger than the rate of fluctuation in demand, the rate of discounting, and other time scales. In this model the capacity process depends on the production rate. The model without this dependency is analyzed in [11] and a limiting problem is obtained by simply replacing the random capacity by its average value. However, for the general model,

<sup>\*</sup> Received by the editors December 5, 1990; accepted for publication (in revised form) September 23, 1991. This research was supported in part by National Science Foundation grant DMS-9002249 and Army Research Office grant DAAL-03-86-K-0171 and the Center for Nonlinear Analysis.

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a straightforward averaging, as it was done in [11], is no longer valid. In fact, the "average" capacity is a function of the production rate and its computation is quite complicated. This dependency also implies that in general the diffusion approximation is not possible. Therefore, we are not able to use the elegant analysis of Kushner [10].

The mathematical analysis of this paper uses the dynamic programming principle and the viscosity solutions of the differential equations. Although our proofs are complicated at times, on the formal level the methodology is straightforward and we wish to emphasize this. An outline of the formal method is as follows: First derive the dynamic programming (Bellman) equation for the full problem. Then let the fluctuation rate of the faster process go to infinity in the equation. Obtain the formal limiting equation by assuming the regularity of the value function. Compute the optimal control problem related to the limiting equation and its optimal solution. This solution in turn generates an asymptotically optimal control for the original model. The asymptotic optimality of this control was recently proved by Zhang and Sethi [15] for the model considered in [11]. Finally, we note that our techniques are related to those in Bensoussan [2] and Saksena, O'Reilly, and Kokotovic [13].

The paper is organized as follows. The stochastic production planning problem is described in § 2. Using this problem as a model, we introduce the general framework in § 3. Section 4 is devoted to the proof of the convergence result. A suboptimal but asymptotically optimal control is constructed in § 5. Finally, a discussion of the convergence rate is given.

2. Production planning. Consider a manufacturing facility in which there are m identical machines that are equally capable of producing n distinct part types. The production must be scheduled to meet a demand that fluctuates randomly. However, we assume that the machines are subject to a Markovian breakdown and repair process. Thus the demand may not be met every time, and the production strategies should take this into account.

Akella and Kumar [1] studied the one-dimensional model (n = m = 1) with a constant demand rate. They explicitly computed the optimal production rate, which is a bang-bang control. They showed that there is a threshold level  $\alpha^* \ge 0$  such that, when the only available machine is in working condition, the production rate is either zero or equal to the full capacity if the inventory is strictly greater than  $\alpha^*$  or less than  $\alpha^*$ . Of course, when the machine is down, the only possible production rate is zero.

The general model we are considering also admits an optimal control, which is bang-bang. However, for large n and m the computation of the threshold levels is complicated. Also the description of the production rate includes not only the threshold levels but the fractions of the capacity devoted to each part. We simplify this model by using its hierarchical structure. As discussed in the Introduction, we assume that the occurrence of machine breakdown and repair process is faster than the other time scales that are relevant to this problem.

We continue with the description of the model. Let an *n*-vector x(t) denote the inventory at time  $t \ge 0$ . For a given production rate (*control*) u(t), the inventory (*state*) satisfies the ordinary differential equation

(2.1) 
$$\frac{d}{dt}x(t) = u(t) - d(t), \quad t > 0,$$

where  $d(t) = (d_1(t), \dots, d_n(t))$  is the *demand* vector. The demand process is assumed to be Markov, taking values in a discrete set  $D \subset (0, \infty)^n$ . The components of the production rate are nonnegative and they are bounded from above by a constant related

to  $\alpha^{\varepsilon}(t)$ , the number of available machines at time t. We assume that  $\alpha^{\varepsilon}(t) \in \{0, 1, \dots, m\}$  is a Markov chain with infinitesimal generator

(2.2) 
$$\frac{1}{\varepsilon} Q^{u(t)} = \left(\frac{1}{\varepsilon} q_{ij}(u(t))\right)_{i,j=0,1,\cdots,m}$$

Note that the generator of  $\alpha^e$  depends on the production rate u(t). Since the machine failures are more likely when the production rate is high, this dependence is a natural one. However, in certain situations one may argue that it is negligible as it was assumed in [11].

The parameter  $\varepsilon > 0$  appearing in the machine availability process is related to the hierarchy in the time scales. Indeed, the mean rate of change of  $\alpha^{\varepsilon}(\cdot)$  is of order  $1/\varepsilon$ , while the rate of change of demand is bounded in  $\varepsilon$ . Hence, for small  $\varepsilon > 0$ , these two time scales are of different order.

The optimization problem is to minimize

(2.3) 
$$J^{\varepsilon}(x, d, i; u) = E_{x, d, i} \int_{0}^{\infty} e^{-t} G(x(t), u(t)) dt$$

over all nonanticipative production processes, u(t), satisfying the machine availability constraint

(2.4) 
$$u(t) \in K(\alpha^{\varepsilon}(t)) \quad \forall t \ge 0,$$

where  $E_{x,d,i}$  denotes the mathematical expectation with initial conditions x(0) = x, d(0) = d, and  $\alpha^{\varepsilon}(0) = i$ . The constraint set is given by

$$K(i) = \left\{ u \in [0,\infty)^n \colon \sum_{k=1}^n u_k \gamma_k \leq i \right\}, \qquad i = 1, \cdots, m,$$

with nonnegative constants  $\gamma_k$ .

Let  $v^{\varepsilon}(x, d, i)$  be the value function

$$v^{\varepsilon}(x, d, i) = \inf_{u(\cdot)} J^{\varepsilon}(x, d, i; u), \qquad x \in \mathbb{R}^n, \quad d \in D, \quad i \in \{0, 1, \cdots, m\}.$$

Then  $v^{\varepsilon}$  is a (viscosity) solution of

$$0 = v^{\varepsilon}(x, d, i) + \sup_{u \in K(i)} \left\{ -(u-d) \cdot D_{x}v^{\varepsilon}(x, d, i) - G(x, u) - \frac{1}{\varepsilon} \sum_{j=0}^{m} q_{ij}(u) [v^{\varepsilon}(x, d, j) - v^{\varepsilon}(x, d, i)] - \sum_{d' \in D} \bar{q}_{dd} [v^{\varepsilon}(x, d', i) - v^{\varepsilon}(x, d, i)] \right\}$$

for all  $x \in \mathbb{R}^n$ ,  $d \in D$ ,  $i \in \{0, 1, \dots, m\}$ , where  $\overline{Q} = (\overline{q}_{dd'})_{d,d' \in D}$  is the infinitesimal generator of  $d(\cdot)$ , and  $D_x$  denotes the gradient in the x-variable.

