Phase Transitions and Generalized Motion by Mean Curvature

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Abstract

We study the limiting behavior of solutions to appropriately rescaled versions of the Allen-Cahn equation, a simplified model for dynamic phase transitions. We rigorously establish the existence in the limit of a phase-antiphase interface evolving according to mean curvature motion. This assertion is valid for all positive time, the motion interpreted in the generalized sense of Evans-Spruck and Chen-Giga-Goto after the onset of geometric singularities.

1. Introduction

Allen and Cahn proposed in [1] the following semilinear parabolic partial differential equation to describe the time evolution of an "order parameter" v determining the phase of a polycrystalline material:

$$(1.1) \qquad v_t - 2\alpha\kappa\,\Delta v + \alpha\,f(v) = 0 \quad \text{in} \quad R^3 \times (0, \infty) \,.$$

Here α is a positive kinetic constant and κ a gradient energy coefficient. The nonlinearity is

$$(1.2) \qquad\qquad\qquad f \equiv F' \,,$$

F denoting the free energy per unit volume. We take F to be a W-shaped potential, whose two wells, of equal depth, correspond to different stable material phases. The mean field Ginzburg-Landau excess free energy is then

$$\int_{R^3} \left[\kappa |Dv|^2 + F(v) \right] dx \,,$$

the term $\kappa|Dv|^2$ corresponding to interfaces between stable regions. See Allen and Cahn [1], Cahn and Hilliard [11], and Caginalp [9], [10], for more explanation. (The partial differential equation (1.1) is related also to the stochas-

tic Ginzburg-Landau model, an equation for first-order phase transitions; see Gunton, San Miguel, and Sahni [29], p. 290.)

We are concerned with asymptotics of the Allen-Cahn equation in the limit $\varepsilon \to 0^+$ for

$$(1.3) \qquad \alpha = \frac{1}{\varepsilon^2}, \qquad \kappa = \frac{\varepsilon^2}{2}, \qquad 2\alpha\kappa = 1 .$$

This represents a rapid rescaling in time and a simultaneous diminution of the gradient energy term. We consequently expect the solution to converge at each point of $R^3 \times (0, \infty)$ to one of the two stable minima of F, creating thereby a sharp interface, the "antiphase boundary", between regions of different phases.

An interesting physical and mathematical problem is determining the motion of this antiphase boundary. In [1] Allen and Cahn propose for the general problem (1.1) the motion by mean curvature rule

$$(1.4) \qquad V = 2\alpha\kappa(\kappa_1 + \kappa_2) ,$$

V denoting the velocity of the interface and κ_1, κ_2 its principal curvatures. In his study of two phase continua, Gurtin (see [30], [31]) has also derived this mean curvature type flow as a model for the motion of the interface, and later Angenent and Gurtin further developed this theory for perfect conductors; see [3]. The asymptotic limit (1.4) is also consistent with the stationary results of Modica in [37], Fonseca and Tartar in [26], Sternberg in [46], [47], etc.: these authors have shown that the Γ-limit of the problem of minimizing the excess free energy is a surface area minimization problem.

Our goal in this paper is a mathematically rigorous verification of the law of motion (1.4) in the asymptotic limit (1.3), for all times $t \geq 0$. This undertaking turns out to be rather subtle mathematically. The big problem is that a surface evolving according to the mean curvature evolution (1.4) can start out smooth and yet later develop singularities. For instance, the boundary of a dumbbell shaped region will after a time "pinch off"; see, for instance, Grayson [28], Sethian [43], etc. From the viewpoint of classical differential geometry it is not so clear if, and how, it may be possible even to define the subsequent evolution of the surface after the onset of singularities.

There have been, to our knowledge, at least three general proposals for interpreting the mean curvature evolution of surfaces past singularities. In [7] Brakke has exploited techniques of geometric measure theory to construct a (generally nonunique) varifold solution. A second proposal for building a generalized mean curvature flow has been suggested by Bronsard and Kohn in [8], DeGiorgi in [17], and others. This is what may be called a "phase field" approach: namely, simply to *define* a generalized geometric motion in terms of asymptotics of the scaled Allen-Cahn equation. To our knowledge this possibility has not heretofore been systematically developed. A third

approach, initially suggested in the physics literature by Ohta, Jasnaw, and Kawasaki (see [38]), for numerical calculations by Sethian (see [42]), Osher and Sethian (see [39]), and, for a first-order model of flame propagation, by Barles (see [4]), represents the evolving surface as the level set of an auxiliary function solving an appropriate nonlinear partial differential equation. This latter suggestion has been extensively developed by Evans and Spruck in [21], [22], [23], and, independently, Chen, Giga, and Goto in [12]. (Chen, Giga, and Goto in [12] consider as well more general geometric motions.) Their analysis makes use of the theory of so-called "viscosity" solutions to fully nonlinear second-order parabolic equations, as developed by Crandall and Lions (see [16]) and Jensen (see [33]). (See also Crandall, Evans, and Lions [14], Lions [35], [36], Ishii [32], Jensen, Lions, and Souganidis [34], etc., etc.; for a detailed overview of the theory of viscosity solutions as well as a complete list of references, see the *User's Guide,* [15]). In [44] Soner has recently recast the definitions, constructions, and uniqueness criterion of [21], [12] into a different and more intrinsic form using the distance function to the surface; this reformulation is an important tool in our analysis below. A general theory for moving fronts using the distance function to the surface is developed in Barles, Soner, and Souganidis; see [6].

The level set approach uniquely defines a generalized mean curvature evolution $\{\Gamma_t\}_{t \geq 0}$, starting with a given compact surface $\Gamma_0 \subset R^n$. This flow exists for all time and agrees with the classical smooth differential geometric flow, so long as the latter exists. The geometric motion may, on the other hand, develop singularities, changing topological type, and exhibit various other geometric pathologies.

In spite of these peculiarities the generalized motion $\{\Gamma_t\}_{t \geq 0}$ seems in many ways a strong candidate for being the "right" way to extend the classical motion past singularities. We and others have consequently been led to conjecture that this generalized mean curvature motion governs asymptotic behavior for solutions of the Allen-Cahn equation (1.1) in the limit (1.3) and thus the phase-field and level-set methods agree. Formal asymptotic expansions suggesting this have been carried out by Caginalp in [9], Fife in [24], Rubinstein, Sternberg, and Keller in [41], Pego in [40], and others. The radial case has been studied by Bronsard and Kohn in [8], and in [18] de Mottoni and Schatzman have given a complete proof for the case of a classical geometric motion. Chen (see [13]) has very recently generalized much of this work and given simpler proofs, as has Korevaar in unpublished work.

All these papers require knowledge that the mean curvature flow be smooth, and consequently fail once geometric irregularities appear. The main accomplishment of this work is consequently our verification that the generalized motion $\{\Gamma_t\}_{t \geq 0}$ does indeed determine the antiphase boundary for all positive time, with the one proviso (discussed in Section 5) that the sets $\{\Gamma_t\}_{t \geq 0}$ do not develop interiors.

This assertion, by the way, provides an independent check on the reasonableness of the level set model of Evans and Spruck and Chen, Giga, and Goto. The generalized motion $\{\Gamma_t\}_{t \geq 0}$ can behave in all sorts of odd ways (cf. Evans and Spruck [21], Section 8) and so it is reassuring to learn $\{\Gamma_t\}_{t \geq 0}$ nevertheless controls asymptotics for the scaled Allen-Cahn equation.

