## A FREE BOUNDARY PROBLEM RELATED TO SINGULAR STOCHASTIC CONTROL: THE PARABOLIC CASE

H.M. Soner ${ }^{1}$<br>S.E. Shreve ${ }^{2}$<br>Department of Mathematics Carnegie Mellon University Pittsburgh, PA 15213

## 1. INTRODUCTION

This paper concerns the existence of classical solutions to the nonlinear partial differential equations.
(1.1) $\max \left\{\frac{\partial}{\partial \mathrm{t}} \mathrm{u}(\mathrm{x}, \mathrm{t})-\Delta \mathrm{u}(\mathrm{x}, \mathrm{t})-\mathrm{h}(\mathrm{x}, \mathrm{t}), \frac{\partial}{\partial \mathrm{x}_{\mathrm{n}}} \mathrm{u}(\mathrm{x}, \mathrm{t})-\mathrm{f}(\mathrm{t})\right\}=0, \mathrm{x} \in \mathbb{R}^{\mathrm{n}}, \mathrm{t}>0$, with a forcing term h which is convex in the $x_{n}$-variable. Under appropriate smoothness and growth conditions on the data, we prove the existence and the uniqueness of polynomially growing, positive, classical solutions to (1.1) for every initial condition

$$
\begin{equation*}
\mathrm{u}(\mathrm{x}, 0)=\mathrm{g}(\mathrm{x}), \mathrm{x} \in \mathrm{R}^{\mathrm{n}} \tag{1.2}
\end{equation*}
$$

which is also convex in the $x_{n}$-variable. Moreover, we obtain the Lipschitz
continuity of the free boundary of the region in which the parabolic equation $u_{t}-\Delta u-h=0$ holds.

The $\mathrm{W}_{10 \mathrm{C}}^{2, \mathrm{~m}}$ regularity in the spatial variables and the boundedness of the time derivative was proved by Chow, Menaldi, Robin [6], Menaldi, Robin [22] and Menaldi, Taksar [23]. They used control-theoretic techniques along with the convexity assumption. Also, the stationary version of (1.1) was recently studied by the authors [27], and part of the present analysis closely follows [27]. As in [27] our approach to (1.1) is to first solve the obstacle problem
(1.3) $\max \left\{\frac{\partial}{\partial t} v(x, t)-\Delta v(x, t)-\frac{\partial}{\partial x_{n}} h(x, t), v(x, t)-f(t)\right\}=0, x \in \mathbb{R}^{n}, t>0$,
with initial condition

$$
\begin{equation*}
\mathrm{v}(\mathrm{x}, 0)=\frac{\partial}{\partial \mathrm{x}_{\mathrm{n}}} \mathrm{~g}(\mathrm{x}), \mathrm{x} \in \mathbb{R}^{\mathrm{n}} \tag{1.4}
\end{equation*}
$$

The convexity assumption on the data enables us to show that

$$
\begin{equation*}
\mathrm{v}(\mathrm{x}, \mathrm{t})=\frac{\partial}{\partial \mathrm{x}_{\mathrm{n}}} \mathrm{u}(\mathrm{x}, \mathrm{t}), \tag{1.5}
\end{equation*}
$$

and we construct $u$ by integrating the above relation. Known regularity results for (1.3), ([3],[9],[11]), with several estimates of the free boundary, yield $u \in C^{2,1}\left(\mathbb{R}^{n} \times(0, \infty)\right)$ (twice continuously differentiable in the spatial variables and once continuously differentiable in the time variable). In the context of one-dimensional stochastic control the connection (1.5) goes back to Bather and Chernoff [2] and has been given probabilistic explanations by

Karatzas and Shreve [19], El-Karoui and Karatzas [7], and analytical derivations by Karatzas [16], Chow, Menaldi and Robin [6], Menaldi and Robin [22]. Without the convexity assumptions, (1.5) is no longer true, and in general (1.1) does not have classical solutions.

The related elliptic problem

$$
\begin{equation*}
\max \{u(x)-\Delta u(x)-h(x),|\nabla u(x)|-1\}=0 \tag{1.6}
\end{equation*}
$$

was studied on a bounded domain by Evans [8] and then by Ishii and Koike [15]. Evans obtained solutions in $\mathrm{W}_{\text {loc }}^{2, \infty}$ via penalization and the maximum principle. In fact this regularity result is sharp in the absence of convexity. However, for (1.6) in two dimensions with a convex forcing term $h$, the authors recently obtained a classical solution [28]. Due to the nonlinearity of the constraint in (1.6), the obstacle problem related to it is more complicated than (1.3) and techniques in [28] are different from the ones employed here and in [27].

Equation (1.1) is the dynamic programming equation related to a singular stochastic control problem. Briefly, the problem is to optimally control an n-dimensional Brownian motion by pushing only along the $(0,0, . .,-1)$ direction. In this context, the solution to (1.1) and (1.2) is the value function for the finite horizon control problem in which $h$ is the running cost, $g$ is the terminal cost and, at time $t, f(t)$ times the displacement caused by the push is equal to its cost. This problem is formulated and solved in Section 7. The reader may refer to Shreve [25] for an introduction to this kind of control problems.

In singular stochastic control literature, the spatial $C^{2}$ regularity of the value function has been called the "principle of smooth fit" by Benes,

Shepp and Witsenhausen [1]. It has been instrumental in the analysis of several one-dimensional problems [7], [12], [13], [14], [17], [18], [19], [26].

The paper is organized as follows: results related to (1.3) are stated in the next section and the Lipschitz continuity and the boundedness of the free boundary are obtained in Sections 3 and 4. Section 5 is devoted to the construction of a classical solution to (1.1) and its uniqueness is proved in Section 6. The related singular stochastic control problem is defined and solved in Section 7. Finally, we analyze a penalization of (1.3) in the Appendix.

## 2. AN OBSTACLE PROBLEM

In this section we study the solutions to equation (1.3) with initial data (1.4). Subscripts $i$ and $t$ denote the differentiation with respect to $x_{i}$ and $t$, respectively. We assume
$h, g$, and $f$ are three times differentiable, non-negative with $\mathrm{f}(\mathrm{t}) \geq 1$ for all $\mathrm{t} \geq 0$. Moreover, these functions, together with their derivatives up to order three, grow at most polynomially as $|\mathrm{x}|$ and $t$ tend to infinity;
there is an $\alpha>0$ such that

$$
g_{n n}(x) \geq \alpha \sum_{i=1}^{n-1}\left|g_{n i}(x)\right|
$$

and

for every $\mathrm{x} \in \mathbb{R}^{\mathrm{n}}, \mathrm{t} \geq 0$; and

$$
\begin{equation*}
g_{n}(x) \leq f(0), \lim _{x_{n} \rightarrow-\infty} g_{n}(x)<f(0) \tag{2.3}
\end{equation*}
$$

for every $\mathbf{x} \in \mathbb{R}^{\mathbf{n}}$.

Theorem 2.1 There is a unique solution v to (1.9) and (1.4) satisfying
(2.4) $\quad \sum_{i, j=1}^{n}\left[|v(x, t)|+\left|v_{i}(x, t)\right|+\left|v_{i j}(x, t)\right|+\left|v_{t}(x, t)\right|\right] \leq K\left(1+|x|^{m}+t^{m}\right)$
with suitable positive constants $K, m$. Henceforth we shall use v to denote the solution of (1.3) and (1.4) satisfying (2.4).

The above regularity result of solutions to (1.3) and (1.4) is now standard in the nonlinear partial differential equations literature. A similar result for the one phase Stefan problem was obtained by Friedman and Kinderlehrer [11], and a modification of their proof yields the above result. Also see [3], [4], [5], [10], [29], [30]. For the sake of completeness, we give the proof in the appendix. To establish notation, we begin the proof here.

Consider the following penalized version of (1.3)
$(2.5)^{\varepsilon}$

$$
v_{t}^{\epsilon}(x, t)-\Delta v^{\epsilon}(x, t)+\beta_{\epsilon}\left(v^{\epsilon}(x, t)-f(t)\right)=h_{n}(x, t)
$$

The penalization term $\beta_{\epsilon}$ is given by $\beta_{\epsilon}(\mathrm{r})=\beta(\mathrm{r} / \epsilon)$ for some smooth
function $\beta$ satisfying

| $(2.6)(\mathrm{i})$ | $\beta(\mathrm{r})=0$, | $\forall \mathrm{r} \leq 0 ;$ |
| :--- | :--- | :--- |
| $(2.6)$ (ii) | $\beta(\mathrm{r})>0$, | $\forall \mathrm{r}>0 ;$ |
| $(2.6)$ (iii) | $\beta(\mathrm{r})=\mathrm{r}-1$, | $\forall \mathrm{r} \geq 2 ;$ |
| $(2.6)(\mathrm{iv})$ | $\beta^{\prime}(\mathrm{r}) \geq 0, \quad \beta^{\prime \prime}(\mathrm{r}) \geq 0$, | $\forall \mathrm{r} \in \mathbb{R}$. |

Let $\mathrm{v}^{\epsilon}$ be the solution to (2.5) ${ }^{\epsilon}$ with initial data (1.4). The following theorem is proved in the Appendix.