We close this section by rewriting (2.5) in a manner which is compatible with the notation of the next section. For  $(x, d, i) \in \mathbb{R}^n \times D \times \{0, 1, \dots, m\}$ , and  $p \in \mathbb{R}^n$ ,  $L \in \mathbb{R}^{|D|}$ ,  $\kappa \in \mathbb{R}^{m+1}$ , define  $H(x, d, i; p, L, \kappa)$  by

(2.6) 
$$H(x, d, i; p, L, \kappa) = \sup_{v \in K(i)} \left\{ -(u-d) \cdot p - G(x, u) - \sum_{j=0}^{m} q_{ij}(u) [\kappa_j - \kappa_i] \right\} - \sum_{d' \in D} \bar{q}_{dd'} [L_{d'} - L_d].$$

Then (2.5) can be rewritten as

$$v^{\varepsilon}(x, d, i) + H\left(x, d, i; D_{x}v^{\varepsilon}(x, d, i), v^{\varepsilon}(x, \cdot, i), \frac{1}{\varepsilon}v^{\varepsilon}(x, d, \cdot)\right) = 0.$$

Finally, we note that the sum of the entries of each row of any infinitesimal generator is zero, i.e.,

$$\sum_{j=0}^{m} q_{ij}(u) = \sum_{d'\in D} \overline{q}_{dd'} = 0, \qquad \forall i \in \{0, \cdots, m\}, \quad d \in D.$$

This implies that

(2.7) 
$$H(x, d, i; p, L+c_1, \kappa+c_2) = H(x, d, i; p, L, \kappa)$$

for any constants  $c_1, c_2 \in (-\infty, \infty)$ , and  $L+c_1$  denotes the vector obtained by adding the constant  $c_1$  to each component of L;  $\kappa + c_2$  is defined similarly.

3. General model. We consider a family of discounted, infinite horizon stochastic optimal control problems indexed by a parameter  $\varepsilon > 0$ , with a state space  $\Sigma = R^n \times D \times Z$ . We take both D and Z to be finite sets. For  $(x, d, i) \in \Sigma$ , let  $v^{\varepsilon}(x, d, i)$  be the value function satisfying the dynamic programming equation

(3.1) 
$$v^{\varepsilon}(x,d,i) + H\left(x,d,i; D_{x}v^{\varepsilon}(x,d,i), v^{\varepsilon}(x,\cdot,i), \frac{1}{\varepsilon}v^{\varepsilon}(x,d,\cdot)\right) = 0 \quad \forall (x,d,i) \in \Sigma,$$

where H is a real valued function of  $\Sigma \times R^n \times R^{|D|} \times R^{|Z|}$ . We will not describe the underlying stochastic model. But the function H is given in terms of the running cost and the dynamics of the state process. In particular H is jointly convex in the last three variables and has the invariance property (2.7). We now make a structural assumption. Fix  $(x, d) \in R^n \times D$ ,  $p \in R^n$ ,  $L \in R^{|D|}$ , and  $\alpha \in R^{|Z|}$ . Consider the nonlinear equation

(3.2) 
$$\alpha_i + H(x, d, i; p, L, \kappa) = 0 \quad \forall i \in \mathbb{Z},$$

where  $\kappa \in \mathbb{R}^{|\mathcal{Z}|}$  is the unknown. Due to the translation invariance (2.7), if  $\kappa$  is a solution of (3.2) then  $\kappa + c$  is a solution for any constant c. So we should search for a unique solution in the quotient space which we call

(3.3) 
$$\mathscr{P}_{|Z|} = \left\{ \kappa \in \mathbb{R}^{|Z|} : \sum_{i \in Z} \kappa_i = 0 \right\}.$$

The translation invariance also yields that the range of the map  $\kappa \mapsto \{H(x, d, i; p, L, \kappa\}_{i \in \mathbb{Z}} \text{ is not equal to } R^{|\mathbb{Z}|}$ . Hence we may only expect (3.2) to have a unique solution  $\kappa \in P_{|\mathbb{Z}|}$  provided that the components of  $\alpha$  satisfy a (possibly nonlinear) scalar equation. More precisely we assume that there are functions

$$(3.4i) H_{av}: R^n \times D \times R^n \times R^{|D|} \times R^{|Z|} \to R,$$

and

such that for all  $(x, d, i) \in \Sigma$ ,  $p \in \mathbb{R}^n$ ,  $L \in \mathbb{R}^{|D|}$ , and  $\alpha \in \mathbb{R}^{|Z|}$ , we have  $A(x, d, \cdot; p, L; \alpha) \in \mathcal{P}_{|Z|}$ , and

(3.5) 
$$\alpha_i + H(x, d, i; p, L, A(x, d, \cdot; p, L; \alpha)) = 0,$$

provided that  $\alpha = \{\alpha_i\}_{i \in \mathbb{Z}}$  satisfies

(3.6) 
$$H_{av}(x, d; p, L; \alpha) = 0.$$

Clearly, the function  $H_{av}$  is not uniquely determined. However, under mild assumptions on the coefficients of the optimization problem, we can show that it is continuous and monotone in  $\alpha$ . Then by multiplying it by (-1) if necessary, we may take it to be nondecreasing in  $\alpha$ . So the following assumption is not restrictive:

(3.7) 
$$A, H_{av}$$
 are continuous and  $H_{av}$  is nondecreasing in  $\alpha$ .

Note that (3.1) is similar to (3.2). However, in (3.2) variables p and L are assumed to be independent of i, and in (3.1)  $p = D_x v^{\varepsilon}(x, d, i)$  and  $L_{d'} = v^{\varepsilon}(x, d', i)$ . However, we expect the dependence of  $v^{\varepsilon}$  on i to be averaged out in the limit  $\varepsilon \to 0$ . So suppose that  $v^{\varepsilon}(x, d, i)$  converges to v(x, d), and

$$\kappa^{\varepsilon}(x, d, j) = \frac{1}{\varepsilon} \left[ v^{\varepsilon}(x, d, j) - \sum_{k \in \mathbb{Z}} v^{\varepsilon}(x, d, k) \right]$$

converges to  $\kappa(x, d, j)$ . Due to the invariance (2.7), we may rewrite (3.1) as

$$v^{\varepsilon}(x, d, i) + H(x, d, i; D_x v^{\varepsilon}(x, d, i), v^{\varepsilon}(x, \cdot, i), \kappa^{\varepsilon}(x, d, \cdot)) = 0.$$

Now let  $\varepsilon$  go to zero. Formally, we obtain

$$v(x, d) + H(x, d, i; D_x v(x, d), v(x, \cdot), \kappa(x, d, \cdot)) = 0 \quad \forall i \in \mathbb{Z}$$

Note that the above equation is a special case of (3.2) with  $p = D_x v(x, d)$  and  $L_{d'} = v(x, d')$ . Hence (3.6) yields

$$\bar{H}_{av}(x, d; D_x v(x, d), v(x, \cdot); v(x, d)) = 0,$$

where for  $x \in \mathbb{R}^n$ ,  $d \in D$ ,  $p \in \mathbb{R}^n$ ,  $L \in \mathbb{R}^{|D|}$ , and a scalar v,

$$H_{\rm av}(x, d; p, L; v) = H_{\rm av}(xd; p, L; \vec{v})$$

with  $\vec{v} = (v, \cdots, v) \in \mathbb{R}^{|Z|}$ .