We have organized this paper by first providing in Section 2 a quick review of the level set approach to mean curvature flow, followed by a detailed analysis of the distance function d to the motion. The key assertion is that d is a supersolution of the heat equation in the region $\{d > 0\}$, in the weak, that is, viscosity sense. This observation is at the heart of Soner's work; see [44]. In Section 3 we build supersolutions of the scaled Allen-Cahn equations out of d and the standing wave solution q of the one-dimensional Allen-Cahn equation. Such a change of variable has already been employed by Gärtner in [27], de Mottoni and Schatzman in [18], [19], [20], Fife and McLeod in [25], Barles, Bronsard, and Souganidis in [5], Rubinstein, Sternberg, and Keller in [41], etc. Our construction is thus deeply motivated by previous work, the new contribution being various adjustments such as cutting off d near Γ_t and adding a small positive term. Finally in Section 4 we extend the maximum principle to our general setting and prove solutions of the scaled Allen-Cahn equation lie everywhere beneath our supersolutions. An analogous assertion for subsolutions completes the proof.

In Section 5 we discuss the possibility the sets $\{\Gamma_t\}_{t \geq 0}$ may develop an interior. We do not know whether our assumptions in fact exclude this possibility. Recently, however, Altschuler, Angenent, and Giga in [2] and Soner and Souganidis in [45] have studied the evolution of surfaces of rotation. In particular they proved that for a large class of rotationally symmetric problems there is no interior. In [2] a complete theory is given for surfaces which are like the dumbbell but with several pinching points. The evolution of the "torus-like" surfaces is carried out in [45]. We are grateful to the referee for so thoroughly reading this paper and providing us with many additional references.

2. The Distance Function to a Generalized Motion
by Mean Curvature

In this section we recall the level set construction in Evans and Spruck (see [21], [22]) and Chen, Giga, and Goto (see [12]) of a generalized evolution by mean curvature, and then study properties of the distance function to the motion.

Given a compact subset $\Gamma_0 \subset R^n$, $n \geq 2$, choose a continuous function $g: R^n \to R$ satisfying

$$(2.1) \qquad \qquad \Gamma_0 = \{x \in R^n \mid g(x) = 0\}$$

and

(2.2) g is constant outside some ball.

We consider then the *mean curvature evolution partial differential equation*

$$(2.3) \quad \begin{cases} u_t = \left(\delta_{ij} - \dfrac{u_{x_i} u_{x_j}}{|Du|^2} \right) u_{x_i x_j} & \text{in} \quad R^n \times (0, \infty) \,, \\ u = g & \text{on} \quad R^n \times \{t = 0\} \,. \end{cases}$$

As explained in [21], this partial differential equation asserts that each level set of u evolves according to mean curvature flow, at least in regions where u is smooth and $Du \neq 0$. In addition, there exists a unique, continuous weak solution of (2.3). See [21], [12] for the relevant definitions, proofs, etc. We accordingly *define* the compact sets

$$(2.4) \quad \Gamma_t \equiv \{ x \in R^n \mid u(x, t) = 0 \} \,, \qquad t \geq 0$$

and call $\{\Gamma_t\}_{t \geq 0}$ the (level set) *generalized motion by mean curvature* starting from Γ_0. Consult [21], Section 5, and [12], Theorem 7.1, for a proof that the definition (2.4) does not depend on the choice of the particular function g verifying (2.1), (2.2).

Let $t^* = \inf\{ t > 0 \mid \Gamma_t = \varnothing \}$ denote the *extinction time*. For each finite time $0 \leq t \leq t^*$, let us set

$$(2.5) \quad d(x, t) \equiv \text{dist}(x, \Gamma_t) \,, \qquad x \in R^n \,,$$

the distance of x to Γ_t in $\mathbf{R}^n$. (*Warning: We later modify this definition, in* (2.30).) Notice that the continuity of u implies Γ_{t^*} is nonempty, and consequently the distance function is defined at t^*. The function d is Lipschitz continuous in the spatial variable x, but may well be discontinuous in the time t. The latter possibility can occur if, say, Γ_t splits into two pieces, one of which evolves into the empty set before the other.

First we verify d is lower semicontinuous and continuous from below (cf. Lemma 7.3 in Soner, [44]).

PROPOSITION 2.1.

(i) *For each $x \in R^n$ and $0 \leq t \leq t^*$,*

$$(2.6) \quad d(x, t) \leq \liminf_{\substack{y \to x \\ s \to t}} d(y, s) \,.$$

(ii) *For each $x \in R^n$ and $0 < t \leq t^*$,*

$$d(x, t) = \lim_{\substack{y \to x \\ s \uparrow t}} d(y, s) \,.$$

Proof:

1. Choose $\{y_k\}_{k=1}^\infty \subset R^n$, $\{s_k\}_{k=1}^\infty \subset [0, t^*]$ so that $y_k \to x$, $s_k \to t$ and

$$d(y_k, s_k) \to \liminf_{\substack{y \to x \\ s \to t}} d(y, s) \ .$$

As Γ_{s_k} is compact and nonempty, there exists a point $z_k \in \Gamma_{s_k}$ for which

$$d(y_k, s_k) = \mathrm{dist}(y_k, \Gamma_{s_k}) = |y_k - z_k| \ , \qquad k = 1, 2, \ldots \ .$$

We extract a subsequence $\{z_{k_j}\}_{j=1}^\infty \subset \{z_k\}_{k=1}^\infty$ and a point $z \in R^n$ so that $z_{k_j} \to z$. As $z_k \in \Gamma_{s_k}$, we have $u(z_k, s_k) = 0$ $(k = 1, \ldots)$; and consequently $u(z, t) = 0$. Thus $z \in \Gamma_t$. Hence

$$\begin{aligned} d(x, t) = \mathrm{dist}(x, \Gamma_t) &\leq |x - z| = \lim_{j \to \infty} |y_{k_j} - z_{k_j}| \\ &= \lim_{j \to \infty} d(y_{k_j}, s_{k_j}) \\ &= \liminf_{\substack{y \to x \\ s \to t}} d(y, s) \ . \end{aligned}$$

This proves assertion (i).

2. To verify property (ii) suppose instead $d(x, t) < \limsup_{\substack{y \to x \\ s \uparrow t}} d(y, s)$ and choose $\{y_k\}_{k=1}^\infty \subset R^n$, $\{s_k\}_{k=1}^\infty \subset [0, t]$ satisfying $y_k \to x$, $s_k \uparrow t$ and $d(y_k, s_k) \to \limsup_{\substack{y \to x \\ s \uparrow t}} d(y, s)$. There exists a number $r \in R$ satisfying

$$(2.7) \qquad d(x, t) < r < d(y_k, s_k)$$

for all sufficiently large k, say $k \geq k_0$. In particular

$$(2.8) \qquad B(y_k, r) \subset [R^n \setminus \Gamma_{s_k}] \ , \qquad k \geq k_0 \ .$$

Now set $B(y_k, r) = \Delta_{s_k}^k$ and let $\{\Delta_s^k\}_{s \geq s_k}$ denote the subsequent evolution of the ball $\Delta_{s_k}^k$ by the mean curvature flow. According to Evans and Spruck (see [21]) (2.8) implies $\Delta_s^k \cap \Gamma_s = \varnothing$ for all times $s \geq s_k$. But a direct computation (see [21], Section 7.1) reveals $\Delta_s^k = B(y_k, r_k(s))$ $(s_k \leq s \leq t)$ for $r_k(s) \equiv (r^2 - 2(n-1)(s - s_k))^{1/2}$. As $\Delta_t^k \cap \Gamma_t = \varnothing$, we deduced $d(y_k, t) \geq r_k(t)$ $(k \geq k_0)$. Now send k to infinity to discover $d(x, t) \geq r$, a contradiction to (2.7).

Next we verify that d is a supersolution of the heat equation off the set $\Gamma = \{d = 0\}$. In what follows, the sub- and supersolutions are interpreted in the "viscosity" sense of Crandall and Lions [16], Lions [35], and Jensen [33].