Theorem 2.2. There are positive constants $K, m$, independent of $\epsilon$, such that $\mathrm{v}^{\epsilon}$ satisfies (2.4) with these constants. Moreover $\mathrm{v}^{\epsilon}$ converges to v uniformly on bounded sets.

## 3. LIPSCHITZ CONTINUITY OF THE FREE BOUNDARY.

In this section we study the spatial boundary of the region

$$
\begin{equation*}
\mathscr{B} \triangleq\left\{(\mathrm{x}, \mathrm{t}) \in \mathbb{R}^{\mathrm{n}} \times(0, \infty): \mathrm{v}(\mathrm{x}, \mathrm{t})<\mathrm{f}(\mathrm{t})\right\} . \tag{3.1}
\end{equation*}
$$

We discover, in Section 4, that for fixed ( $\mathrm{x}_{1}, . ., \mathrm{x}_{\mathrm{n}-1}$ ) and $\mathrm{t}>0$, the region $\mathscr{B}$ is bounded above in the $\mathrm{x}_{\mathrm{n}}$-coordinate. However, this upper bound may approach infinity as $\mathfrak{t}$ tends to zero. We show here that the boundary of $\mathscr{C}$ admits a lipschitz continuous parametrization. Our method is to obtain a uniform Lipschitz estimate for parametrizations of the boundaries of an approximating sequence of regions

$$
\begin{equation*}
\mathscr{C}_{\epsilon} \triangleq\left\{(\mathrm{x}, \mathrm{t}) \in \mathbb{R}^{\mathrm{n}} \times(0, \infty): \mathrm{v}^{\mathcal{E}}(\mathrm{x}, \mathrm{t})<\mathrm{f}(\mathrm{t})\right\} . \tag{3.2}
\end{equation*}
$$

Lemma 3.1. For $\epsilon>0$, there is a continuously differentiable function $q^{\epsilon}: \mathbb{R}^{\mathrm{n}-1} \times(0, \infty) \rightarrow \mathbb{R}$ such that

$$
\begin{equation*}
\mathscr{C}_{\epsilon}=\left\{\left(\mathrm{y}, \mathrm{x}_{\mathrm{n}}, \mathrm{t}\right) \in \mathbb{R}^{\mathrm{n}-1} \times \mathbb{R} \times(0, \infty): \mathrm{x}_{\mathrm{n}}<\mathrm{q}^{\epsilon}(\mathrm{y}, \mathrm{t})\right\} \tag{3.3}
\end{equation*}
$$

Proof. Differentiate $(2.5)^{\epsilon}$ with respect to $X_{n}$ to obtain

$$
\begin{equation*}
v_{n t}^{\epsilon}(x, t)+\beta_{\epsilon}^{\prime}\left(v^{\epsilon}(x, t)-f(t)\right) v_{n}^{\epsilon}(x, t)-\Delta v_{n}^{\epsilon}(x, t)=h_{n n}(x, t) \tag{3.4}
\end{equation*}
$$

Since $\beta_{\epsilon}^{\prime}$ is bounded, $\beta_{\epsilon}^{\prime}$ and the initial condition $v_{n}^{\epsilon}(x, 0)=g_{n n}(x)$ are non-negative and $h_{n n} \geq \alpha>0$, the maximum principle yields that for each $\mathrm{t}>0, \mathrm{v}_{\mathrm{n}}^{\epsilon}(\mathrm{x}, \mathrm{t})$ is bounded below by a positive constant, uniformly in x . Thus $q^{\epsilon}(y, t) \triangleq \inf \left\{x_{n} \mid v^{\epsilon}\left(y, x_{n}, t\right) \geq f(t)\right\}$ is finite. Since the boundary of $\mathscr{E}_{\epsilon}$ in $\mathbb{R}^{n} \times(0, \infty)$ is the zero level curve of the function $v^{\epsilon}(x, t)-f(t)$, the differentiability of $q^{\epsilon}$ is a direct consequence of the implicit function theorem and the strict positivity of $v_{n}{ }^{\epsilon}$. $\square$

We proceed to obtain a uniform Lipschitz estimate of $q^{\epsilon}$. We need several estimates of the derivatives of $\mathrm{v}^{\epsilon}$.

Lemma 3.2. There is a positive constant $K$, independent of $\epsilon$, such that

$$
\begin{equation*}
\left|v_{i}^{\epsilon}(x, t)\right| \leq K \quad v_{n}^{\epsilon}(x, t) \tag{3.5}
\end{equation*}
$$

for every $x \in \mathbb{R}^{n}, t>0, i=1, \ldots, n-1$.

Proof. Differentiate $(2.5)^{\epsilon}$ with respect to $x_{i}$ :

$$
\begin{equation*}
\mathrm{v}_{\mathrm{it}}^{\epsilon}(\mathrm{x}, \mathrm{t})+\beta_{\epsilon}^{\prime}\left(\mathrm{v}^{\epsilon}(\mathrm{x}, \mathrm{t})-\mathrm{f}(\mathrm{t})\right) \mathrm{v}_{\mathrm{i}}^{\epsilon}(\mathrm{x}, \mathrm{t})-\Delta \mathrm{v}_{\mathrm{i}}^{\epsilon}(\mathrm{x}, \mathrm{t})=\mathrm{h}_{\mathrm{ni}}(\mathrm{x}, \mathrm{t}) \tag{3.6}
\end{equation*}
$$

Assumption (2.2) yields that $\left|v_{i}^{\epsilon}(x, 0)\right|=\left|g_{n i}(x)\right| \leq \frac{1}{\alpha} g_{n \mathrm{n}}(x)=\frac{1}{\alpha} v_{n}(x, 0)$. Also, $\left|h_{n i}(x, t)\right| \leq \frac{1}{\alpha} h_{n n}(x, t)$. Hence the maximum principle, together with (3.4) and (3.6), implies (3.5) with $K=1 / \alpha$.

Lemma 3.3. There are positive constants $K, m$, independent of $\epsilon$, such that

$$
\begin{equation*}
\left|v_{t}^{\epsilon}(x, t)-f^{\prime}(t)\right| \leq K \frac{e^{t}}{t}\left(1+|x|^{2 m}\right) v_{n}^{\epsilon}(x, t) \tag{3.7}
\end{equation*}
$$

for every $\mathrm{x} \in \mathbb{R}^{\mathrm{n}}, \mathrm{t}>0$.

Proof. Theorem 2.2 and assumption (2.1) yield the existence of a positive constant $C$ and an integer $m>1$ such that

$$
\begin{equation*}
\left|v_{t}^{\epsilon}(x, t)-f^{\prime}(t)\right| \leq e^{t}\left(C+\frac{1}{4}|x|^{2 m}\right) \tag{3.8}
\end{equation*}
$$

For $\mathrm{x}_{0} \in \mathbb{R}^{\mathrm{n}}$ we define an auxiliary function $\phi$ by

$$
\phi(x, t)=t\left(v_{t}^{\epsilon}(x, t)-f^{\prime}(t)\right)-e^{t}\left|x-x_{0}\right|^{2 m}
$$

It should be noted that $\phi$ actually depends on $x_{0}$ but this dependence is suppressed in the notation. We calculate directly that

$$
\begin{aligned}
I(x, t) & \triangleq \phi_{t}(x, t)+\beta_{\epsilon}^{\prime}(\cdots) \phi(x, t)-\Delta \phi(x, t)= \\
& =t\left[v_{t t}^{\epsilon}(x, t)-\Delta v_{t}^{\epsilon}(x, t)+\beta_{\epsilon}^{\prime}(\cdots)\left(v_{t}^{\epsilon}(x, t)-f^{\prime}(t)\right)\right]-t f^{\prime \prime}(t)
\end{aligned}
$$

$$
\begin{aligned}
& +e^{\mathrm{t}}\left[-\left|\mathrm{x}-\mathrm{x}_{0}\right|^{2 \mathrm{~m}}+2 \mathrm{~m}(2 \mathrm{~m}+\mathrm{n}-2)\left|\mathrm{x}-\mathrm{x}_{0}\right|^{2(\mathrm{~m}-1)}-\beta_{\epsilon}^{\prime}(\cdots)\left|\mathrm{x}-\mathrm{x}_{0}\right|^{2 \mathrm{~m}}\right] \\
& +\left[\mathrm{v}_{\mathrm{t}}^{\epsilon}(\mathrm{x}, \mathrm{t})-\mathrm{f}^{\prime}(\mathrm{t})\right]
\end{aligned}
$$

where $(\cdots)$ denotes $v^{\epsilon}(x, t)-f(t)$. Use (3.8), equation (2.5) ${ }^{\epsilon}$ and the positivity of $\beta_{\epsilon}^{\prime}(\cdots)$ to obtain

$$
\begin{aligned}
& I(x, t) \leq t\left[h_{n t}(x, t)-f^{\prime \prime}(t)\right] \\
& \quad+e^{t}\left[-\left|x-x_{0}\right|^{2 m}+2 m(2 m+n-2)\left|x-x_{0}\right|^{2(m-1)}\right] \\
& \quad+e^{t}\left(C+\frac{1}{4}|x|^{2 m}\right) \\
& =t\left[h_{n t}(x, t)-f^{\prime \prime}(t)\right] \\
& \\
& \quad+e^{t}\left[-\frac{1}{2}\left|x-x_{0}\right|^{2 m}+2 m(2 m+n-2)\left|x-x_{0}\right|^{2(m-1)}\right] \\
& \\
& \quad+\frac{1}{4} e^{t}\left[-2\left|x-x_{0}\right|^{2 m}+|x|^{2 m}\right]+C e^{t} .
\end{aligned}
$$