In the next section, we will show that  $v^e$  converges to a solution of the above equation. Since  $\bar{H}_{av}$  is convex in the last three variables,  $\bar{H}_{av} = 0$  is the dynamic programming equation of an optimal control problem with state space  $R^n \times D$ . Therefore v is the value function of this problem. The connection between the equation  $\bar{H}_{av} = 0$  and the optimal control problem will be clarified in Examples 3.1-3.3, below.

Our final assumption is a strong monotonicity condition on  $H_{av}$ . For each  $i \in \mathbb{Z}$ ,  $\kappa^i \in \mathbb{R}^{|\mathbb{Z}|}$ , set

$$\alpha_i = -H(x, d, i; p, L, \kappa^i), \qquad i \in \mathbb{Z}.$$

Since  $\kappa^i$  may depend on *i*, we can *not* conclude that (3.6) holds. However, we assume that

(3.8i) 
$$H_{av}(x, d, i; p, L; \alpha) \leq 0$$
 (or  $\geq 0$ , respectively)

whenever there is  $\kappa \in \mathbb{R}^{|Z|}$  such that for all  $i, j \in Z$ ,

(3.8ii) 
$$\kappa_i^i \leq \kappa_i^i + [\kappa_i - \kappa_i]$$
 (or  $\geq$ , respectively).

We now give two examples to clarify the above hypothesis.

*Example* 3.1. Consider (2.5) with n = m = 1,  $D = \{d_0\}$ , and  $q_{01}(u) = -q_{00} = \lambda > 0$ ,  $q_{10}(u) = -q_{11}(u) = \mu(u) \ge 0$ . Then  $K(0) = \{0\}$ ,  $K(1) = [0, 1/\gamma_1]$  and the Hamiltonian H

in (2.6) has the form

$$H(x, 0; p, \kappa) = d_0 p - G(x, 0) - \lambda[\kappa_1 - \kappa_0],$$
  
$$H(x, 1; p, \kappa) = \sup_{0 \le u \le 1/\gamma_1} \{-up - G(x, u) + \mu(u)[\kappa_1 - \kappa_0]\} + d_0 p.$$

Equation (3.2) is equivalent to

(3.9i) 
$$\alpha_0 + d_0 p - G(x, 0) - \lambda[\kappa_1 - \kappa_0] = 0,$$

(3.9ii) 
$$\alpha_1 + \sup_{0 \le u \le 1/\gamma_1} \{ -(u - d_0)p - G(x, u) + \mu(u)[\kappa_1 - \kappa_0] \} = 0.$$

Suppose that for a given  $(\alpha_0, \alpha_1)$  we have a solution  $(\kappa_0, \kappa_1) \in \mathscr{P}_2$  solves (3.9). Then (3.9i) yields

(3.10) 
$$\kappa_1 - \kappa_0 = \frac{1}{\lambda} \left[ \alpha_0 + d_0 p - G(x, 0) \right].$$

Since  $(\kappa_0, \kappa_1) \in \mathcal{P}_2$ ,  $\kappa_0 + \kappa_1 = 0$ . Therefore,

$$\kappa_1 = A(x, 1; p, \alpha) = \frac{1}{2\lambda} [\alpha_0 + d_0 p - G(x, 0)],$$
  

$$\kappa_0 = A(x, 0; p, \alpha) = -\kappa_1.$$

Observe that we used only (3.9i) to obtain the above formula. The other equation, (3.9ii), will be used to compute  $H_{av}$ . Indeed, using (3.10) in (3.9ii), we arrive at

$$H_{\rm av}(x; p; \alpha) = 0,$$

where

$$H_{\rm av}(x; p; \alpha_0, \alpha_1) = \alpha_1 + \sup_{0 \le u \le 1/\gamma_1} \left\{ -(u - d_0)p - G(x, u) + \frac{\mu(u)}{2\lambda} [\alpha_0 + d_0p - G(x, 0)] \right\}.$$

To verify (3.8), suppose that (3.9i) holds with  $\kappa^0 = (\kappa_0^0, \kappa_1^0)$  and (3.9ii) holds with  $\kappa^1 = (\kappa_0^1, \kappa_1^1)$ . Then (3.10) holds with  $\kappa_1^0 - \kappa_0^0$  on the left-hand side. Also suppose that (3.8ii) holds. Then

$$\kappa_1^0 - \kappa_0^0 \leq \kappa_1^1 - \kappa_0^1.$$

Using the above inequality and (3.10) in (3.9ii), we obtain  $H_{av}(x, p; \alpha_0, \alpha_1) \leq 0$ . Hence (3.8i) holds.

In this example, the optimal control problem related to the limiting equation  $\bar{H}_{av} = 0$  is to minimize

$$\int_0^\infty \exp\left(-t - \int_0^t \frac{\mu(u(s))}{2\lambda} \, ds\right) \left[ G(x(t), u(t)) + \frac{\mu(u(t))}{2\lambda} \, G(x(t), 0) \right] \, dt$$

subject to

$$\frac{d}{dt}x(t) = u(t) - \left(1 + \frac{\mu(u(t))}{2\lambda}\right)d_0, \qquad t > 0$$

and  $u(t) \in [0, 1/\gamma_1], t \ge 0$ .

Example 3.2. Again consider (2.5) with n = 1, m = 2,  $D = \{d_0\}$ , and

$$Q(u) = \begin{bmatrix} -\lambda_1 & \lambda_1 & 0\\ \mu_1(u) & -[\mu_1(u) + \lambda_2] & \lambda_2\\ 0 & \mu_2(u) & -\mu_2(u) \end{bmatrix}.$$

Then  $K(0) = \{0\}$ ,  $K(1) = [0, 1/\gamma_1]$ ,  $K(2) = [0, 2/\gamma_1]$ . Suppose that for a given  $x, p, \alpha \in \mathbb{R}^3$ ,  $K \in \mathcal{P}_3$  solves (8.2). Then a computation similar to the previous case yields

$$\kappa_{1} - \kappa_{0} = \frac{1}{\lambda_{1}} [\alpha_{0} + d_{0}p - G(x, 0)],$$
  

$$\kappa_{2} - \kappa_{1} = \frac{1}{\lambda_{2}} \left[ \alpha_{1} + \sup_{0 \le u_{1} \le 1/\gamma_{1}} \{-(u_{1} - d_{0})p - G(x, u_{1}) + \mu_{1}(u_{1})[\kappa_{1} - \kappa_{0}]\} \right],$$
  

$$\alpha_{2} + \sup_{0 \le u_{2} \le 2/\gamma_{1}} \{-(u_{2} - d_{0})p - G(x, u_{2}) + \mu_{2}(u_{2})[\kappa_{2} - \kappa_{1}]\} = 0.$$