THEOREM 2.2. *Let d be the distance function, as above. Then*

(2.9) $$d_t - \Delta d \geqq 0 \quad in \quad \{d > 0\} \subset R^n \times (0, t^*) \; .$$

If Γ is a smooth evolution via mean curvature, a direct calculation (cf. [22]) verifies $d_t - \Delta d \geq 0$ off Γ, at least in the region near Γ where d is smooth. The point now is that d is in fact globally supersolution of the heat equation, although we must interpret this statement in the weak, i.e., viscosity solution, sense, since d need not be smooth, or even continuous.

Proof:

1. Fix a test function $\phi \in C^\infty(R^n \times (0, \infty))$ and suppose

(2.10) $$d - \phi \text{ has a minimum at a point } (x_0, t_0) \in R^n \times (0, t^*) \; ,$$

where

(2.11) $$d(x_0, t_0) > 0 \; .$$

We must demonstrate

(2.12) $$\phi_t - \Delta \phi \geqq 0 \quad \text{at} \quad (x_0, t_0) \; .$$

2. Adding if necessary a constant to ϕ we may assume

(2.13) $$\phi(x_0, t_0) = d(x_0, t_0) \equiv \delta > 0 \; .$$

Owing to (2.10) and (2.12) we have

(2.14) $$d(x, t) \geqq \phi(x, t) \; , \qquad x \in R^n \; , \; 0 < t < t^* \; .$$

Choose $z_0 \in \Gamma_{t_0}$ so that

(2.15) $$d(x_0, t_0) = |x_0 - z_0| = \delta \; .$$

Upon rotating coordinates we may assume

(2.16) $$x_0 = z_0 + \delta e_n \; ,$$

where $e_n = (0, \dots, 0, 1)$. Set

(2.17) $$\psi(x, t) \equiv \phi(x + x_0 - z_0, t) - \delta \; , \qquad x \in \mathbb{R}^n \; , \; t > 0 \; .$$

Then

(2.18) $$\psi(z_0, t_0) = 0 .$$

3. We now *claim*

(2.19) $$\{\psi > 0\} \subseteq \{d > 0\} .$$

To verify this inclusion select any point $(x, t) \in R^n \times (0, t^*)$ where $\psi(x, t) > 0$. Then (2.14), (2.17) force $d(x + x_0 - z_0, t) \geqq \phi(x + x_0 - z_0, t) > \delta$. Now if $d(x, t) = 0$, then $\delta < d(x + x_0 - z_0, t) - d(x, t) \leqq |x_0 - z_0| = \delta$, a contradiction. Assertion (2.19) is proved.

4. For use later, let us pause to verify

(2.20) $$D\phi(x_0, t_0) = e_n ,$$

and

(2.21) $$\phi_{x_n x_n}(x_0, t_0) \leqq 0 .$$

Indeed, (2.13), (2.14) imply

$$\phi(x, t_0) - \phi(x_0, t_0) \leqq d(x, t_0) - d(x_0, t_0) \leqq |x - x_0| , \qquad x \in R^n .$$

Consequently $|D\phi(x_0, t_0)| \leqq 1$. On the other hand, let us consider next the scalar function $\Phi(s) = \phi(z_0 + se_n, t_0)$ $(s > 0)$. By (2.14) we have

$$\Phi(s) \leqq d(z_0 + se_n, t_0) \leqq \delta ,$$

since $z_0 \in \Gamma_{t_0}$. In addition $\Phi(\delta) = \phi(z_0 + \delta e_n, t_0) = d(x_0, t_0) = \delta$. Thus

$$\Phi'(\delta) = 1 , \qquad \Phi''(\delta) \leqq 0 ;$$

that is, $\phi_{x_n}(x_0, t_0) = 1$, $\phi_{x_n x_n}(x_0, t_0) \leqq 0$.

5. We return now to the main task at hand, verifying the inequality (2.12). Replacing u by $|u|$ if necessary, we may assume

$$u \geqq 0 \quad \text{in} \quad R^n \times [0, \infty) .$$

(Recall from Evans and Spruck [21], Section 2.4, that $|u|$ is also a solution of the mean curvature evolution partial differential equation.) Thus $\{d > 0\} = \{u > 0\}$; whence (2.19) implies

(2.22) $$\{\psi > 0\} \subseteq \{u > 0\} .$$

We next build a continuous function $\Psi: [0, \infty) \to [0, \infty)$ such that

$$(2.23) \qquad \Psi(0) = 0 , \quad \Psi(z) > 0 \quad \text{if} \quad z > 0$$

and

$$(2.24) \qquad \psi(z, t) \leqq \Psi(u(z, t)) \quad \text{for all} \quad (z, t) \quad \text{near} \quad (z_0, t_0) .$$

To carry out this construction, define the compact sets

$$E_k \equiv \left\{ x \in R^n, 0 < t < t^* \mid \psi(x, t) \geqq \frac{1}{k}, |x - x_0| \leqq 1, |t - t_0| \leqq 1 \right\}$$

for $k = 1, \ldots$. Write $\beta_k = \inf_{E_k} u$. Owing to (2.22) $\beta_1 \geqq \cdots \geqq \beta_k \geqq \beta_{k+1} \ldots > 0$. Furthermore $\lim_{k \to \infty} \beta_k = 0$, since $u(z_0, t_0) = \psi(z_0, t_0) = 0$. Pass to a subsequence $\{\beta_{k_j}\}_{j=1}^{\infty} \subset \{\beta_k\}_{k=1}^{\infty}$ satisfying $\beta_{k_j} > \beta_{k_{j+1}}$ $(j = 1, \ldots)$ and define $\Psi: [0, \infty) \to \mathbb{R}$ by

$$\begin{cases} \Psi(\beta_{k_{j+1}}) = \dfrac{1}{k_j} , & j = 1, \ldots , \\ \Psi \text{ linear on } [\beta_{k_{j+1}}, \beta_{k_j}] . \end{cases}$$

Then if $(x, t) \in E_{k_{j+1}} \setminus E_{k_j}$,

$$\Psi(u(x, t)) \geqq \Psi(\beta_{k_{j+1}}) = \frac{1}{k_j} > \psi(x, t) .$$

Thus (2.24) is valid on the set

$$\bigcup_{j=1}^{\infty} E_{k_{j+1}} \setminus E_{k_j} = \left\{ 0 < \psi < \frac{1}{k_1} \right\} .$$

Since (2.24) is trivial on $\{\psi \leqq 0\}$, we deduce (2.24) is valid for all points near (z_0, t_0).

6. Now $\Psi(u)$ is a solution of the mean curvature partial differential equation according to Evans and Spruck (see [21], Section 2.4). As (2.18) and (2.24) imply

$$\Psi(u) - \psi \text{ has a local minimum at } (z_0, t_0) ,$$

we have

$$\psi_t - \left(\delta_{ij} - \frac{\psi_{x_i} \psi_{x_j}}{|D\psi|^2} \right) \psi_{x_i x_j} \geqq 0 \quad \text{at} \quad (z_0, t_0) .$$

Now

$$\psi_t(z_0, t_0) = \phi_t(x_0, t_0) \ , \quad D\psi(z_0, t_0) = D\phi(x_0, t_0) \ , \quad D^2\psi(z_0, t_0) = D^2\phi(x_0, t_0) \ ,$$

according to (2.17). Thus (2.20), (2.21) force

$$\phi_t - \Delta\phi = \phi_t - \left(\delta_{ij} - \frac{\phi_{x_i}\phi_{x_j}}{|D\phi|^2}\right)\phi_{x_i x_j} - \phi_{x_n x_n} \geqq 0 \quad \text{at} \quad (x_0, t_0) \ .$$

This is inequality (2.12).