We estimate the above terms by using the assumption (2.2) and the inequalities

$$
\begin{equation*}
-\frac{1}{2} \mathrm{r}^{\mathrm{p}+2}+\frac{\mathrm{a}}{2} \mathrm{p}^{\mathrm{p}} \leq \frac{1}{\mathrm{p}}\left(\frac{\mathrm{ap}}{\mathrm{p}+2}\right)^{\frac{\mathrm{p}}{2}+1}, \quad \forall \mathrm{r}, \mathrm{p}, \mathrm{a} \geq 0 \tag{3.9}
\end{equation*}
$$

and

$$
-2\left|x-x_{0}\right|^{2 m}+|x|^{2 m} \leq\left|x_{0}\right|^{2 m} 2\left(2^{\frac{1}{2 m-1}}-1\right)^{-(2 m-1)} .
$$

The latter inequality is obtained by observing that its left-hand side is maximized over $x$ by $x=\left(1-2^{-\frac{1}{2 m-1}}\right)^{-1} x_{0}$. Hence, we have

$$
\mathrm{I}(\mathrm{x}, \mathrm{t}) \leq \frac{\mathrm{t}}{\alpha} \mathrm{~h}_{\mathrm{nn}}(\mathrm{x}, \mathrm{t})+\mathrm{e}^{\mathrm{t}}\left[\mathrm{C}_{\mathrm{m}}+\mathrm{B}_{\mathrm{m}}\left|\mathrm{x}_{0}\right|^{2 \mathrm{~m}}\right]
$$

where $\mathrm{C}_{\mathrm{m}}=\mathrm{C}+2^{2 \mathrm{~m}-1}(\mathrm{~m}-1)^{(\mathrm{m}-1)}(2 \mathrm{~m}+\mathrm{n}-2)^{\mathrm{m}}$ and $B_{m}=\frac{1}{2}\left(2^{\frac{1}{2 \mathrm{~m}-\mathrm{I}}}-1\right)^{-(2 \mathrm{~m}-1)}$.
Since $h_{n n}(x, t) \geq \alpha$, the above inequality yields

$$
\begin{align*}
I(x, t) & \leq \frac{1}{\alpha} h_{n n}(x, t)\left[t+\left(C_{m}+B_{m}\left|x_{0}\right|^{2 m}\right) e^{t}\right]  \tag{3.10}\\
& \leq h_{n n}(x, t) \frac{1}{\alpha}\left[T+\left(C_{m}+B_{m}\left|x_{0}\right|^{2 m}\right) e^{T}\right], \forall t \leq T
\end{align*}
$$

Recall that $\mathrm{I}(\mathrm{x}, \mathrm{t})=\phi_{\mathrm{t}}(\mathrm{x}, \mathrm{t})+\beta_{\epsilon}^{\prime}(\cdots) \phi(\mathrm{x}, \mathrm{t})-\Delta \phi(\mathrm{x}, \mathrm{t})$. Hence (3.4) shows that $\psi(\mathrm{x}, \mathrm{t}) \triangleq \phi(\mathrm{x}, \mathrm{t})-\frac{1}{\alpha}\left[\mathrm{~T}+\left(\mathrm{C}_{\mathrm{m}}+\mathrm{B}_{\mathrm{m}}\left|\mathrm{x}_{0}\right|^{2 \mathrm{~m}}\right) \mathrm{e}^{\mathrm{T}} \mathrm{Jv}_{\mathrm{n}}^{\epsilon}(\mathrm{x}, \mathrm{t})\right.$ satisfies

$$
\psi_{t}(x, t)+\beta_{\epsilon}^{\prime}(\cdots) \psi(x, t)-\Delta \psi(x, t) \leq 0 .
$$

The maximum principle now implies

$$
\begin{aligned}
& \sup _{\substack{x \in \mathbb{R}^{n} \\
0 \leq t \leq T}}\left\{\phi(x, t)-\frac{1}{\alpha}\left[T+\left(C_{m}+B_{m}\left|x_{0}\right|^{2 m}\right) e^{T}\right] v_{n}^{\epsilon}(x, t)\right\} \\
\leq & \sup _{x \in \mathbb{R}^{n}}\left\{\phi(x, 0)-\frac{1}{\alpha}\left[T+\left(C_{m}+B_{m}\left|x_{0}\right|^{2 m}\right) e^{T}\right] v_{n}^{\epsilon}(x, 0)\right\} \leq 0 .
\end{aligned}
$$

Therefore

$$
T\left(v_{t}^{\epsilon}\left(x_{0}, T\right)-f^{\prime}(T)\right)=\phi\left(x_{0}, T\right) \leq \frac{1}{\alpha}\left[T+\left(C_{m}+B_{m}\left|x_{0}\right|^{2 m}\right) e^{T}\right] v_{n}^{\epsilon}\left(x_{0}, T\right)
$$

Set $K=\max \left\{B_{m}, \max \left\{\mathrm{Te}^{-T}+C_{m}: T>0\right\}\right\} / \alpha$. Then the above inequality implies that

$$
v^{\epsilon}(x, t)-f^{\prime}(t) \leq K \frac{e^{t}}{t}\left(1+|x|^{2 m}\right) v_{n}^{\epsilon}(x, t)
$$

To prove the reverse inequality we consider the auxiliary function

$$
\bar{\phi}(x, t) \triangleq-t\left[v_{t}^{\epsilon}(x, t)-f^{\prime}(t)\right]-e^{t}\left|x-x_{0}\right|^{2 m}
$$

and proceed exactly as before.

Lemma 3.4. There are positive constants $K, m$, independent of $\epsilon$, such that

$$
\begin{gather*}
\left|\nabla q^{\epsilon}(y, t)\right| \leq K  \tag{3.11}\\
\left|q_{t}^{\epsilon}(y, t)\right| \leq  \tag{3.12}\\
K{\frac{e^{t}}{t}\left[1+|y|^{2 m}+\left|q^{\epsilon}(y, t)\right|^{2 m}\right]}_{\substack{\inf \\
\epsilon>0}} q^{\epsilon}(y, t)>-\infty \tag{3.13}
\end{gather*}
$$

for all $\mathbf{y} \in \mathbb{R}^{\mathrm{n}-1}, \mathrm{t}>0$.

Proof. Use two characterizations, (3.2) and (3.3) of $\mathscr{C}_{\epsilon}$ to obtain the two
different expressions for the unit normal vector $2^{\epsilon}(x, t)$ at a boundary point $(\mathrm{x}, \mathrm{t}) \in \partial \mathscr{E}_{\epsilon}$,

$$
\begin{aligned}
z^{\epsilon}(x, t) & =\frac{\left(\nabla v^{\epsilon}(x, t), v_{t}^{\epsilon}(x, t)-f^{\prime}(t)\right)}{\left[\left|\nabla v^{\epsilon}(x, t)\right|^{2}+\left(v_{t}^{\epsilon}(x, t)-f^{\prime}(t)\right)^{2}\right]^{1 / 2}} \\
& =\frac{\left(-\nabla q^{\epsilon}(\bar{x}, t), \quad 1,-q_{t}^{\epsilon}(\bar{x}, t)\right)}{\left[\left|\nabla q^{\epsilon}(\bar{x}, t)\right|^{2}+1+\left|q_{t}^{\epsilon}(\bar{x}, t)\right|^{2}\right]^{1 / 2}}
\end{aligned}
$$

where for any $x=\left(x_{1}, \ldots, x_{n}\right)$,

$$
\begin{equation*}
\overline{\mathrm{x}}=\left(\mathrm{x}_{1}, \ldots, \mathrm{x}_{\mathrm{n}-1}\right) \in \mathbb{R}^{\mathrm{n}-1} \tag{3.14}
\end{equation*}
$$

The above identity yields that for any $(\mathrm{x}, \mathrm{t}) \in \partial \mathscr{C}_{\epsilon}$,

$$
\begin{gathered}
\mathrm{v}_{\mathrm{n}}^{\epsilon}(\mathrm{x}, \mathrm{t})=\frac{\left[\left|\nabla \mathrm{v}^{\epsilon}(\mathrm{x}, \mathrm{t})\right|^{2}+\left(\mathrm{v}_{\mathrm{t}}^{\epsilon}(\mathrm{x}, \mathrm{t})-\mathrm{f}^{\prime}(\mathrm{t})\right)^{2}\right]^{1 / 2}}{\left[\left|\nabla \mathrm{q}^{\epsilon}(\bar{x}, \mathrm{t})\right|^{2}+1+\left(\mathrm{q}_{\mathrm{t}}^{\epsilon}(\overline{\mathrm{x}}, \mathrm{t})\right)^{2}\right]^{1 / 2}} \\
\mathrm{v}_{\mathrm{i}}^{\epsilon}(\mathrm{x}, \mathrm{t})=-\mathrm{q}_{\mathrm{i}}^{\epsilon}(\overline{\mathrm{x}}, \mathrm{t}) \mathrm{v}_{\mathrm{n}}^{\epsilon}(\mathrm{x}, \mathrm{t}), \mathrm{i}=1, \ldots, \mathrm{n}-1
\end{gathered}
$$

and $v_{t}^{\epsilon}(\mathrm{x}, \mathrm{t})-\mathrm{f}^{\prime}(\mathrm{t})=-\mathrm{q}_{\mathrm{t}}^{\epsilon}(\overline{\mathrm{x}}, \mathrm{t}) \mathrm{v}_{\mathrm{n}}^{\epsilon}(\mathrm{x}, \mathrm{t})$. Combine these identities with (3.5) and (3.7) to arrive at (3.11) and (3.12).