Hence

$$H_{av}(x; p; \alpha) = \sup_{\substack{0 \le u_1 \le 1/\gamma_1 \\ 0 \le u_2 \le 2/\gamma_1}} \left\{ -f(u_1, u_2)p - g(x, u_1, u_2) + \alpha_2 + \frac{\mu_2(u_2)}{\lambda_2} \left[ \alpha_1 + \alpha_0 \frac{\mu_1(u_1)}{\lambda_1} \right] \right\},$$

where

$$f(u_1, u_2) = (u_2 - d_0) + \frac{\mu_2(u_1)}{\lambda_2} \left[ (u_1 - d_0) - \frac{\mu_1(u_1)}{\lambda_1} d_0 \right],$$
$$g(x, u_1, u_2) = G(x, u_2) + \frac{\mu_2(u_2)}{\lambda_2} \left[ G(x, u_1) + \frac{\mu_1(u_1)}{\lambda_1} G(x, 0) \right].$$

The corresponding control problem is similar to that described in the Example 3.1.

*Example* 3.3. Again consider (2.5) with n = 1,  $D = \{d_0\}$  and Q(u) = Q is an irreducible  $(m+1) \times (m+1)$  stochastic matrix. Then, (3.2) has the form

(3.11) 
$$\alpha_i = -\sup_{0 \le u \le i/\gamma_i} \{-(u-d_0) \cdot p - G(x, u)\} + (Q\kappa)_i, \quad i \in \{0, 1, \cdots, m\}.$$

Since Q is irreducible, there is a positive vector  $\nu \in \mathbb{R}^{m+1}$  such that  $\nu_i > 0$ ,  $\sum_i \nu_i = 1$ , and  $(\nu Q)_i = 0$  for all *i*. Multiply the above equation by  $\nu_i$  and sum over *i* to obtain

$$\sum_{i=0}^{m} \left[ \alpha_{i} \nu_{i} + \sup_{0 \leq u_{i} \leq i/\gamma_{1}} \left\{ -\nu_{i} (u_{i} - d_{0}) p - \nu_{i} G(x, u_{i}) \right\} \right] = 0.$$

A straightforward algebraic manipulation gives

$$H_{\rm av}(x, p; \alpha) = \sum_{i=0}^{m} \alpha_i \nu_i + \sup_{0 \le u \le \bar{v}} \{-(u-d_0)p - \bar{G}(x, u)\},\$$

where

$$\bar{v} = \sum_{i=0}^{m} i\nu_i / \gamma_i,$$
$$\bar{G}(x, u) = \inf\left\{\sum_{i=0}^{m} \nu_i G(x, u_i) \colon u_j \in K(j) \text{ and } \sum_{i=0}^{m} \nu_i u_i = u\right\}.$$

To verify (3.8), suppose that (3.11) holds with  $\kappa^i \in \mathbb{R}^{m+1}$ , and  $\kappa^i$ 's satisfy (3.8ii). Multiply (3.11) by  $\nu_i$ , sum over *i*, and then use the formula for  $H_{av}$  to obtain

$$H_{\mathrm{av}}(x, p; \alpha) = \sum_{i,j=0}^{m} \nu_i q_{ij} \kappa_j^i.$$

Now use (3.8ii) and the nonnegativity of  $q_{ij}$  for  $i \neq j$ , to obtain

$$\sum_{i,j=0}^{\sum} \nu_i q_{ij} \kappa_j^i \leq \sum_{i,j=0}^{m} \nu_i q_{ij} [\kappa_j^j + (\kappa_j - \kappa_i)]$$
$$= \sum_{j=0}^{m} (\kappa_j^j + \kappa_j) \left[ \sum_{i=0}^{m} \nu_i q_{ij} \right] - \sum_{i=0}^{m} (\nu_i \kappa_i) \left[ \sum_{j=0}^{m} q_{ij} \right]$$
$$= 0.$$

The corresponding optimal control problem is simple, and it is described in Example 5.1.

These examples can easily be generalized to obtain the following lemma.

LEMMA 3.1. Suppose that H is as in (2.4) and G(x, u) is convex in the u-variable, and either  $\alpha^{\varepsilon}(t)$  is a birth-death process, i.e.,

(3.12) 
$$q_{ij}(u) = \begin{cases} \mu_i(u), & j = i - 1, \quad i = 1, \cdots, m, \\ \lambda_i, & i = j - 1, \quad j = 1, \cdots, m, \\ -[\lambda_i + \mu_i(u)], & if \ i = j = 0, \cdots, m, \\ 0, & otherwise, \end{cases}$$

with  $\mu_i(u) \ge 0$ ,  $\lambda_i > 0$ , or Q(u) = Q for all u and Q is irreducible. Then the assumptions (3.5), (3.6), and (3.8) are satisfied.

The convergence results under the second hypothesis is first obtained in [11]. These results are then improved in [15]. The asymptotic analysis of  $v^{\epsilon}$  under the first set of assumptions, however, is not covered in the previous studies. In this case, the parameter  $\lambda_i$  is the machine repair rate when i-1 machine are operating. It is natural to assume that  $\lambda_i$  is independent of the production rate. The quantity  $\mu_i(u)$  is the machine failure rate when i machines are operating with a production rate of u, and, in general,  $\mu_i$  is a function of the production rate.

4. Convergence. In this section we study the limiting behavior of  $v^{\varepsilon}$  as  $\varepsilon$  tends to zero. In whatever follows we always assume the structural assumptions (3.5)-(3.8). However, to obtain convergence results we need to impose some uniform estimates on  $v^{\varepsilon}$ . In this section we assume that there are K,  $\nu \ge 0$ , independent of  $\varepsilon$ , such that for all  $\varepsilon \in (0, 1]$ ,  $(x, d, i) \in \Sigma$ ,

(4.1i) 
$$|v^{\varepsilon}(x, d, i)| \leq K(1+|x|^{\nu}),$$

(4.1ii) 
$$\frac{1}{|x-y|} |v^{\varepsilon}(x, d, i) - v^{\varepsilon}(y, d, i)| \leq K(1+|x|^{\nu}), \quad 0 < |y-x| \leq 1,$$

(4.1iii) 
$$|v^{\varepsilon}(x, d, i) - v^{\varepsilon}(x, d, j)| \leq \varepsilon K (1+|x|^{\nu}), \quad j \in \mathbb{Z}.$$

The inequality (4.1ii) is a uniform Lipschitz estimate. If the function  $v^{\varepsilon}$  is continuously differentiable in the x-variable, then (4.1ii) is equivalent to the uniform boundedness of the gradient, i.e.,

(4.2) 
$$\sup_{\varepsilon \in (0,1]} |D_x v^{\varepsilon}(x, d, i)| \leq K (1+|x|^{\nu}).$$

The estimate (4.1iii) is related to the scaling used in the equation (3.1). Notice that in (3.1) the vector  $(1/\varepsilon)v^{\varepsilon}(x, d, \cdot)$  appears. So intuitively we expect the differences  $|v^{\varepsilon}(x, d, i) - v^{\varepsilon}(x, d, j)|/\varepsilon$  to be locally bounded as assumed in (4.1iii). Note that the translation invariance (2.7) is the reason why we do not expect  $v^{\varepsilon}/\varepsilon$  to be bounded.