Our proof has a geometric interpretation. In view of (2.17), (2.20) the set $\{\psi = 0\}$ is a smooth hypersurface S near (z_0, t_0), and owing to (2.18), (2.19) this (smooth) surface is tangent to the (possibly nonsmooth) set Γ at (z_0, t_0). It then follows from the definition of a solution for the mean curvature evolution partial differential equation that

$$\psi_t - \left(\delta_{ij} - \frac{\psi_{x_i}\psi_{x_j}}{|D\psi|^2}\right)\psi_{x_i x_j} \geqq 0 \quad \text{at} \quad (z_0, t_0) \ .$$

This means that the velocity of S at (z_0, t_0) is greater than or equal to $((n-1)$ times) the mean curvature of S at (z_0, t_0). This interpretation is related to observations in Soner; see [44], Section 14A.

Remark. In fact d is a supersolution of the heat equation all the way up to time t^*. In other words,

$$(2.25) \qquad\qquad d_t - \Delta d \geqq 0 \quad \text{in} \quad \{d > 0\} \subset R^n \times (0, t^*] \ .$$

To verify this, we assume that for a ϕ as above

$$d - \phi \quad \text{has a minimum at a point} \quad (x_0, t_0)$$

with $t_0 = t^*$ and $d(x_0, t_0) > 0$.

Upon modifying ϕ if necessary, we may assume that $d - \phi$ has a strict minimum at (x_0, t_0). Finally, given $\varepsilon > 0$ we write

$$\psi^\varepsilon(x, t) \equiv \phi(x, t) + \frac{\varepsilon}{t - t^*} \ , \qquad x \in R^n, \quad 0 < t < t^* \ .$$

Since d is lower semicontinuous and $\phi^\varepsilon = -\infty$ on $\{t = t^*\}$, $d - \phi^\varepsilon$ has a minimum at a point $(x_\varepsilon, t_\varepsilon) \in R^n \times (0, t^*)$ with

$$(2.26) \qquad\qquad x_\varepsilon \to x_0 \quad \text{and} \quad t_\varepsilon \to t_0 = t^* \quad \text{as} \quad \varepsilon \to 0 \ .$$

Since $d(x_0, t_0) > 0$ and d is lower semicontinuous, we have $d(x_\varepsilon, t_\varepsilon) > 0$ for sufficiently small ε. Consquently Theorem 2.2 implies

$$\phi_t^\varepsilon - \Delta\phi^\varepsilon \geqq 0 \quad \text{at} \quad (x_\varepsilon, t_\varepsilon) \ .$$

Now

$$\phi_t^\varepsilon(x, t) = \phi_t(x, t) - \frac{\varepsilon}{(t - t^*)^2} \leqq \phi_t(x, t) \ .$$

Thus

$$\phi_t - \Delta\phi \geqq 0 \quad \text{at} \quad (x_\varepsilon, t_\varepsilon) \ .$$

Now let $\varepsilon \to 0$.

We conclude this section by *modifying our notation*, as follows. We henceforth assume Γ_0 is the boundary of a bounded, open set $U \subset R$, and choose a continuous function g so that

$$(2.27) \qquad g(x) \begin{cases} > 0 & \text{if} \quad x \in U \\ = 0 & \text{if} \quad x \in \Gamma_0 \\ < 0 & \text{if} \quad x \in R^n - \overline{U} \end{cases} \ .$$

We solve the mean curvature partial differential equation (2.3), and then define

$$(2.28) \qquad I_t \equiv \{x \in R^n \mid u(x, t) > 0\}$$

and $t \geqq 0$

$$(2.29) \qquad O_t \equiv \{x \in R^n \mid u(x, t) < 0\} \ .$$

In view of (2.4) and (2.27) we may informally regard I_t as the "inside" and O_t as the "outside" of the evolution at time t. We also write

$$(2.30) \qquad I \equiv \{(x, t) \in R^n \times (0, \infty) \mid u(x, t) > 0\}$$

and

$$(2.31) \qquad O \equiv \{(x, t) \in R^n \times (0, \infty) \mid u(x, t) < 0\} \ .$$

Let us now change notation, hereafter writing

$$(2.30') \qquad d(x, t) = \begin{cases} \text{dist}(x, \Gamma_t) & \text{if} \quad x \in I_t \\ 0 & \text{if} \quad x \in \Gamma_t \\ -\text{dist}(x, \Gamma_t) & \text{if} \quad x \in O_t \end{cases} \ .$$

for $x \in R^n$, $0 \leqq t \leqq t^*$. We henceforth call d the *signed distance function*.

We recast Theorem 2.2 into the new notation:

THEOREM 2.3. *Let d be the signed distance function, as above. Then*

(2.31′) $d_t - \Delta d \geqq 0 \quad in \quad I \cap (R^n \times (0, t^*])$

and

(2.32) $d_t - \Delta d \leqq 0 \quad in \quad O \cap (R^n \times (0, t^*])$.

Remark. Thus, formally at least,

$$d_t - \Delta d = 0 \quad \text{on} \quad \Gamma .$$

This is consistent with the classical observation that the signed distance function solves the heat equation *on* a smooth surface evolving by mean curvature motion.

Remark. In Soner (see [44]) a set-valued map $\{C_t\}_{t \geq 0}$ is called a viscosity solution of the mean curvature flow problem if both (2.31) and (2.32) hold. Hence the above theorem establishes a connection between the level set of solutions of Evans and Spruck and Chen, Giga, and Goto, and that constructed in [44]. In particular, these two definitions coincide if $\partial I_t = \partial O_t$ for all $t \neq t^*$. A more detailed discussion of this point is given in [44], Section 11.

3. Supersolutions

We intend next to utilize the signed distance function d to build sub- and supersolutions of the Allen-Cahn partial differential equation.

For definiteness let us take the free energy per unit volume F to be the quartic

(3.1) $F(z) = \dfrac{1}{2} \left(z^2 - 1 \right)^2 , \qquad z \in R ,$

so that

(3.2) $f(z) = F'(z) = 2 \left(z^3 - z \right) , \qquad z \in R .$

(Our arguments, however, are still valid without significant change if F is

any W-shaped potential, whose two wells are of equal depth.) For this free energy the ordinary differential equation

(3.3)
$$\begin{cases} q''(s) = f(q(s)) , & s \in \mathbb{R} \\ \lim_{s \to \pm\infty} q(s) = \pm 1 \end{cases}$$

has an explicit standing wave solution

$$q(s) = \tanh(s) = \frac{e^{2s} - 1}{e^{2s} + 1} , \quad s \in R .$$

We record for later use the equalities

(3.4)
$$\begin{cases} q'(s) = \text{sech}^2(s) = \dfrac{4}{(e^s + e^{-s})^2} & s \in R . \\ q''(s) = -2\,\text{sech}^2(s)\tanh(s) \end{cases}$$

Next fix $0 < \delta \ll 1$ and consider a smooth auxiliary function $\eta: R \to R$ satisfying

(3.5)
$$\begin{cases} \eta(z) = -\delta , & -\infty < z \leq \delta/4 \\ \eta(z) = z - \delta , & z \geq \delta/2 \\ 0 \leq \eta' \leq C , \quad |\eta''| \leq \dfrac{C}{\delta} , \end{cases}$$

where C is a constant, independent of δ.

Remark. Since we intend to construct a *super* solution of the scaled Allen-Cahn equation, we need to redefine d on the set $\{d < 0\}$, when according to Theorem 2.3 d is a *subsolution* of the heat equation. This is the reason for introducing the auxiliary function η.

Suppose $\{\Gamma_t\}_{t\geq 0}$ is a generalized motion by mean curvature, and d is the corresponding signed distance function.

LEMMA 3.1. *There exists a constant C, independent of δ, such that*

(3.6)
$$\eta(d)_t - \Delta\eta(d) \geq -\frac{C}{\delta} \quad in \quad R^n \times (0, t^*]$$

and

(3.7)
$$\eta(d)_t - \Delta\eta(d) \geq 0 \quad in \quad \left\{ d > \frac{\delta}{2} \right\} \subseteq R^n \times (0, t^*] .$$

Proof:

1. Take $\phi \in C^\infty(R^n \times (0, \infty))$ and assume $\eta(d) - \phi$ has a strict minimum at point $(x_0, t_0) \in R^n \times (0, t^*)$.