We continue by obtaining a lower bound for $q^{\epsilon}(y, t)$. First consider the equation

$$
\begin{equation*}
V_{t}(x, t)-\Delta V(x, t)=h_{n}(x, t) \tag{3.15}
\end{equation*}
$$

with initial data $V(x, 0)=g_{n}(x)$. Since $v^{\epsilon}(x, t)$ is a subsolution to the above equation, the maximum principle yields that

$$
\begin{equation*}
v^{\epsilon}(x, t) \leq V(x, t) \tag{3.16}
\end{equation*}
$$

Differentiate (3.15) with respect to $x_{n}$ to obtain $V_{n t}(x, t)-\Delta V_{n}(x, t)=$ $h_{n n}(x, t) \geq \alpha$ and $V_{n}(x, 0)=g_{n n}(x) \geq 0$. Hence,

$$
\begin{equation*}
V_{n}(x, t) \geq \alpha t \tag{3.17}
\end{equation*}
$$

for all $x \in \mathbb{R}^{n}, t \geq 0$, and consequently $Q(y, t) \triangleq \inf \left\{x_{n}: V\left(y, x_{n}, t\right) \geq f(t)\right\}$ is finite for every $y \in \mathbb{R}^{n-1}, t>0$. However (3.16) yields $q^{\epsilon}(y, t) \geq Q(y, t)$.

For a positive integer $m$, we define $q^{\epsilon, m}$ by $q^{\epsilon, m}(y, t)=$ $\min \left\{q^{\epsilon}(y, t), m\right\}$. The previous lemma shows that for each $m, q^{\epsilon, m}$ is locally Lipschitz continuous, uniformly in $\epsilon$. By using a diagonal argument we may choose a subsequence, denoted by $\epsilon$ again, along which $q^{\epsilon, m}$ converges to a Lipschitz continuous function $q^{m}$ for each $m$. Finally let $\mathrm{q}(\mathrm{y}, \mathrm{t})$ be the increasing limit of $\mathrm{q}^{\mathrm{m}}(\mathrm{y}, \mathrm{t})$. This limit may take the value $+\infty$, but this possibility is ruled out in the next section. Indeed, the local boundedness of q proved in Theorem 4.2, together with the local Lipschitz continuity of $q^{m}$, implies local Lipschitz continuity of $q$ on $\mathbb{R}^{n-1} \times(0, \infty)$.

The following "sharper" lower bound for $q(y, t)$ shall be used in Section 5.

Lemma 3.5. For each $T>0$, there is a positive constant $K(T)$ such that

$$
\begin{equation*}
\inf _{0 \leq t \leq T} q(y, t) \geq-K(T)-K \sum_{i=1}^{n-1}\left|y_{i}\right| \tag{3.18}
\end{equation*}
$$

where $\mathrm{K} \geq 0$ is as in (9.11).

Proof. In view of (3.11) and (3.17), it suffices to show that

$$
\inf _{0 \leq t \leq T} Q(0, t)>-\infty
$$

for every $T>0$. Observe that $Q(y, t)$ is the zero level curve of $V-f$ and $V_{n}>0$ on $\mathbb{R}^{\mathrm{n}} \times(0, \infty)$. Hence by the implicit function theorem, Q is continuous on $\mathbb{R}^{\mathrm{n}} \times(0, \infty)$. Moreover, due to the assumption (2.3) and the fact that $V(x, 0)=g_{n}(x)$,

$$
\lim _{x_{n} \rightarrow-\infty} V\left(y, x_{n}, 0\right)<f(0), y \in \mathbb{R}^{n-1}
$$

Hence, $\liminf _{t \rightarrow 0} Q(y, t)>-\infty$.

Corollary 3.6. We have

$$
\begin{equation*}
\mathscr{C}=\left\{\left(\mathrm{y}, \mathrm{x}_{\mathrm{n}}, \mathrm{t}\right) \in \mathbb{R}^{\mathrm{n}-1} \times \mathbb{R}_{\times}(0, \infty): \mathrm{x}_{\mathrm{n}}<\mathrm{q}(\mathrm{y}, \mathrm{t})\right\} \tag{3.19}
\end{equation*}
$$

In particular, $q$ is independent of the subsequence along which the limit is taken.

Proof. It suffices to show that $\mathscr{C}^{\mathrm{m}}=\mathscr{\mathscr { P }}^{\mathrm{m}}$, where

$$
\begin{aligned}
& \mathscr{C}^{m} \triangleq \mathscr{C} \cap\left\{\left(y, x_{n}, t\right) \in \mathbb{R}^{n-1} \times \mathbb{R} \times(0, \infty): x_{n}<m\right\} \\
& \mathscr{H}^{m} \triangleq\left\{\left(y, x_{n}, t\right) \in \mathbb{R}^{n-1} \times \mathbb{R} \times(0, \infty): \quad x_{n}<q^{m}(y, t)\right\}
\end{aligned}
$$

for every $m$. Let $\left(\tilde{y}, \tilde{x}_{n}, t\right) \in \mathscr{H}^{m}$ be given. Since $q^{\epsilon, m}$ converges to $q^{m}$ uniformly on bounded subsets, there are $\epsilon_{0}>0$ and a neighborhood $\mathcal{N}$ of $\left(\tilde{y}, \tilde{x}_{n}, \tilde{f}\right)$ such that

$$
\mathrm{x}_{\mathrm{n}}<\mathrm{q}^{\epsilon, \mathrm{m}}(\mathrm{y}, \mathrm{t}), \quad \forall\left(\mathrm{y}, \mathrm{x}_{\mathrm{n}}, \mathrm{t}\right) \in \mathscr{N}, 0<\varepsilon \leq \epsilon_{0} .
$$

Therefore $\mathrm{v}^{\epsilon}(\mathrm{x}, \mathrm{t})<\mathrm{f}(\mathrm{t})$ and $\mathrm{v}_{\mathrm{t}}^{\epsilon}(\mathrm{x}, \mathrm{t})-\Delta \mathrm{v}^{\epsilon}(\mathrm{x}, \mathrm{t})=\mathrm{h}_{\mathrm{n}}(\mathrm{x}, \mathrm{t})$ for all $(\mathrm{x}, \mathrm{t}) \in \mathscr{N}$ and $0<\epsilon \leq \epsilon$. By letting $\epsilon$ go to zero we obtain that $v(x, t) \leq f(t)$ and $\mathrm{v}_{\mathrm{t}}(\mathrm{x}, \mathrm{t})-\Delta \mathrm{v}(\mathrm{x}, \mathrm{t})=\mathrm{h}_{\mathrm{n}}(\mathrm{x}, \mathrm{t})$ for all $(\mathrm{x}, \mathrm{t}) \in \mathscr{N}$. Differentiating the last equation with respect to $\mathrm{x}_{\mathrm{n}}$ and then using the positivity of $\mathrm{h}_{\mathrm{nn}}$ and the non-negativity of $\mathrm{v}_{\mathrm{n}}$, we obtain that $\mathrm{v}_{\mathrm{n}}>0$ on $\mathscr{N}$. Hence $\mathrm{v}<\mathrm{f}$ on $\mathcal{N}$ and consequently $\left(\tilde{y}, \tilde{x}_{n}, \mathfrak{t}\right) \in \mathscr{C}^{m}$.