In the production planning examples, these estimates are always satisfied. Indeed consider the cases discussed in Lemma 3.1 and assume that

(4.3) 
$$|G(x, u)| + \frac{1}{|x-y|} |G(x, u) - G(y, u)| \le K(1+|x|^{\nu})$$

for all x, y,  $u \in \mathbb{R}^n$ ,  $0 < |x - y| \le 1$ . Then for the second case of Lemma 3.1, the estimates (4.1) are proved in [11]; see Lemma 2.1 in [11]. A very similar proof yields these estimates also in the first case of Lemma 3.1.

Using (4.1) and the Ascoli-Arzela theorem we construct a sequence, denoted by  $\varepsilon$  again, and locally Lipschitz continuous function v(x, d) such that  $v^{\varepsilon}(x, d, i)$  converges to v(x, d) uniformly on compact subsets on  $\Sigma$ . As we discussed in § 3, formally v solves the limiting equation

(4.4) 
$$\bar{H}_{av}(x, d; D_x v(x, d), v(x, \cdot); v(x, d)) = 0, \quad (x, d) \in \mathbb{R}^n \times D.$$

Recall that for  $(x, d; p, L) \in \mathbb{R}^n \times D \times \mathbb{R}^n \times \mathbb{R}^{|D|}$  and a scalar v,

$$H_{\mathrm{av}}(x, d; p, L; v) \coloneqq H_{\mathrm{av}}(x, d; p, L; \vec{v}),$$

with  $\vec{v} = (v, v, \dots, v) \in \mathbb{R}^{|Z|}$ . We will show below that v indeed is a solution of (4.4).

In general, v is not differentiable and the equation (4.4) must be interpreted in the viscosity sense. We refer the reader to Crandall and Lions [5]; Crandall, Evans, and Lions [4]; Lions [12]; Soner [14]; Fleming, Sethi, and Soner [6]; and [11] for the definition and the properties of the viscosity solutions of (3.1) or (4.4).

THEOREM 4.1 (Stability). Assume (4.1), and that  $v^{\varepsilon}$  is a viscosity solution of (3.1). Suppose that (4.4) has a unique viscosity solution v satisfying (4.1). Then  $v^{\varepsilon}$  converges to v uniformly on compact subsets of  $\mathbb{R}^n \times D$ , as  $\varepsilon$  tends to zero.

**Proof.** Let  $\bar{v}(x, d)$  be the limit of  $v^{\varepsilon_m}(x, d, i)$  for some sequence  $\varepsilon_m \to 0$ . Let  $\psi(x, d)$  be a continuously differentiable function and for  $d \in D$ , let  $x_0 \in \mathbb{R}^n$  be the strict maximum of  $\bar{v}(\cdot, d) - \psi(\cdot, d)$  on  $\mathbb{R}^n$ . To show that  $\bar{v}$  is a viscosity subsolution of (4.4), we must verify the inequality

(4.5) 
$$\bar{H}_{av}(x_0, d; D_x\psi(x_0, d), \bar{v}(x_0, \cdot); \bar{v}(x_0, d) \leq 0.$$

Consider the map  $x \mapsto v^{\varepsilon_m}(x, d, i) - \psi(x, d)$ . Since  $x_0$  is a strict maximizer, there are  $x_m(i) \in \mathbb{R}^n$  converging to  $x_0$  and maximizing the above map locally in the x variable. Then the viscosity property of  $v^m \coloneqq v^{\varepsilon_m}$  yields

$$v^{m}(x_{m}(i), d, i) + H\left(x_{m}(i), d, i; D_{x}\psi(x_{m}(i), d), v^{m}(x_{m}(i), \cdot, i), \frac{1}{\varepsilon_{m}}v^{m}(x_{m}(i), d, \cdot)\right) \leq 0.$$

Since  $v^m$  converges to  $\bar{v}$  and  $x_m(i)$  converges to  $x_0$ , there is a sequence  $K_m \to 0$  such that

(4.6) 
$$\bar{v}(x_0, d) + H(x_0, d, i; p_0, \bar{v}(x_0, \cdot), \kappa^{m,i}) \leq K_m,$$

where  $p_0 = D_x \psi(x_0, d)$  and

$$\kappa_j^{m,i} = \frac{1}{\varepsilon_m} v^m(x_m(i), d, j), \qquad x \in \mathbb{R}^n, \quad i, j \in \mathbb{Z}.$$

Since  $x_m(i)$  is a local maximizer of  $v^m(\cdot, d, i) - \psi(\cdot, d)$ , we have

$$v^m(x_m(j), d, j) - \psi(x_m(j), d) \ge v^m(x_m(i), d, j) - \psi(x_m(i), d)$$

for all  $i, j \in \mathbb{Z}$ . Set  $\kappa_j = \psi(x_m(j), d)$ . Then

$$\kappa_j^{m,i} \leq \kappa_j^{m,j} + (\kappa_i - \kappa_j)$$

for every  $i, j \in \mathbb{Z}$ . Hence, (3.8) implies that

$$H_{\rm av}(x_0, d; p_0, \bar{v}(x_0, \cdot); \beta) \leq 0,$$

where

$$\beta(i) = -H(x_0, d, i; p_0, \bar{v}(x_0, \cdot), \kappa^{m,i}), \quad i \in \mathbb{Z}.$$

Also (4.6) yields that  $\beta(i) \ge \overline{v}(x_0, d) - K_m$  for every  $i \in \mathbb{Z}$ . Hence, the monotonicity of  $H_{av}$  yields that

 $\bar{H}_{av}(x_0, d; p_0, \bar{v}(x_0, \cdot); \bar{v}(x_0, d) - K_m) \leq H_{av}(x_0, d; p_0, \bar{v}(x_0, \cdot); \beta) \leq 0 \quad \forall m.$ 

Now let *m* go to infinity and use the fact that  $K_m \rightarrow 0$  to obtain (4.5). Hence  $\bar{v}$  is a viscosity subsolution. Similarly we can show that it is also a viscosity supersolution, and therefore a solution. Since (4.4) has a unique viscosity solution *v* satisfying (4.1),  $\bar{v} = v$ .  $\Box$ 

COROLLARY 4.1. Assume the hypothesis of Lemma 3.1, and (4.3). Then  $v^{\varepsilon}$  converges uniformly on compact subsets of  $\Sigma$ , as  $\varepsilon$  tends to zero.