2. Assume first $d(x_0, t_0) > 0$. Fix $\varepsilon > 0$, write

$$\eta_\varepsilon(z) = \eta(z) + \varepsilon z , \qquad z \in R ,$$

and set

$$\rho_\varepsilon \equiv (\eta_\varepsilon)^{-1} .$$

Then $\eta_\varepsilon(d)$ is lower semicontinuous near (x_0, t_0) and thus $\eta_\varepsilon(d) - \phi$ has a minimum at a point $(x_\varepsilon, t_\varepsilon) \in R^n \times (0, t^*)$, with

(3.8) $$x_\varepsilon \to x_0 , \quad t_\varepsilon \to t_0 \quad \text{as} \quad \varepsilon \to 0 .$$

Adding a constant to ϕ if necessary we may assume $\eta_\varepsilon(d) - \phi = 0$ at $(x_\varepsilon, t_\varepsilon)$. Thus $\eta_\varepsilon(d) \geqq \phi$ and so

(3.9) $$d \geqq \rho_\varepsilon(\phi) \equiv \psi^\varepsilon$$

for all (x, t) near $(x_\varepsilon, t_\varepsilon)$, with equality at $(x_\varepsilon, t_\varepsilon)$. Since $d(x_0, t_0) > 0$ and d is lower semicontinuous near (x_0, t_0)

$$d(x_\varepsilon, t_\varepsilon) > 0$$

for all small $\varepsilon > 0$. According to (3.9) and Theorem 2.2

$$\psi_t^\varepsilon - \Delta\psi^\varepsilon \geqq 0 \quad \text{at} \quad (x_\varepsilon, t_\varepsilon) ;$$

that is,

(3.10) $$\rho_\varepsilon'(\phi)(\phi_t - \Delta\phi) - \rho_\varepsilon''(\phi)|D\phi|^2 \geqq 0 \quad \text{at} \quad (x_\varepsilon, t_\varepsilon) .$$

Now

$$\frac{\rho_\varepsilon''(\phi)}{\rho_\varepsilon'(\phi)} = -\eta_\varepsilon''(\psi^\varepsilon)\rho_\varepsilon'(\phi)^2$$

and so (3.10) yields

(3.11) $$\phi_t - \Delta\phi \geqq -\eta_\varepsilon''(\psi^\varepsilon)\,\rho_\varepsilon'(\phi)^2|D\phi|^2 = -\eta''(\psi^\varepsilon)\,|D\psi^\varepsilon|^2 \geqq -\frac{C}{\delta}$$

by (3.5) at $(x_\varepsilon, t_\varepsilon)$. We employed in this calculation the bound $|D\psi^\varepsilon| \leqq 1$, which follows from (3.9). Sending $\varepsilon \to 0$ we deduce

(3.12) $$\phi_t - \Delta\phi \geqq -\frac{C}{\delta} \quad \text{at} \quad (x_0, t_0) .$$

3. Assume next $d(x_0, t_0) \leq 0$. Since d is continuous from below, we have $\eta(d) \equiv -\delta$ on the set $\{|x - x_0| \leq \sigma, \, t_0 - \sigma \leq t \leq t_0\}$ for some $\sigma > 0$. Thus

$$\phi_t(x_0, t_0) \geq 0, \qquad D^2\phi(x_0, t_0) \leq 0$$

and so

$$(3.13) \qquad \phi_t - \Delta\phi \geq 0 \qquad \text{at} \qquad (x_0, t_0).$$

4. If $\eta(d) - \phi$ has a minimum at a point (x_0, t^*), we argue using the Remark after Theorem 2.2. Assertion (3.6) is proved.

5. To prove (3.7), suppose $d(x_0, t_0) > \delta/2$. Then for small $\varepsilon > 0$, $d(x_\varepsilon, t_\varepsilon) > \delta/2$. By (3.5) we conclude that $\eta''(\psi^\varepsilon) = 0$ at $(x_\varepsilon, t_\varepsilon)$. Using this in (3.11), we arive at (3.7).

Our intention next is to build using q and d a supersolution of the scaled Allen-Cahn equation. For this let us take constants $\alpha, \beta > 0$ (to be selected later) and write

$$(3.14) \qquad w^\varepsilon(x, t) \equiv q\left(\frac{\eta(d(x, t)) + \alpha t}{\varepsilon}\right) + \varepsilon\beta, \quad x \in R^n, \quad 0 \leq t \leq t^*.$$

Since the cut-off function η depends on the parameter δ, so does the above function w^ε. For notational simplicity, however, we suppress this dependence in the notation.

THEOREM 3.2. *There exist constants $\alpha = \alpha(\delta) > 0$, $\beta = \beta(\delta) > 0$, and $\varepsilon_0 = \varepsilon_0(\delta) > 0$ such that*

$$(3.15) \qquad w_t^\varepsilon - \Delta w^\varepsilon + \frac{1}{\varepsilon^2}f(w^\varepsilon) \geq 0 \qquad in \qquad R^n \times (0, t^*]$$

for all $0 < \varepsilon \leq \varepsilon_0$. In addition $\alpha, \beta = 0(\delta)$ as $\delta \to 0$.

Proof:

1. As usual choose $\phi \in C^\infty(R^n \times (0, \infty))$ and suppose

$$(3.16) \qquad w^\varepsilon - \phi \text{ has a minimum at } (x_0, t_0) \in R^n \times (0, t^*]$$

with

$$(3.17) \qquad w^\varepsilon - \phi = 0 \qquad \text{at} \qquad (x_0, t_0).$$

We must demonstrate

(3.18) $$\phi_t - \Delta\phi + \frac{1}{\varepsilon^2} f(\phi) \geqq 0 \quad \text{at} \quad (x_0, t_0) \,,$$

provided ε is sufficiently small, depending only on δ but not on ϕ.

2. Write

$$q^{-1}(z) \equiv \frac{1}{2} \log\left(\frac{1+z}{1-z}\right) \,, \qquad -1 < z < 1$$

and set $\phi(x,t) \equiv \varepsilon q^{-1}(\phi(x,t) - \varepsilon\beta)$. This function is defined near (x_0, t_0), since $-1 < \phi(x_0, t_0) - \varepsilon\beta = q\left(\frac{\eta(d)+\alpha t}{\varepsilon}\right) < 1$. Owing to (3.4), (3.14), (3.16), (3.17)

(3.19) $$\eta(d) - (\psi - \alpha t) \text{ has a minimum at } (x_0, t_0) \,,$$

with $\eta(d) - (\psi - \alpha t) = 0$ at (x_0, t_0).

According to Lemma 3.1 we have

(3.20) $$\psi_t - \Delta\psi \geqq \alpha - \frac{C}{\delta} \quad \text{at} \quad (x_0, t_0) \,,$$

and

(3.21) $$\psi_t - \Delta\psi \geqq \alpha \quad \text{at} \quad (x_0, t_0) \,, \quad \text{if} \quad d(x_0, t_0) > \frac{\delta}{2} \,.$$

3. Since

$$\phi \equiv q\left(\frac{\psi}{\varepsilon}\right) + \varepsilon\beta \,,$$

we can compute

(3.22)
$$\phi_t - \Delta\phi + \frac{1}{\varepsilon^2} f(\phi)$$
$$= \frac{1}{\varepsilon} q'\left(\frac{\psi}{\varepsilon}\right)(\psi_t - \Delta\psi) - \frac{1}{\varepsilon^2} q''\left(\frac{\psi}{\varepsilon}\right)|D\psi|^2 + \frac{1}{\varepsilon^2} f\left(q\left(\frac{\psi}{\varepsilon}\right) + \varepsilon\beta\right)$$
$$= \frac{1}{\varepsilon} q'\left(\frac{\psi}{\varepsilon}\right)(\psi_t - \Delta\psi) + \frac{1}{\varepsilon^2} q''\left(\frac{\psi}{\varepsilon}\right)\left(1 - |D\psi|^2\right) + \frac{1}{\varepsilon^2}\left[f\left(q\left(\frac{\psi}{\varepsilon}\right)\right.\right.$$
$$\left.\left. + \varepsilon\beta\right) - f\left(q\left(\frac{\psi}{\varepsilon}\right)\right)\right]$$

at the point (x_0, t_0). We utilized the ordinary differential equation (3.3) to derive the last equality.