To prove the reverse inclusion, let ( $\tilde{\mathrm{y}}, \tilde{\mathrm{x}}_{\mathrm{n}}, \mathfrak{f}$ ) be an element of $\mathscr{C}^{\mathrm{m}}$. Then $v\left(\tilde{y}, \tilde{x}_{n}, \mathfrak{t}\right)<f(t)$ and the convergence of $v^{\epsilon}$ to $v$ implies that $\tilde{x}_{\mathrm{n}}<\mathrm{q}^{\left.\epsilon, \mathrm{m}_{(\tilde{y}}, \tilde{\varepsilon}\right)}$ for all sufficiently small $\epsilon$. Letting $\epsilon$ go to zero, we conclude that $\left(\tilde{y}, \tilde{x}_{\mathrm{n}}, \tilde{f}\right) \in$ closure $\left(\mathscr{C}^{\mathrm{m}}\right)$. Since $\mathscr{B}^{\mathrm{m}}$ is an open set, $\mathscr{C}^{\mathrm{m}} \underline{\sqsubseteq}$ interior (closure $\left(\mathscr{\mathscr { C }}^{\mathrm{m}}\right.$ ). But the right-hand side is equal to $\mathscr{\mathscr { H }} \mathrm{m}$, due to the Lipschitz continuity of $q^{m}$.

## Remarks

1. In one space dimension, Van Moerbeke obtained the smoothness of the free boundary under quite different structural assumptions [29], [30]. Van

Moerbeke uses an integral equation satisfied by the parametrization of the free boundary to obtain local existence, and then he obtains the global result by deep stochastic analysis. In the multi-dimensional case it is also possible to obtain an integral equation for the parametrization. However it seems to be of little use because of its very complicated structure.
2. In the stationary case [27], the authors proved the smoothness of the free boundary by applying theorems of Caffarrelli [5] and Kinderlehrer and Nirenberg [20], after its Lipschitz continuity was established. However, the results of Section 2 of [5] are not directly applicable to the problem under investigation.

## 4. AN UPPER BOUND FOR $q$

We start by analyzing the zero-level curve of $v(x, t)+1$. Let

$$
\begin{equation*}
\hat{\mathrm{q}}(\mathrm{y}, \mathrm{t})=\inf \left\{\mathrm{x}_{\mathrm{n}}: \quad \mathrm{v}\left(\mathrm{y}, \mathrm{x}_{\mathrm{n}}, \mathrm{t}\right) \geq-1\right\} . \tag{4.1}
\end{equation*}
$$

Because $\lim _{x_{n} \rightarrow-\infty} V\left(y, x_{n}, t\right)=-\infty(\operatorname{see}(3.17))$, and

$$
v(x, t)=\lim _{\epsilon \downarrow 0} v^{\epsilon}(x, t) \leq V(x, t)
$$

(see (3.16)), we know that $\hat{q}(y, t)>-\infty$.

Lemma 4.1. For $T>0$ there is a constant $C(T)$ such that

$$
\begin{equation*}
\hat{\mathrm{q}}(\mathrm{y}, \mathrm{t}) \leq \mathrm{C}(\mathrm{~T}) \quad(|\mathrm{y}|+1) \tag{4.2}
\end{equation*}
$$

for all $y \in \mathbb{R}^{n-1}$ and $t \in[0, T]$.

Proof. Consider the linear equation with initial condition

$$
\begin{equation*}
\underline{V}_{\mathrm{t}}(\mathrm{x}, \mathrm{t})-\Delta \underline{\mathrm{y}}(\mathrm{x}, \mathrm{t})=-\left(\mathrm{h}_{\mathrm{n}}(\mathrm{x}, \mathrm{t})\right)^{-}, \quad \underline{\mathrm{v}}(\mathrm{x}, 0)=-\left(\mathrm{g}_{\mathrm{n}}(\mathrm{x})\right)^{-}, \tag{4.3}
\end{equation*}
$$

where $-(a)^{-} \triangleq \min \{a, 0\}$. Since $f(t) \geq 1$ and $\underline{V}(x, t) \leq 0, \underline{V}$ is a subsolution to $(2.5)^{\epsilon}$ for each $\epsilon>0$. Hence Lemma 8.1 of the appendix yields that $\underline{V}(\mathrm{x}, \mathrm{t}) \leq \mathrm{v}^{\epsilon}(\mathrm{x}, \mathrm{t})$ for every $\epsilon>0$ and therefore

$$
\begin{equation*}
\underline{v}(x, t) \leq v(x, t) . \tag{4.4}
\end{equation*}
$$

Moreover,

$$
\begin{aligned}
& \underline{\mathrm{V}}(\mathrm{x}, \mathrm{t})=-\int_{\mathbb{R}^{\mathrm{n}}}\left\{\int_{0}^{\mathrm{t}}\left[\mathrm{~h}_{\mathrm{n}}(\mathrm{x}+\sqrt{\mathrm{s}} \mathrm{z}, \mathrm{t}-\mathrm{s})\right]^{-} \mathrm{ds}\right. \\
& \left.\quad+\left[g_{\mathrm{n}}(\mathrm{x}+\sqrt{\mathrm{t}} \mathrm{z})\right]^{-}\right\} \cdot(4 \pi)^{-\mathrm{n} / 2} \exp \left(-|z|^{2} / 4\right) \mathrm{d} \mathrm{z}
\end{aligned}
$$

Also the assumption (2.2) yields that

$$
\lim _{x_{n} \rightarrow \infty}\left[\left(\left(h_{n}\left(y, x_{n}, t\right)\right)^{-}, \quad\left(g_{n}\left(y, x_{n}\right)\right)^{-}\right]=(0,0)\right.
$$

for every $y \in \mathbb{R}^{n-1}$ and $t \geq 0$. A simple application of the monotone convergence theorem yields that $\lim _{x_{n} \rightarrow \infty} \underline{V}\left(\mathrm{y}, \mathrm{x}_{\mathrm{n}}, \mathrm{t}\right)=0$ for every $\mathrm{y} \in \mathbb{R}^{\mathrm{n}-1}$ and $t \geq 0$. This together with (4.4) implies that $\hat{\mathrm{q}}(\mathrm{y}, \mathrm{t})<\infty$. We claim that there is $\tilde{\mathrm{C}}(\mathrm{T})$, such that
(4.5)

$$
\hat{\mathrm{q}}(0, \mathrm{t}) \leq \tilde{\mathrm{C}}(\mathrm{~T}), \forall \mathrm{t} \in[0, \mathrm{~T}] .
$$

Indeed, $v_{n}(y, \hat{q}(y, t), t)>0$ for $y \in \mathbb{R}^{n-1}$ and $t>0$, because in a neighborhood of $(y, \hat{q}(y, t), t)$ we have $v_{n t}-\Delta v_{n}=h_{n n}>0$. By the implicit function theorem, $\hat{q}$ is smooth, in particular continuous, on $\mathbb{R}^{\mathrm{n}-1} \times(0, \infty)$. We also have the inequality at the initial point

$$
\lim _{t \downarrow 0} \sup \hat{q}(0, t) \leq \sup \left\{x_{n} \in \mathbb{R}: g_{n}\left(0, x_{n}\right) \leq-1\right\}<\infty
$$

The above inequality together with the continuity of $\hat{q}$ on the open domain $\mathbb{R}^{\mathrm{n}-1} \times(0, \infty)$ is enough to arrive at (4.5).

Proceeding exactly as in the proof of Lemma 3.4, we can show that

$$
\sup _{y \in \mathbb{R}^{n-1}, t>0}|\nabla \hat{q}(y, t)|=C<\omega
$$

Now let $C(T)=\max \{\hat{C}(T), C\}$.
$\square$

Theorem 4.2. The function $q(y, t)$ is locally bounded in $(y, t) \in \mathbb{R}^{n-1} \times(0, \infty)$.

Proof. Fix $T>0$ and then chose a positive constant $C \geq 1$ satisfying

$$
\begin{align*}
& \hat{\mathrm{q}}(\mathrm{y}, \mathrm{t}) \leq \mathrm{C}(|\mathrm{y}|+1), \quad \forall \mathrm{y} \in \mathbb{R}^{\mathrm{n}-1}, \quad \mathrm{t} \in[0, \mathrm{~T}]  \tag{4.6}\\
& \mathrm{g}_{\mathrm{n}}\left(\mathrm{y}, \mathrm{x}_{\mathrm{n}}\right) \geq-1, \forall \mathrm{y} \in \mathbb{R}^{\mathrm{n}-1}, \mathrm{x}_{\mathrm{n}} \geq \mathrm{C}(|\mathrm{y}|+1)
\end{align*}
$$

Define

$$
k(y, t) \triangleq \inf \left\{x_{n}: h_{n}\left(y, x_{n}, t\right)-f^{\prime}(t) \geq 0\right\}
$$

and note that because $h_{n n} \geq \alpha>0, k(0, t)$ is bounded uniformly in $t \in[0, T]$ and $k$ is differentiable. From (2.2) and the argument used to prove (3.11), we can show that for some constant $C$, which also satisfies (4.6), (4.7), we have

$$
k(y, t) \leq C(|y|+1) \quad \forall(y, t) \in \mathbb{R}^{n-1} \times[0, T] .
$$

It follows that

$$
\begin{gather*}
h_{n}\left(y, x_{n}, t\right)-f^{\prime}(t) \geq \alpha\left[x_{n}-C(|y|+1)\right]+h_{n}(y, C(|y|+1), t)-f^{\prime}(t)  \tag{4.8}\\
\geq \alpha\left[x_{n}-C(|y|+1)\right] \quad \forall y \in \mathbb{R}^{n-1}, x_{n} \geq C(|y|+1), t \in[0, T]
\end{gather*}
$$