*Proof.* We have argued that (4.3) implies the estimates (4.1). Also the uniqueness of viscosity solutions of (4.4) satisfying (4.1) follows from the classical techniques of Crandall, Evans, Lions [4].

5. Asymptotically optimal controls. In this section we outline a procedure of constructing suboptimal controls by using the limiting equation (4.4). We will show that under certain assumptions, the difference between the value function and the performance of the controls that we construct converges to zero in the limit  $\varepsilon \rightarrow 0$ . Before we describe the procedure for the general case, we discuss two examples.

*Example* 5.1. Consider the case described in Example 3.3. Let v(x) be the unique viscosity solution of the limit equation,

$$0 = \bar{H}_{av}(x, D_x v(x), v(x))$$
  
=  $v(x) + \sup_{0 \le u \le \bar{u}} \{-(u - d_0) D_x v(x) - \bar{G}(x, u)\}.$ 

Then v(x) is the value function of a deterministic optimal control problem. Indeed,

$$v(x) = \inf \int_0^\infty e^{-t} \,\tilde{G}(x(t), u(t)) \,dt,$$

subject to constraints x(0) = x, (2.1) with  $d(t) \equiv d_0$ , and  $0 \le u(t) \le \overline{v}$  for all  $t \ge 0$ . Suppose that v is differentiable. For each x, pick

$$u^*(x) \in \operatorname{argmax} \{-(u-d_0)D_xv(x) - \bar{G}(x, u): 0 \le u \le \bar{v}\}.$$

If

(5.1) 
$$\frac{d}{dt}x(t) = u^*(x(t)) - d_0, \qquad t > 0,$$

has a solution, then it is elementary to show that  $\tilde{u}(t) = u^*(x(t))$  is optimal. So suppose that this is the case. Then we construct a feedback control for the  $\varepsilon > 0$  problem by setting

$$u^{*,\varepsilon}(x, i) = iu^*(x), \qquad x \in (-\infty, \infty), \qquad i = 0, 1, \cdots, m.$$

Then we expect  $u^{*,\varepsilon}$  to perform close to the optimal control. Indeed, Zhang and Sethi [15] have shown that

$$\lim_{\varepsilon\to 0} |J^{\varepsilon}(x, i; u^{*,\varepsilon}) - v^{\varepsilon}(x, i)| = 0,$$

provided that (5.1) has a unique solution.

*Example* 5.2. Now we return to Example 3.2. As in the previous example suppose that the solution v of the limit equation is differentiable, i.e.,

(5.2) 
$$\sup_{\substack{0 \le u_1 \le 1/\gamma_1 \\ 0 \le u_2 \le 2/\gamma_2}} \{l(u_1, u_2)v(x) - f(u_1, u_2)D_xv(x) - g(x, u_1, u_2)\} = 0,$$

where f, g are as in Example 3.2, and

$$l(u_1, u_2) = 1 + \frac{\mu_2(u_2)}{\lambda_2} \left[ 1 + \frac{\mu_1(u_2)}{\lambda_1} \right].$$

Let  $u_1^*(x)$ ,  $u_2^*(x)$  be a maximizer in (5.2). Clearly, the sequence  $u^*(x, 0) = 0$ ,  $u^*(x, 1) = u_1^*(x)$ , and  $u^*(x, 2) = u_2^*(x)$  satisfies the machine availability constraint and is a candidate for an asymptotically optimal control. We will show in Theorem 5.1 below that this is indeed the case, provided that  $u^*$  has certain properties.

To motivate the construction in the general framework, we will derive a property of  $u^*$  next. Set

$$p(x)=D_xv(x),$$

and

$$A(x, i) = A(x, i; p(x), \vec{v}(x)), \quad i = 0, 1, 2;$$

recall that  $\vec{v}(x) = (v(x), v(x), v(x))$ . Then by (3.5), we have

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$$0 = v(x) + H(x, i; p(x), A(x, \cdot))$$
  
=  $v(x) + \sup_{0 \le u \le i/\gamma_1} \{-(u - d_0)p(x) - G(x, u) - (Q(u)A(x, \cdot))(i)\}, \quad i = 0, 1, 2.$ 

Then it is straightforward to show that  $u^*(x, i)$  maximizes the expression in the above equation. We will use this description of  $u^*$  in the discussion of the general problem.

In general, the Hamiltonian H has the form

(5.3) 
$$H(\xi; p, L, \kappa) = \sup_{u \in K(\xi)} \left\{ -\mathscr{L}^{u,\xi}(p, L, \kappa) - G(\xi, u) \right\}$$

for  $\xi \in \Sigma$ ,  $p \in \mathbb{R}^n$ ,  $L \in \mathbb{R}^{|D|}$ ,  $\kappa \in \mathbb{R}^{|Z|}$ , a set  $K(\xi) \subset U$ , a function G of  $\Sigma \times U$ , and a family of *linear* operators  $\mathscr{L}^{u,\xi}(p, L, \kappa)$ , which are invariant under scalar translations of L,  $\kappa$ . In the notation of § 2, for example,  $U = [0, \infty)^n$ ,  $K(\xi) = K(i)$ ,

(5.4) 
$$\mathscr{L}^{u,\xi}(p,L,\kappa) = (u-d) \cdot p + (\bar{Q}L)(d) + (Q(u)\kappa)(i)$$

for  $\xi = (x, d, i) \in \Sigma$ . Assume that v is differentiable and set

$$p(x, d) = D_x v(x, d),$$
  $A(x, d, i) = A(x, d, i; p(x, d), v(x, \cdot); \vec{v}(x, d)).$ 

Then choose  $u^*(\xi) \in K(\xi)$  such that

(5.5) 
$$-\mathcal{L}^{u^{*}(\xi),\xi}(p(x,d),v(x,\cdot),A(x,d,\cdot)) - G(\xi,u^{*}(\xi)) \\ = H(\xi;p(x,d),v(x,\cdot),A(x,d,\cdot)).$$

It is known that not every feedback control yields a well-defined state process. However, we assume that  $u^*$  is indeed related to a well-defined state process. Let  $J^{\varepsilon,*}(\xi)$  be the value of the pay-off functional. Then  $J^{\varepsilon,*}$  formally solves

$$(5.6)^{\varepsilon} \quad J^{\varepsilon,*}(\xi) - \mathscr{L}^{u^{*}(\xi),\xi}\left(D_{x}J^{\varepsilon,*}(\xi), J^{\varepsilon,*}(x,\cdot,i), \frac{1}{\varepsilon}J^{\varepsilon,*}(x,d,\cdot)\right) - G(\xi, u^{*}(\xi)) = 0.$$

The definition of  $u^*$  implies that the formal limit of  $(5.6)^{\varepsilon}$  is (4.4). So we expect  $J^{\varepsilon,*}$  to converge to v; the unique (viscosity) solution of (4.4). However the coefficients of  $(5.6)^{\varepsilon}$  are not necessarily smooth, and the procedure of § 4 may not apply to this case. Still a convergence theorem holds if  $(5.6)^{\varepsilon}$  has a comparison principle, which we define next.