We now must estimate the various terms in (3.22).

Case 1: $d(x_0, t_0) > \frac{\delta}{2}$.

In this situation $d > \delta/2$ near (x_0, t_0) and so $\eta(d) = d - \delta$ near (x_0, t_0). Then (3.19) implies

$$|D\psi(x_0, t_0)| = 1 ,$$

as in (2.20). Thus (3.21) and (3.22) yield

$$(3.23) \quad \begin{aligned} \phi_t - \Delta\phi + \frac{1}{\varepsilon^2} f(\phi) &\geq q'\left(\frac{\psi}{\varepsilon}\right)\frac{\alpha}{\varepsilon} + \frac{f'(q(\psi/\varepsilon))\varepsilon\beta + O(\varepsilon^2)}{\varepsilon^2} \\ &= \frac{1}{\varepsilon}\left[q'\left(\frac{\psi}{\varepsilon}\right)\alpha + f'\left(q\left(\frac{\psi}{\varepsilon}\right)\right)\beta\right] + O(1) . \end{aligned}$$

Fix $0 < \gamma < 1$ so that

$$\inf_{\gamma \leq |z| \leq 1} f'(z) \equiv a_1 > 0 .$$

Then set

$$\inf_{|q(s)| \leq \gamma} q'(s) \equiv a_2 > 0 ,$$

and define

$$(3.24) \qquad \alpha = \frac{\delta}{4t^*} , \qquad \beta = a_2\alpha\left[2\|f'\|_{L^\infty((-1,1))}\right]^{-1} .$$

(We shall need these explicit choices to handle Case 2, below.) There are now two possibilities:

Subcase 1: $|q(\frac{\psi}{\varepsilon})| \geq \gamma$.

Then (3.23) implies

$$\phi_t - \Delta\phi + \frac{1}{\varepsilon^2} f(\phi) \geq \frac{a_1\beta}{\varepsilon} + O(1) \geq 0 \quad \text{at} \quad (x_0, t_0)$$

if ε is small enough, depending on δ.

Subcase 2: $|q(\frac{\psi}{\varepsilon})| \leq \gamma$.

Then (3.23) implies

$$\begin{aligned} \phi_t - \Delta\phi + \frac{1}{\varepsilon^2} f(\phi) &\geq \frac{1}{\varepsilon}\left[a_2\alpha - \|f'\|_{L^\infty}\beta\right] + O(1) \\ &= \frac{a_2\alpha}{2\varepsilon} + O(1) \geq 0 \quad \text{at} \quad (x_0, t_0) \end{aligned}$$

for small ε, depending on δ.

Both subcases therefore yield (3.18).

Case 2: $d(x_0, t_0) \leqq \frac{\delta}{2}$.

We use the same choices of α and β as in the previous case. In this situation $\eta(d) \leqq -\delta/2$ and so

$$\eta(d) + \alpha t_0 \leqq -\frac{\delta}{2} + \alpha t^* \leqq -\frac{\delta}{4} \, ,$$

according to (3.24). Hence (3.19) yields the inequality

(3.25) $$\psi \leqq -\frac{\delta}{4} \quad \text{at} \quad (x_0, t_0) \, .$$

Statement (3.19) and the definition (3.5) of η imply also $|D\psi| \leq C$ at (x_0, t_0).
We then compute utilizing (3.20), (3.22)

(3.26)
$$\phi_t - \Delta\phi + \frac{1}{\varepsilon^2} f(\phi) \geqq \frac{1}{\varepsilon} \left[q'\left(\frac{\psi}{\varepsilon}\right)\alpha + f'\left(q\left(\frac{\psi}{\varepsilon}\right)\right)\beta \right]$$
$$+ O(1) - \frac{C}{\varepsilon\delta} q'\left(\frac{\psi}{\varepsilon}\right) - \frac{C}{\varepsilon^2}\left| q''\left(\frac{\psi}{\varepsilon}\right)\right| \, .$$

But since $q'' \geq 0$ on $(-\infty, 0]$, (3.25) and (3.4) force

$$\frac{C}{\varepsilon\delta} q'\left(\frac{\psi}{\varepsilon}\right) \leqq \frac{C}{\varepsilon\delta} q'\left(-\frac{\delta}{4\varepsilon}\right) \leqq \frac{C}{\varepsilon\delta} \exp\{-\delta/2\varepsilon\} = o(1) \quad \text{as} \quad \varepsilon \to 0 \, .$$

Similarly

$$\frac{C}{\varepsilon^2}\left| q''\left(\frac{\psi}{\varepsilon}\right)\right| \leqq \frac{C}{\varepsilon^2} \exp\{-\delta/2\varepsilon\} = o(1) \quad \text{as} \quad \varepsilon \to 0 \, .$$

We analyze the remaining terms on the right-hand side of (3.26) as in the two subcases of Case 1.
The conclusion is

$$\phi_t - \Delta\phi + \frac{1}{\varepsilon^2} f(\phi) \geqq 0 \quad \text{at} \quad (x_0, t_0)$$

for all $0 < \varepsilon \leq \varepsilon_0(\delta)$, $\varepsilon_0(\delta)$ sufficiently small. As the constant appearing in the above argument is independent of ϕ, the choice of $\varepsilon_0(\delta)$ does not depend on ϕ.

4. Asymptotics for the Allen-Cahn Equation

We at last turn to the scaled Allen-Cahn equation

(4.1$_\varepsilon$)
$$\begin{cases} v_t^\varepsilon - \Delta v^\varepsilon + \dfrac{1}{\varepsilon^2} f(v^\varepsilon) = 0 & \text{in} \quad \mathbb{R}^n \times (0, \infty) \\ v^\varepsilon = h^\varepsilon & \text{on} \quad \mathbb{R}^n \times \{t = 0\} \, , \end{cases}$$

the cubic f given by (3.2) and the initial function h^ε described below.

We intend to prove $v^\varepsilon \to 1$ in a region $I \in R^n \times [0, \infty)$, $v^\varepsilon \to -1$ in another region $O \subset R^n \times [0, \infty)$, the "interface" Γ between I (the "inside") and O (the "outside") being a generalized motion governed by mean curvature.

To induce this behavior, however, we must choose special initial functions. More specifically, let Γ_0 henceforth denote the smooth boundary of a bounded, connected open set $U \subset R^n$. Let d_0 be the signed distance function to Γ_0, and set

$$(4.2) \qquad h^\varepsilon(x) \equiv q\left(\frac{d_0(x)}{\varepsilon}\right), \qquad x \in R^n .$$

Thus h^ε is approximately equal to 1 within U, is approximately equal to -1 within $R^n \setminus \overline{U}$, and has a transition layer of width $O(\varepsilon)$ across the surface Γ_0. Moreover, by the maximum principle, $-1 < v_\varepsilon < 1$ in $R^n \times [0, \infty)$. The analysis of more general initial functions is given in [5].

We shall show that v^ε roughly maintains this form at later times, the transition layer forming across the generalized motion by mean curvature starting with Γ_0. To this end, we choose a continuous function Γ_t, I_t, O_t, I, O by (2.4), (2.28)–(2.31).