It is elementary to construct a smooth function $\eta$ satisfying

$$
\begin{gather*}
\eta(\mathrm{y}) \geq \mathrm{C}(|\mathrm{y}|+1)  \tag{4.9}\\
|\nabla \eta(\mathrm{y})| \leq \sqrt{2} \mathrm{C},|\Delta \eta(\mathrm{y})| \leq 2 \mathrm{C}^{2} \tag{4.10}
\end{gather*}
$$

We continue by constructing a subsolution to (1.3). First consider the following equation with arbitrary constants $K>\beta \geq 0$, and C as in (4.6)-(4.10):
(4.11) $\max \left\{\phi(\mathrm{r})-\left(2 \mathrm{C}^{2}+1\right) \phi^{\prime \prime}(\mathrm{r})+2 \mathrm{C}^{2} \phi^{\prime}(\mathrm{r})-\mathrm{K}, \phi(\mathrm{r})-\beta\right\}=0, \forall \mathrm{r}>0$, with boundary condition $\phi(0)=0$. An explicit formula for the solution $\phi$ is
(4.12) $\phi(r)=\left\{\begin{array}{lc}\beta, & r \geq r_{0} \\ K-\frac{(K-\beta)}{2 C^{2}+2} e^{r-r_{0}}-\frac{(K-\beta)\left(2 C^{2}+1\right)}{2 C^{2}+2} e^{\frac{-r}{2 C^{2}+I_{0}}}, 0 \leq r \leq r_{0},\end{array}\right.$
where $r_{0}$ is the solution to the transcendental equation

$$
\frac{(K-\beta)}{2 C^{2}+2} e^{-r_{0}}+\frac{(K-\beta)\left(2 C^{2}+1\right)}{2 C^{2}+2} e^{\frac{\mathrm{I}_{0}}{2 C^{2}+1}}=K
$$

The following properties of $\phi$ rather than the explicit formula for it will be used in the analysis:
(4.13) (i)
$\phi, \phi^{\prime}$ are Lipschitz continuous,
(4.13) (ii)

$$
\phi(\mathrm{r}) \geq 0, \phi^{\prime}(\mathrm{r}) \geq 0, \quad \phi^{\prime \prime}(\mathrm{r}) \leq 0, \quad \forall \mathrm{r} \geq 0
$$

Set
(4.14) (i)

$$
\alpha^{*} \triangleq 1-\mathrm{f}(0)+\max _{0 \leq \mathrm{t} \leq \mathrm{T}} \mathrm{f}(\mathrm{t})
$$

(4.14) (ii)

$$
\mathrm{K} \triangleq 2\left(\mathrm{f}(0)+\alpha^{*}\right) \mathrm{e}^{-\mathrm{T}} \mathrm{~T}^{-1}
$$

(4.14) (iii)

$$
\beta \triangleq \mathrm{K} / 2
$$

(4.14) (iv) $\mathrm{R}(\mathrm{x}) \triangleq \mathrm{x}_{\mathrm{n}}-\eta(\overline{\mathrm{x}})-\left(\mathrm{f}(0)+\alpha^{*}\right)(2 \mathrm{~T}+1)(\alpha \mathrm{T})^{-1}, \quad \forall \mathrm{x} \in \mathbb{R}^{\mathrm{n}}$,
where $\overline{\mathrm{x}}=\left(\mathrm{x}_{1}, . . \mathrm{x}_{\mathrm{n}-1}\right)$ when $\mathrm{x}=\left(\mathrm{x}_{1}, \ldots, \mathrm{x}_{\mathrm{n}}\right)$. Finally define

$$
\Omega \triangleq\{(x, t): 0<t \leq T, R(x)>0\}
$$

and define $w$ on $\bar{\Omega}$ by

$$
\begin{equation*}
\mathrm{w}(\mathrm{x}, \mathrm{t}) \triangleq \mathrm{t} \mathrm{e}^{\mathrm{t}} \phi(\mathrm{R}(\mathrm{x}))-\alpha^{*}+\mathrm{f}(\mathrm{t})-\mathrm{f}(0) \tag{4.15}
\end{equation*}
$$

where $\phi$ is as in (4.12) with K and $\beta$ given in (4.14).
We claim that w is a subsolution of (1.3) on the region $\Omega$. Indeed, using (4.15) and (4.11), we calculate that

$$
\mathrm{w}_{\mathrm{t}}-\Delta \mathrm{w}=\mathrm{t} \mathrm{e}^{\mathrm{t}}\left[\phi-\left(1+|\nabla \eta|^{2}\right) \phi^{\prime \prime}+(\Delta \eta) \phi^{\prime}\right]+\mathrm{e}^{\mathrm{t}} \phi+\mathrm{f}^{\prime}
$$

Using (4.13) (ii) with (4.10), then (4.11) and (4.14), we arrive at,

$$
\begin{align*}
\mathrm{w}_{\mathrm{t}} & -\Delta \mathrm{w} \leq \mathrm{te}^{\mathrm{t}}\left[\phi-\left(1+2 \mathrm{C}^{2}\right) \phi^{\prime \prime}+2 \mathrm{C}^{2} \phi^{\prime}\right]+\mathrm{e}^{\mathrm{t}} \phi+\mathrm{f}^{\prime} \leq \mathrm{te}^{\mathrm{t}} \mathrm{~K}+\mathrm{e}^{\mathrm{t}} \beta+\mathrm{f}^{\prime}  \tag{4.16}\\
& =\mathrm{te}^{\mathrm{t}} \frac{2\left(\mathrm{f}(0)+\alpha^{*}\right)}{\mathrm{Te}}+\mathrm{e}^{\mathrm{t}} \frac{\left(\mathrm{f}(0)+\alpha^{*}\right)}{\mathrm{Te}}+\mathrm{f}^{\prime} \leq \frac{\left(\mathrm{f}(0)+\alpha^{*}\right)(2 \mathrm{~T}+1)}{\mathrm{T}}+\mathrm{f}^{\prime}
\end{align*}
$$

Also, for $(\mathrm{x}, \mathrm{t}) \in \Omega,(4.8)$ and (4.9) imply that

$$
\mathrm{h}_{\mathrm{n}}(\mathrm{x}, \mathrm{t})-\mathrm{f}^{\prime}(\mathrm{t}) \geq \alpha\left[\mathrm{x}_{\mathrm{n}}-\eta(\overline{\mathrm{x}})\right] \geq \frac{1}{\mathrm{~T}}\left(\mathrm{f}(0)+\alpha^{*}\right)(2 \mathrm{~T}+1)
$$

Substitute the above inequality into (4.16) to obtain

$$
\begin{equation*}
w_{t}(x, t)-\Delta w(x, t) \leq h_{n}(x, t), \forall(x, t) \in \Omega \tag{4.17}
\end{equation*}
$$

For $(x, t) \in \Omega$,

$$
\begin{align*}
& \mathrm{w}(\mathrm{x}, \mathrm{t})=\mathrm{te}  \tag{4.18}\\
& \mathrm{t} \phi(\mathrm{R}(\mathrm{x}))-\alpha^{*}+\mathrm{f}(\mathrm{t})-\mathrm{f}(0) \leq \mathrm{te}^{\mathrm{t}} \beta-\alpha^{*}+\mathrm{f}(\mathrm{t})-\mathrm{f}(0) \\
& \leq \frac{\mathrm{te}}{\mathrm{Te}}\left(\mathrm{f}(0)+\alpha^{*}\right)-\alpha^{*}+\mathrm{f}(\mathrm{t})-\mathrm{f}(0) \leq \mathrm{f}(\mathrm{t})
\end{align*}
$$

This shows that $w$ is a subsolution of (1.3) on $\Omega$.
We now show that $w \leq v$ on the parabolic boundary of $\Omega$. Due to (4.7),

$$
\begin{equation*}
\mathrm{w}(\mathrm{x}, 0)=-\alpha^{*} \leq-1 \leq \mathrm{g}_{\mathrm{n}}(\mathrm{x})=\mathrm{v}(\mathrm{x}, 0), \forall(\mathrm{x}, 0) \in \Omega \tag{4.19}
\end{equation*}
$$

From the definition of $R$, we see that $R(x)=R(\bar{x}, 0)+x_{n}$ for all $x \in \mathbb{R}^{n}$. In particular, $(y,-R(y, 0)) \in \partial \Omega$ for each $y \in \mathbb{R}^{n-1}$. Now (4.1), (4.6), (4.9) and the negativity of $v_{n}$ imply that

$$
\begin{align*}
& w(y,-R(y, 0), t)=t e^{t} \phi(0)-\alpha^{*}+f(t)-f(0)  \tag{4.20}\\
& \quad=-\alpha^{*}+f(t)-f(0) \leq-1 \\
& \quad=v(y, \hat{q}(y, t), t) \leq v(y,-R(y, 0), t)
\end{align*}
$$

In the last inequality we have used the fact that

$$
\hat{\mathrm{q}}(\mathrm{y}, \mathrm{t}) \leq \mathrm{C}(|\mathrm{y}|+1) \leq \eta(\mathrm{y}) \leq \eta(\mathrm{y})+\left(\mathrm{f}(0)+\alpha^{*}\right)(2 \mathrm{~T}+1)(\alpha \mathrm{T})^{-1}=-\mathrm{R}(\mathrm{y}, 0)
$$

Taken together, (4.19) and (4.20) imply that $w \leq v$ on the parabolic
boundary of $\Omega$. Hence the maximum principle, Lemma 8.5, yields that $w(x, t) \leq v(x, t), \forall(x, t) \in \Omega$.