DEFINITION 5.1. We say that  $(5.6)^{\varepsilon}$  has a comparison principle if any viscosity subsolution w of  $(5.6)^{\varepsilon}$  satisfying (4.1i) is less than or equal to any viscosity supersolution  $\tilde{w}$  of  $(5.6)^{\varepsilon}$  satisfying (4.1i).

If, for example,  $\mathscr{L}^{u,\xi}$  is as in (5.4), then  $(5.6)^{\varepsilon}$  has a comparison principle for a large class of  $u^*(\cdot)$ . This class, in particular, includes the Lipschitz continuous functions.

We start our convergence proof with a lemma, which is due to Souganidis.

LEMMA 5.1. Suppose that the unique viscosity solutions v of (4.4) and  $v^{\varepsilon}$  of (3.1) are convex, continuously differentiable in the x-variable, and satisfy (4.1). Then  $D_x v^{\varepsilon}(x, d, i)$  converges to  $D_x v(x, d)$  uniformly on compact subsets of  $\Sigma$ , as  $\varepsilon \to 0$ .

*Proof.* Pick  $\varepsilon_n \to 0$ ,  $x_n \to x$  such that  $p_n = D_x v^{\varepsilon_n}(x_n, d, i)$  converges to p. First, the convexity of  $v^{\varepsilon_n}(\cdot, d, i)$  yields

$$v^{\varepsilon_n}(x_n+y, d, i) - v^{\varepsilon_n}(x_n, d, i) \ge p_n \cdot y, \quad \forall y$$

Let *n* go to infinity to obtain

$$v(x+y, d) - v(x, d) \ge p \cdot y \quad \forall y.$$

Then the differentiability of v implies that  $p = D_x v(x, d)$ .  $\Box$ 

For  $\xi = (x, d, i) \in \Sigma$ , set

$$A^{\varepsilon}(\xi) = A(\xi; D_{x}v^{\varepsilon}(\xi), v^{\varepsilon}(x, \cdot, i); v^{\varepsilon}(x, d, \cdot)).$$

Equation (3.1) and the translation invariance (2.7) yield that

$$A^{\varepsilon}(\xi) = \frac{1}{\varepsilon} \left[ v^{\varepsilon}(\xi) - \sum_{j \in Z} v^{\varepsilon}(x, d, j) \right].$$

Using the definition of A, we rewrite (3.1) as

(5.7) 
$$v^{\varepsilon}(\xi) + H(\xi; D_{x}v^{\varepsilon}(\xi), v^{\varepsilon}(x, \cdot, i), A^{\varepsilon}(x, d, \cdot)) = 0.$$

Set

$$K^{\varepsilon}(\xi) = v^{\varepsilon}(\xi) - \mathscr{L}^{u^{*}(\xi),\xi}(D_{x}v^{\varepsilon}(\xi), v^{\varepsilon}(x,\cdot,i), A^{\varepsilon}(x,d,\cdot)) - G(\xi, u^{*}(\xi)).$$

In view of (5.3) and (5.7),  $K^{\varepsilon}(\xi) \leq 0$ .

LEMMA 5.2. Suppose that  $(5.6)^{e}$  has a comparison principle. Let  $J^{e}$  be the viscosity solution of  $(5.6)^{e}$  satisfying (4.1). Then

$$(5.8) v^{\varepsilon} \leq J^{\varepsilon}.$$

*Proof.* Since  $K^{\varepsilon} \leq 0$ ,  $v^{\varepsilon}$  is a subsolution of  $(5.6)^{\varepsilon}$ . Therefore, the comparison principle yields  $v^{\varepsilon} \leq J^{\varepsilon}$ .  $\Box$ 

LEMMA 5.3. Assume the hypothesis of Theorem 4.1 and Lemma 5.1. Then  $K^{\varepsilon}$  converges to zero, as  $\varepsilon \rightarrow 0$ .

*Proof.* This follows from the continuity of A, (3.7), the convergence of  $v^{\varepsilon}$  and  $D_x v^{\varepsilon}$ , and the definition of  $\mathcal{L}^{u^*}$ .  $\Box$ 

The above result indicates that the difference  $J^{\varepsilon} - v^{\varepsilon}$  should converge to zero. To prove this convergence, we assume that there are  $\bar{K}$ , K,  $\nu$ ,  $\bar{\nu} \ge 0$  such that for all  $\xi \in \Sigma$ ,

(5.9) 
$$|A(\xi; p, L; \alpha)| \leq \bar{K}(1+|x|^{\bar{\nu}}) \quad \forall |p|+|L|+|\alpha| \leq K(1+|x|^{\nu}),$$

and

(5.10) 
$$\sup_{u \in K(\xi)} |\mathcal{L}^{u,\xi}(p,L,\kappa)| \leq \bar{K}(1+|p|+|L|+|\kappa|) \quad \forall p \in \mathbb{R}^n.$$

LEMMA 5.4. Assume (5.9), (5.10). Then for every  $\tilde{K}$ ,  $\tilde{v} \ge 0$  there is a continuously differentiable function  $\eta(x) \ge 1$ , such that

(5.11) 
$$\frac{1}{2}\eta(x) - \mathscr{L}^{u,\xi}(D_x\eta(x),0,0) \ge 0 \quad \forall \xi \in \Sigma, \quad u \in K(\xi),$$

and

(5.12) 
$$\eta(x) \ge \tilde{K}(1+|x|^{\tilde{\nu}}).$$

*Proof.* Let  $\eta(x) = C + \tilde{K}|x|^a$  for some  $C \ge \tilde{K}$  and  $a = \max\{2, \tilde{\nu}\}$ . We will show that, for an appropriate choice of C,  $\eta$  satisfies (5.11) and (5.12). Indeed,

$$D_x\eta(x)=a\tilde{K}x|x|^{a-2}$$

and (5.10) yields

$$\left|\mathscr{L}^{u,\xi}(D_{x}\eta(x),0,0)\right| \leq \bar{K}(1+[a\tilde{K}]|x|^{a-1}).$$

Therefore,

$$\frac{1}{2}\eta(x) - \mathscr{L}^{u,\xi}(D_x\eta(x), 0, 0) \ge (\frac{1}{2}C - \bar{K}) + \tilde{K}|x|^{a-1}(\frac{1}{2}|x| - a\bar{K}).$$

It is now elementary to show that the right-hand side of the above inequality is positive for every x if C is sufficiently large.  $\Box$ 

THEOREM 5.1. Assume the hypothesis of Theorem 4.1, Lemma 5.1, and (5.9), (5.10). Then  $J^{\varepsilon} - v^{\varepsilon}$  converges to zero. In particular,  $u^*$  is asymptotically optimal.