THEOREM 4.1. *We have*

$$(4.3) \qquad v^\varepsilon \to 1 \quad uniformly\ on\ compact\ subsets\ of \quad I$$

and

$$(4.4) \qquad v^\varepsilon \to -1 \quad uniformly\ on\ compact\ subsets\ of \quad O .$$

Remark. Assertions (4.3), (4.4) provide a great deal, but by no means all, of the desired information about the limiting behavior of the $\{v^\varepsilon\}_{\varepsilon>0}$. We note in particular it *is not known whether the "interface" Γ can develop an interior:* see the discussion following in Section 5. It would also be interesting to see to what extent other initial data (not having the form (4.2)) could be handled.

Proof:

1. As Γ_0 is smooth, we may choose g to be smooth, with $|Dg| = 1$ near Γ_0. Thus if $\delta > 0$ is small enough the set

$$(4.5) \qquad \Gamma_0^\delta \equiv \{x \in R^n \mid g(x) = d_0(x) = -2\delta\}$$

is smooth. We let

$$(4.6) \qquad \Gamma_t^\delta = \{x \in R^n \mid u(x, t) = -2\delta\} , \qquad t \geqq 0$$

be the generalized evolution starting with Γ_0^δ, and take d^δ to denote the signed distance function to Γ_t^δ, d_0^δ being the signed distance function to Γ_0^δ. Let t_δ^* be the extinction time for $\{\Gamma_t^\delta\}_{t \geq 0}$.

Choose $\eta(\cdot)$ as in Section 3 and set

$$(4.7) \qquad w^{\varepsilon,\delta}(x,t) \equiv q\left(\frac{\eta\left(d^\delta(x,t)\right) + \alpha t}{\varepsilon}\right) + \varepsilon\beta \,,$$

α and β are given by (3.24), with t_δ^* replacing t^*. Then for $0 < \varepsilon < \varepsilon_0(\delta)$ we have

$$(4.8) \qquad w_t^{\varepsilon,\delta} - \Delta w^{\varepsilon,\delta} + \frac{1}{\varepsilon^2} f(w^{\varepsilon,\delta}) \geq 0 \quad \text{in} \quad R^n \times (0, t_\delta^*) \,.$$

2. We first claim

$$(4.9) \qquad w^{\varepsilon,\delta}(x, 0) \geq h^\varepsilon(x) \,, \qquad x \in R^n \,.$$

To verify this inequality it suffices in view of (4.2) to prove

$$\eta\left(d_0^\delta(x)\right) \geq d_0(x) \,, \qquad x \in R^n \,.$$

Now owing to (4.5) $d_0^\delta(x) \geq d_0(x) + 2\delta$; and so $\eta(d_0^\delta(x)) \geq \eta(d_0(x) + 2\delta)$ $(x \in R^n)$. It is therefore enough to show

$$(4.10) \qquad d_0(x) \leq \eta(d_0(x) + 2\delta) \,, \qquad x \in R^n \,.$$

But if $d_0(x) \geq -3\delta/2$, then $d_0(x) + 2\delta \geq \delta/2$; whence

$$\eta(d_0(x) + 2\delta) = d_0(x) + \delta \geq d_0(x) \,.$$

On the other hand, if $d_0(x) \leq -(3/2)\delta$, (4.10) is obvious as $\eta \geq -\delta$.

3. Now write

$$(4.11) \qquad w \equiv e^{-\lambda t} w^{\varepsilon,\delta} \,, \qquad \lambda > 0 \,.$$

We next claim

$$(4.12) \qquad w_t - \Delta w + \lambda w + \frac{e^{-\lambda t}}{\varepsilon^2} f\left(e^{\lambda t} w\right) \geq 0 \quad \text{in} \quad R^n \times (0, t_\delta^*] \,.$$

To check this, select as always $\phi \in C^\infty(R^n \times (0, \infty))$ and assume

$$w - \phi \text{ has a minimum at a point } (x_0, t_0) \in \mathbb{R}^n \times (0, t_\delta^*]$$

with $w - \phi = 0$ at (x_0, t_0). Then

$$e^{-\lambda t} w^{\varepsilon, \delta} = w \geqq \phi \quad \text{in} \quad R^n \times (0, t_\delta^*] \, ,$$

with equality at (x_0, t_0). Hence

$$w^{\varepsilon, \delta} \geqq \psi \quad \text{in} \quad R^n \times (0, t_\delta^*] \, ,$$

with equality at (x_0, t_0), for $\psi \equiv e^{\lambda t} \phi$. Assertion (4.8) then implies

$$\psi_t - \Delta \psi + \frac{1}{\varepsilon^2} f(\psi) \geqq 0 \quad \text{at} \quad (x_0, t_0) \, .$$

We rewrite the last inequality to read

$$\phi_t - \Delta \phi + \lambda \phi + \frac{e^{-\lambda t}}{\varepsilon^2} f(e^{\lambda t} \phi) \geqq 0 \quad \text{at} \quad (x_0, t_0) \, .$$

This establishes (4.12).

4. We hereafter set

$$\lambda = \lambda_\varepsilon \equiv \frac{2 \| f' \|_{L^\infty((-1,1))}}{\varepsilon^2} \, .$$

Then for each t the mapping

(4.13) $z \mapsto \lambda z + \dfrac{e^{-\lambda t}}{\varepsilon^2} f(e^{\lambda t} z) \quad$ is strictly increasing.

5. We now assert

(4.14) $$w^{\varepsilon, \delta} \geqq v^\varepsilon \quad \text{in} \quad R^n \times [0, t_\delta^*] \, .$$

Indeed if not, then

$$w^{\varepsilon, \delta} < v^\varepsilon \quad \text{somewhere in} \quad R^n \times [0, t_\delta^*]$$

and consequently

$$w < v \quad \text{somewhere in} \quad R^n \times [0, t_\delta^*] \, ,$$

for $w = e^{-\lambda t} w^{\varepsilon, \delta}$, $v = e^{-\lambda t} v^\varepsilon$. The function w is lower semicontinuous. In addition

$$w \geqq v \quad \text{on} \quad R^n \times [t = 0] \, .$$

Hence, by perturbing w if necessary, we may assume that there exists a point $(x_0, t_0) \in R^n \times (0, t_\delta^*]$ such that

$$(4.15) \qquad (w - v)(x_0, t_0) = \min_{R^n \times [0, t_\delta^*]} (w - v) \equiv b < 0 \ .$$

Indeed such a point always exists because

$$\lim_{|x| \to \infty} w \geqq e^{-\lambda t}(-1 + \varepsilon \beta) > \lim_{|x| \to \infty} v = -e^{-\lambda t} \ .$$

But we do not need this exact characterization of v later.

Now (4.1_ε) yields

$$(4.16) \qquad v_t - \Delta v + \lambda v + \frac{e^{-\lambda t}}{\varepsilon^2} f\left(e^{\lambda t} v\right) = 0 \quad \text{in} \quad R^n \times (0, \infty) \ .$$

If

$$(4.17) \qquad \phi \equiv v + b \ ,$$

then $\phi \in C^\infty(R^n \times [0, \infty))$ and (4.15) says

$$w - \phi \quad \text{has a minimum at} \quad (x_0, t_0)$$

with $w - \phi = 0$ at (x_0, t_0). According to Step 3 above, we conclude

$$(4.18) \qquad \phi_t - \Delta \phi + \lambda \phi + \frac{e^{-\lambda t}}{\varepsilon^2} f\left(e^{\lambda t} \phi\right) \geqq 0$$

at (x_0, t_0). However since $b < 0$, $\phi < v$. Consequently (4.13), (4.17), (4.18) imply

$$v_t - \Delta v + \lambda v + \frac{e^{-\lambda t}}{\varepsilon^2} f\left(e^{\lambda t} v\right) > 0$$

at (x_0, t_0). This contradicts (4.16) and thereby proves (4.14).