Recall that $\phi\left(\mathrm{r}_{0}\right)=\beta$. Hence the above inequality used at the point ( $\left.y, r_{0}-R(y, 0), T\right)$ gives

$$
\begin{aligned}
& f(T) \geq \mathrm{v}\left(\mathrm{y}, \mathrm{r}_{0}-\mathrm{R}(\mathrm{y}, 0), \mathrm{T}\right) \geq \mathrm{w}\left(\mathrm{y}, \mathrm{r}_{0}-\mathrm{R}(\mathrm{y}, 0), \mathrm{T}\right) \\
& =\mathrm{Te} \phi\left(\mathrm{r}_{0}\right)-\alpha^{*}+\mathrm{f}(\mathrm{~T})-\mathrm{f}(0)=\mathrm{Te} \mathrm{~T}^{\mathrm{T}} \beta-\alpha^{*}+\mathrm{f}(\mathrm{~T})-\mathrm{f}(0)=\mathrm{f}(\mathrm{~T})
\end{aligned}
$$

Hence $v\left(y, r_{0}-R(y, 0), T\right)=f(T)$, and consequently $q(y, T) \leq r_{0}-R(y, 0)$. $\square^{\square}$

Remark. A similar result was obtained by Karatzas [17], Section 7, in the one-dimensional special case of $h(x, t)=x^{2}$.
5. $\mathrm{C}^{2,1}$ REGULARITY OF u

Let $U(x, t), V(x, t)$ be the polynomially growing solutions of

$$
\begin{equation*}
U_{t}(x, t)-\Delta U(x, t)=h(x, t), x \in \mathbb{R}^{n}, t>0 \tag{5.1}
\end{equation*}
$$

$$
\begin{equation*}
V_{t}(x, t)-\Delta V(x, t)=h_{n}(x, t), x \in \mathbb{R}^{n}, t>0 \tag{5.2}
\end{equation*}
$$

with initial conditions

$$
\begin{align*}
& \mathrm{U}(\mathrm{x}, 0)=\mathrm{g}(\mathrm{x}), \mathrm{x} \in \mathbb{R}^{\mathrm{n}}  \tag{5.3}\\
& \mathrm{~V}(\mathrm{x}, 0)=\mathrm{g}_{\mathrm{n}}(\mathrm{x}), \mathrm{x} \in \mathbb{R}^{\mathrm{n}} \tag{5.4}
\end{align*}
$$

For $x \in \mathbb{R}^{n}$ and $t \geq 0$, define $u(x, t)$ by

$$
\begin{equation*}
\mathrm{u}(\mathrm{x}, \mathrm{t})=\mathrm{U}(\mathrm{x}, \mathrm{t})+\int_{-\infty}^{\mathrm{x}_{\mathrm{n}}}[\mathrm{v}(\overline{\mathrm{x}}, \xi, \mathrm{t})-\mathrm{V}(\overline{\mathrm{x}}, \xi, \mathrm{t})] \mathrm{d} \xi \tag{5.5}
\end{equation*}
$$

where as in the previous sections $\overline{\mathrm{x}}=\left(\mathrm{x}_{1}, \ldots, \mathrm{x}_{\mathrm{n}-1}\right)$ and v is the solution of (1.3) and (1.4). The integrability of $v-V$, required by (5.5), is a part of the following theorem.

Theorem 5.1. The function $u(x, t)$ is well-defined, twice continuously differentiable in the spatial variables and once continuously differentiable in the $t$-variable. Moreover, it is a solution to (1.1) and (1.2).

We need the following lemma in the proof of the above theorem. For positive constants $K^{*}, T$, set

$$
\Omega\left(\mathrm{K}^{*}, \mathrm{~T}\right) \triangleq\left\{(\mathrm{x}, \mathrm{t}) \in \mathbb{R}^{\mathrm{n}} \times(0, \mathrm{~T}): \mathrm{x}_{\mathrm{n}}<-\mathrm{K}^{*} \sum_{\mathrm{i}=1}^{\mathrm{n}-1}\left|\mathrm{x}_{\mathrm{i}}\right|\right\}
$$

Lemma 5.2. Suppose that a continuous function $\varphi$, defined on all of $\mathbb{R}^{\mathrm{n}} \times[0, \mathrm{~T})$, satisfies the following with suitable positive constants $K^{*}, C^{*}, T$ and $m \geq 1$ :
(5.6) (i)

$$
\varphi \in \mathrm{C}^{2,1}\left(\Omega\left(\mathrm{~K}^{*}, \mathrm{~T}\right)\right)
$$

$$
\begin{equation*}
\varphi_{\mathrm{t}}(\mathrm{x}, \mathrm{t})-\Delta \varphi(\mathrm{x}, \mathrm{t})=0, \quad \forall(\mathrm{x}, \mathrm{t}) \in \Omega\left(\mathrm{K}^{*}, \mathrm{~T}\right) \tag{5.6}
\end{equation*}
$$

(5.6) (iii)

$$
\varphi(x, 0)=0, \quad \forall x \in \mathbb{R}^{n}
$$

(5.6) (iv)

$$
|\varphi(x, t)| \leq C^{*}\left(|x|^{2 m}+1\right), \forall(x, t) \in \Omega\left(K^{*}, T\right)
$$

Then, for every $y \in \mathbb{R}^{\mathrm{n}-1}$, $t \in[0, \mathrm{~T}]$, the function $x_{n}{ }^{+} \varphi\left(y, x_{n}, t\right)$ is absolutely integrable on any interval of the form $(-\infty, a]$.

Proof. Set

$$
\begin{gathered}
\mathrm{C}_{\mathrm{m}} \triangleq \mathrm{C}^{*}\left[1+\left(\mathrm{K}^{*}\right)^{2}(\mathrm{n}-1)\right]^{\mathrm{m}} \\
\mathrm{~A} \triangleq 1+(\mathrm{n}-1)\left(\mathrm{K}^{*}\right)^{2}+2 \mathrm{~m}\left(2 \mathrm{~m}+\mathrm{n}-3+2 \mathrm{~K}^{*}(\mathrm{n}-1)\right)
\end{gathered}
$$

and

$$
\begin{align*}
\psi\left(y, x_{n}, t\right) & \triangleq C_{m} e^{A t}\left(|y|^{2 m}+1\right)  \tag{5.7}\\
& \cdot \sum_{J \subset\{1, \ldots, n-1\}}\left[\exp \left(x_{n}+K^{*} \sum_{i \in J} y_{i}-K^{*} \Sigma y_{j}\right)\right] .
\end{align*}
$$

We directly calculate that

$$
\begin{aligned}
\psi\left(y, x_{n}, t\right)-\Delta \psi\left(y, x_{n}, t\right) & =\underset{J \subset\{1, \ldots, n-1\}}{\Sigma} C_{m} e^{A t} e^{(\ldots)}\left[A\left(|y|^{2 m}+1\right)-\right. \\
& -2 m(2 m+n-3)|y|^{2(m-1)}-\left(|y|^{2 m}+1\right)\left(1+(n-1)\left(K^{*}\right)^{2}\right) \\
& \left.-4 m K^{*}\left\{\sum_{i \in J} y_{i}|y|^{2(m-1)}-\sum_{i \neq J} y_{i}|y|^{2(m-1)}\right\}\right]
\end{aligned}
$$

$$
\begin{aligned}
\geq & C_{m} e^{A t} J \subset\{1, \ldots, n-1\} \\
& e^{(\ldots)}\left\{\left[A-1-(n-1)\left(K^{*}\right)^{2}\right]\left(|y|^{2 m}+1\right)\right. \\
& \left.-2 m(2 m+n-3)|y|^{2(m-1)}-4 m K^{*}(n-1)|y|^{2 m-1}\right\}
\end{aligned}
$$

where $(\ldots)=x_{n}+K^{*} \sum_{i \in J} y_{i}-K^{*} \sum_{i \notin J} y_{i}$. Since both $|y|^{2(m-1)}$ and $|y|^{2 m-1}$ are bounded from above by $|y|^{2 m}+1$, the definition of $A$ yields that the above expression is non-negative. Hence

$$
\psi_{\mathrm{t}}(\mathrm{x}, \mathrm{t})-\Delta \psi(\mathrm{x}, \mathrm{t}) \geq 0, \forall(\mathrm{x}, \mathrm{t}) \in \Omega\left(\mathrm{K}^{*}, \mathrm{~T}\right) .
$$