*Proof.* Since  $v^{\varepsilon}$  satisfies (4.1), (5.9) implies that

$$|A^{\varepsilon}(\xi)| \leq \bar{K}(1+|x|^{\bar{\nu}}).$$

In view of (5.10), the above estimate together with (4.1) yields

$$|K^{\varepsilon}(\xi)| \leq \tilde{K}(1+|x|^{\tilde{\nu}-1})$$

for some  $\tilde{K}$ ,  $\tilde{\nu} \ge 1$ . Let  $\eta$  be as in Lemma 5.4. Then (5.12) implies that  $K^{\varepsilon}(\xi)/\eta(x)$  converges to zero uniformly on  $\Sigma$ , as  $\varepsilon \to 0$ . Set

$$k^{\varepsilon} = \inf \left\{ \left| K^{\varepsilon}(\xi) / \eta(x) \right| : \xi \in \Sigma \right\}$$

and

$$w^{\varepsilon}(\xi) = v^{\varepsilon}(\xi) + 2k^{\varepsilon}\eta(x), \qquad \xi = (x, d, i) \in \Sigma.$$

Then, the linearity of  $\mathscr{L}^{u,\xi}$ , the definition of  $K^{\varepsilon}$ , and (5.11) yield

$$w^{\varepsilon}(\xi) - \mathcal{L}^{u^{*}(\xi),\xi} \left( D_{x} w^{\varepsilon}(\xi), w^{\varepsilon}(x, \cdot, i), \frac{1}{\varepsilon} w^{\varepsilon}(x, d, \cdot) \right) - G(\xi, u^{*}(\xi))$$
$$= K^{\varepsilon}(\xi) + 2k^{\varepsilon} [\eta(x) - \mathcal{L}^{u^{*}(\xi),\xi}(D_{x}\eta(x), 0, 0)]$$
$$\geq K^{\varepsilon}(\xi) + k^{\varepsilon} \eta(x) \geq 0.$$

Hence  $w^{\varepsilon}$  is a supersolution of  $(5.6)^{\varepsilon}$ . Consequently, the comparison principle implies that  $J^{\varepsilon} \leq w^{\varepsilon}$ . Now let  $\varepsilon$  go to zero and use the convergence of  $K^{\varepsilon}$  to zero together with (5.8) to obtain the convergence of  $J^{\varepsilon}$  to v.  $\Box$ 

*Remark.* If the operator  $\mathscr{L}^{u^*(\xi),\xi}(p, L, k)$  is continuous in  $\xi$ , then  $K^{\varepsilon}(\xi)$  converges to zero uniformly on compact subsets of  $\Sigma$ . Therefore  $w^{\varepsilon}$ , defined as in the above proof, converges to v uniformly on compact sets. Consequently, the conclusion of Theorem 5.1 holds uniformly on compact subsets of  $\Sigma$ .

Example 5.3. Consider the problem described in §2 with n = 2, m = 1,  $D = \{(\delta, 1)\}$ ,  $\gamma_1 = \beta/K$ ,  $\gamma_2 = 1/K$ ,  $\mu_1(u) = \mu[\nu u_1 + u_2]$ ,  $\lambda_0 = 1$ , and  $G(x, y, u) = \alpha |x| + |y|$ , where all the parameters are positive. Set

$$E = K\mu + 1,$$
$$\lambda = K\mu\nu + \beta.$$

Case 1.  $\lambda \leq E\alpha$ . In this case, the optimal feedback control is

$$u^{*}(x, y, 1) = \begin{cases} (0, 0) & \text{if } x > 0, y > 0, \\ (0, 1) & \text{if } x > 0, y = 0, \\ (0, K) & \text{if } x > 0, y < 0, \\ (E\delta, K - \lambda\delta)/[1 + \mu\delta(\beta - \nu)] & \text{if } x = 0, y < 0, \\ (K/\beta, 0) & \text{if } x < 0, \\ (\delta, 0) & \text{if } x = 0, y > 0, \\ (\delta, 1) & \text{if } x = 0, y = 0. \end{cases}$$

Case 2.  $\lambda \ge E\alpha$ . Then

$$u^{*}(x, y, 1) = \begin{cases} (0, 0) & \text{if } x > 0, y > 0, \\ (0, 1) & \text{if } x > 0, y = 0, \\ (0, K) & \text{if } y < 0, \\ (K - E, \lambda) / [\mu \nu + (1 - \mu)\beta] & \text{if } x < 0, y = 0, \\ (K / \beta, 0) & \text{if } x < 0, y > 0, \\ (\delta, 0) & \text{if } x = 0, y > 0, \\ (\delta, 1) & \text{if } x = 0, y = 0. \end{cases}$$

The value function is continuously differentiable in either case, and a comparison principle for  $(5.6)^{\varepsilon}$  holds.

The above strategies differ only on the fourth quadrant. This is expected because in the other quadrants at most one of the products is in shortage, and then the optimal policy is to produce the product in shortage, if there is one, in full capacity. However, in the fourth quadrant of each of the products is in shortage, and therefore a priority rule is needed. The above calculations provide just this. In the first case, the optimal strategy is to produce x in full capacity. Hence in this case, the first product has priority over the second one. In the second case, this priority changes. So we may sum the above findings into the following rule:

If  $\lambda \leq E\alpha$ , produce the first product in full capacity if there is any shortage of it regardless the inventory level of the second product. If  $\lambda \geq E\alpha$ , reverse this rule.

6. Convergence rate. The exact rate at which  $|v^{\varepsilon} - v|$  converges to zero is an interesting question. Recently Zhang and Sethi [15] obtained the rate  $\varepsilon^{1/2}$  or  $\varepsilon^{1/4}$  under different assumptions for the case described in Example 3.3. Also, they show, with an

explicit example, that in general  $\varepsilon^{1/2}$  is the best rate. However, when the limit function is continuously differentiable we expect that the function  $(v^{\varepsilon}(x, d, i) - v(x, d))/\varepsilon$  is uniformly bounded in  $\varepsilon$  on every compact subset of  $\Sigma$ . A similar result was proved in [7].

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