6. Utilizing (4.14) and the definition (4.7) of the auxiliary function $w^{\varepsilon, \delta}$, we discover

$$(4.19) \qquad q\left(\frac{\eta\left(d^\delta(x, t)\right) + \alpha t}{\varepsilon}\right) + \varepsilon \beta \geqq v^\varepsilon(x, t)$$

for $x \in R^n$, $0 \leqq t \leqq t_\delta^*$. Now if

$$\eta\left(d^\delta(x, t)\right) + \alpha t \leqq -\delta + \alpha t_\delta^*$$

$$\leqq -\frac{3}{4}\delta \quad \text{by (3.24) (with } t_\delta^* \text{ replacing } t^*) \ .$$

Thus

$$\lim_{\varepsilon \to 0} q \left(\frac{\eta \left(d^{\delta}(x,t) \right) + \alpha t}{\varepsilon} \right) + \varepsilon \beta = -1 \;.$$

In view of (4.19) we have

$$\lim_{\varepsilon \to 0} v^{\varepsilon}(x,t) = -1 \;,$$

uniformly on $O^{\delta} \equiv \{(x,t) \in R^n \times [0, t_{\delta}^*] \,|\, u(x,t) < -2\delta\}$, for sufficiently small δ. In particular,

(4.20) $$\lim_{\varepsilon \to 0} v^{\varepsilon}(x,t) = -1$$

uniformly on compact subsets of O^{δ} for sufficiently small δ. Since

$$O = \bigcup_{\delta > 0} O^{\delta} \;,$$

the proof of (4.4) is now complete. A similar argument proves (4.3).

5. Uniqueness?

In this concluding section we elaborate upon the remark following Theorem 4.1. Let us return to the scaled Allen-Cahn partial differential equation and calculate the time derivative of the scaled excess free energy:

$$\frac{d}{dt} \int_{R^n} \left[\frac{\varepsilon}{2} |Dv^{\varepsilon}|^2 + \frac{1}{\varepsilon} F(v^{\varepsilon}) \right] dx = \int_{R^n} \left[\varepsilon Dv^{\varepsilon} \cdot Dv_t^{\varepsilon} + \frac{1}{\varepsilon} f(v^{\varepsilon}) v_t^{\varepsilon} \right] dx$$

$$= \int_{R^n} v_t^{\varepsilon} \left(-\varepsilon \Delta v^{\varepsilon} + \frac{1}{\varepsilon} f(v^{\varepsilon}) \right) dx$$

$$= -\varepsilon \int_{R^n} (v_t^{\varepsilon})^2 \, dx \leq 0 \;.$$

Thus

(5.1)
$$\sup_{0 \leq t \leq T} \int_{R^n} \left[\frac{\varepsilon}{2} |Dv^{\varepsilon}|^2 + \frac{1}{\varepsilon} F(v^{\varepsilon}) \right] dx + \varepsilon \int_0^T \int_{R^n} (v_t^{\varepsilon})^2 \, dx \, dt$$

$$\leq \int_{R^n} \left[\frac{\varepsilon}{2} |Dh^{\varepsilon}| + \frac{1}{\varepsilon} F(h^{\varepsilon}) \right] dx \leq C < \infty \;,$$

in view of the special form (4.2) for the initial function h^ε. Since this inequality implies

$$\int_0^T \int_{R^n} F(v^\varepsilon)\, dx\, dt \leqq O(\varepsilon)$$

as $\varepsilon \to 0$ for each $T > 0$, we deduce

(5.2)
$$(v^\varepsilon)^2 \to 1 \text{ a.e. in } R^n \times [0, \infty)\,.$$

In addition, if we set $G(z) \equiv (z^3/3) - z$ and write

$$\tilde{v}^\varepsilon = G(v^\varepsilon)\,,$$

we have (cf. Bronsard and Kohn, [8])

$$\int_{R^n} |D\tilde{v}^\varepsilon|\, dx = \int_{R^n} \left|(v^\varepsilon)^2 - 1\right| |Dv^\varepsilon|\, dx$$

$$\leqq \int_{R^n} \left[\frac{\varepsilon}{2}|Dv^\varepsilon|^2 + \frac{F(v^\varepsilon)}{\varepsilon}\right]\, dx \leqq C < \infty$$

and

$$\int_0^T \int_{R^n} |\tilde{v}_t^\varepsilon|\, dx\, dt = \int_0^T \int_{R^n} \left|(v^\varepsilon)^2 - 1\right| |v_t^\varepsilon|\, dx\, dt$$

$$\leqq \int_0^T \int_{R^n} \left[\varepsilon (v_t^\varepsilon)^2 + \frac{1}{2\varepsilon}F(v^\varepsilon)\right]\, dx\, dt \leqq CT < \infty\,.$$

Thus $\{\tilde{v}^\varepsilon\}_{\varepsilon > 0}$ is bounded in $BV(R^n \times (0, T))$ for each $T > 0$, and so is precompact in $L^1_{\text{loc}}(R^n \times (0, T))$. It follows that $\{v^\varepsilon\}_{\varepsilon > 0}$ is precompact in $L^1_{\text{loc}}(R^n \times (0, T))$. Consequently, passing if necessary to a subsequence, we have

(5.3)
$$v^{\varepsilon_j} \to \pm 1 \quad \text{in} \quad R^n \times [0, \infty)\,.$$

Our Theorem 4.1 augments this simple fact with the assertion

(5.4)
$$v^\varepsilon \to 1 \quad \text{in} \quad I\,, \qquad v^\varepsilon \to -1 \quad \text{in} \quad O\,.$$

However we do not know

$$\Gamma \equiv R^n \times [0, \infty) \setminus [I \cup 0]$$

has $(n+1)$-dimensional Lebesgue measure zero, and consequently (5.4) does *not* imply (5.2), (5.3). The problem is that the sets $\{\Gamma\}_{t \geq 0}$ could conceivably develop an interior for times $t^* \geqq t \geqq t_*$, t_* denoting the first time the

classical evolution by mean curvature has a singularity. See [21], Section 8, for an example of a nonsmooth 1-dimensional compact set $\Gamma_0 \subset R^2$ for which Γ_t has an interior for times $t > 0$.

On the other hand, Evans and Spruck (see [23]) have recently proved for smooth Γ_0 that

$$H^{n-1}(\Gamma_t^*) < \infty , \qquad t \geqq 0 ,$$

where H^{n-1} is $(n-1)$-dimensional Hausdorff measure and $\Gamma_t^* = \partial \Gamma_t$. Thus Γ_t has positive n-dimensional Lebesgue measure if and only if Γ_t has an interior. Finally, Barles, Soner, and Souganidis (see [6]) give a general, but by no means sharp, geometric condition which guarantees no interior.

Now if in fact $\mathrm{int}(\Gamma_t) \neq \varnothing$ in R^n for some time $t_* \leqq t \leqq t^*$, then $\mathrm{int}(\Gamma) \neq \varnothing$ in $R^n \times [0, \infty)$. In this case assertion (5.3) tells us that for some subsequence $v^{\varepsilon_j} \to \pm 1$ a.e. within Γ, whereas (5.4) provides no information at all regarding v^ε inside Γ.

Should this be possible, it seems most likely that the regions when $v^{\varepsilon_j} \to 1$ and $v^{\varepsilon_j} \to -1$ would be separated by an "interface" evolving by mean curvature in the sense of Soner; see [44]. Such a motion is generally nonunique. And perhaps different subsequences correspond to different interfaces, or the initial profile picks the particular interface to which the solutions converge. At present it is unclear whether these circumstances can arise and, if so, how the solutions v^ε of the scaled Allen-Cahn equation would behave within the interior of Γ.

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