Moreover, using the definition of $\psi$ and (5.6) (iv) we obtain

$$
\begin{aligned}
\psi\left(y,-K \sum_{i=1}^{* n-1}\left|y_{i}\right|, t\right) & \geq C_{m} e^{A t}\left(|y|^{2 m}+1\right) \\
& \geq C^{*}\left[\left(1+\left(K^{*}\right)^{2}(n-1)\right)^{m}\left(|y|^{2 m}+1\right)\right] \\
& \geq\left|\varphi\left(y,-K^{*} \sum_{i=1}^{n-1}\left|y_{i}\right|, t\right)\right|
\end{aligned}
$$

for all $\mathrm{y} \in \mathbb{R}^{\mathrm{n}-1}, \mathrm{t} \in[0, \mathrm{~T}]$. Also, $\psi(\mathrm{x}, 0) \geq 0=|\varphi(\mathrm{x}, 0)|$. Since $\psi$ is growing at most polynomially on $\Omega\left(\mathrm{K}^{*}, \mathrm{~T}\right)$ and it dominates $|\varphi|$ on the parabolic boundary of $\Omega\left(\mathrm{K}^{*}, \mathrm{~T}\right)$, the maximum principle yields that

$$
\psi(\mathrm{x}, \mathrm{t}) \geq|\varphi(\mathrm{x}, \mathrm{t})|, \forall(\mathrm{x}, \mathrm{t}) \in \Omega\left(\mathrm{K}^{*}, \mathrm{~T}\right)
$$

## Proof of Theorem 5.1.

Fix $T>0$ and let $K(T)$ be the constant appearing in (3.18).

Define

$$
\varphi\left(y, x_{n}, t\right)=v\left(y, x_{n}-K(T), t\right)-V\left(y, x_{n}-K(T), t\right) .
$$

Due to the estimate (3.18), equations (1.3) and (5.2), boundary conditions (1.4) and (5.4), and the polynomial growth estimate (2.4), $\varphi$ satisfies the hypothesis of the previous lemma. Hence $u(x, t)$ is well-defined. Similarly $\varphi_{\mathfrak{t}}, \varphi_{\mathrm{i}}, \varphi_{\mathrm{ij}}$ satisfy the hypothesis of Lemma 5.2 for each $\mathrm{i}, \mathrm{j}=1, ., \mathrm{n}$. Hence

$$
\begin{equation*}
u_{t}(x, t)=U_{t}(x, t)+\int_{-\infty}^{x_{n}}\left[v_{t}(\bar{x}, \xi, t)-V_{t}(\bar{x}, \xi, t)\right] d t \tag{5.8}
\end{equation*}
$$

$$
\begin{equation*}
u_{i j}(x, t)=U_{i j}(x, t)+\int_{-\infty}^{x_{n}}\left[v_{i j}(\bar{x}, \xi, t)-V_{i j}(\bar{x}, \xi, t)\right] d t, i, j=1, \ldots, n \tag{5.9}
\end{equation*}
$$

In view of (5.1), (5.2), we have

$$
u_{t}(x, t)-\mid u(x, t)-h(h, t)=\int_{-\infty}^{x_{n}}\left[v_{t}(\bar{x}, \xi, t)-\Delta v(\bar{x}, \xi, t)-h_{n}(\bar{x}, \xi, t)\right] d \xi
$$

Now using (1.3) and the fact that $u_{n}=v$, we conclude that $u$ is a solution of (1.1). Also $u$ satisfies (1.2).

To complete the proof of the theorem, it suffices to show that the integral terms in (5.8) and (5.9) are continuous in ( $x, t$ ). For $\delta>0$, approximate the integral in (5.8) by

$$
\begin{aligned}
\mathrm{F}^{\delta}(\mathrm{x}, \mathrm{t}) & =\int_{-\infty}^{(\mathrm{q}(\overline{\mathrm{x}}, \mathrm{t})-\delta) \wedge \mathrm{x}_{\left.\mathrm{n}_{\left[\mathrm{v}_{\mathrm{t}}(\bar{x}, \xi, \mathrm{t})\right.}-\mathrm{v}_{\mathrm{t}}(\overline{\mathrm{x}}, \xi, \mathrm{t})\right] \mathrm{dt}}} \begin{aligned}
& \mathrm{x}_{\mathrm{n}} \\
&+\int_{\mathrm{q}(\overline{\mathrm{x}}, \mathrm{t}) \wedge \mathrm{x}_{\mathrm{n}}}\left[\mathrm{v}_{\mathrm{t}}(\overline{\mathrm{x}}, \xi, \mathrm{t})-\mathrm{v}_{\mathrm{t}}(\overline{\mathrm{x}}, \xi, \mathrm{t})\right] \mathrm{dt}
\end{aligned}
\end{aligned}
$$

On $x_{n}>q(\bar{x}, t), v_{t}=f^{\prime}$, and hence the second integral is continuous. The continuity of the first integral follows from the parabolic regularity. Hence $F^{\delta}$ is continuous for each positive $\delta$ and

$$
\left|u_{t}(x, t)-F^{\delta}(x, t)-U_{t}(x, t)\right| \leq \int_{q(\bar{x}, t)-\delta}^{q(\bar{x}, t)}\left[\left|v_{t}(\bar{x}, \xi, t)\right|+\left|V_{t}(\bar{x}, \xi, t)\right|\right] d \xi
$$

Therefore, $F^{\delta}+U_{t}$ converges to $u_{t}$ uniformly on bounded subsets, and $u_{t}$ is a continuous function. The continuity of $u_{i j}$ is proved similarly. a

## 6. UNIQUENESS

We start with a comparison result. This proof is related to the uniqueness proof of Evans [8].

Lemma 6.1. Suppose that $\mathfrak{u}, \overline{\mathrm{u}} \in \mathrm{C}^{2,1}\left(\mathbb{R}^{\mathrm{n}} \times(0, \infty)\right) \cap C\left(\mathbb{R}^{\mathrm{n}} \times(0, \infty)\right)$ are sub and supersolutions to (1.1) and (1.2). Further assume that for any $T>0$ there are positive constants $C^{*}, m$ such that for all $t \in[0, \mathrm{~T}]$,
(6.1) $\max \{\underline{u}(\mathrm{x}, \mathrm{t}), 0\}+\max \{-\overline{\mathrm{u}}(\mathrm{x}, \mathrm{t}), 0\} \leq \mathrm{C}^{*}\left(1+|\mathrm{x}|^{2 \mathrm{~m}}\right), \forall \mathrm{x} \in \mathbb{R}^{\mathrm{n}}$,
and

$$
\begin{equation*}
\overline{\mathrm{u}}_{\mathrm{n}}(\mathrm{x}, \mathrm{t})<\mathrm{f}(\mathrm{t}) \text { whenever } \mathrm{x}_{\mathrm{n}}<-\mathrm{C}^{*}(1+|\overline{\mathrm{x}}|) . \tag{6.2}
\end{equation*}
$$

Then

$$
\begin{equation*}
\underline{u}(x, t) \leq \bar{u}(x, t) . \tag{6.3}
\end{equation*}
$$

Proof. Fix $\mathrm{T}>0$. Consider the auxiliary function

$$
\begin{equation*}
\phi^{\delta}(\mathrm{x}, \mathrm{t}) \triangleq \mathrm{e}^{-\mathrm{t}}[(1-\delta) \underline{\mathrm{u}}(\mathrm{x}, \mathrm{t})-\overline{\mathrm{u}}(\mathrm{x}, \mathrm{t})]-\delta\left[\sum_{\mathrm{i}=1}^{\mathrm{n}-1} \xi\left(\delta \mathrm{x}_{\mathrm{i}}\right)+\xi\left(\delta\left[\mathrm{x}_{\mathrm{n}}+\eta(\overline{\mathrm{x}})\right]\right)\right] \tag{6.4}
\end{equation*}
$$

where $\delta>0$ is a small parameter,

$$
\xi(r) \triangleq e^{|r|}-|r|-\frac{1}{2} r^{2}, \quad \forall r \in(-\infty, \infty)
$$

and $\eta$ is a smooth function satisfying
(6.5) (i)

$$
\eta(\mathrm{y}) \geq \mathrm{C}^{*}(1+|\mathrm{y}|), \forall \mathrm{y} \in \mathbb{R}^{\mathrm{n}-1}
$$

(6.5) (ii)

$$
\sup _{y \in \mathbb{R}^{\mathrm{n}-1}}|\nabla \eta(\mathrm{y})|+\left\|\mathrm{D}^{2} \eta(\mathrm{y})\right\|<\infty
$$

with the constant $\mathrm{C}^{*}$ appearing in the hypothesis of the lemma. Since $\xi$ is exponentially growing, $\phi^{\delta}$ achieves its maximum over $\mathbb{R}^{n} \times[0, T]$, say at $\left(\mathrm{x}^{*}, \mathrm{t}^{*}\right)$. If $\mathrm{t}^{*}=0$, then $\phi^{\delta}\left(\mathrm{x}^{*}, \mathrm{t}^{*}\right) \leq 0$. Otherwise,

$$
\overline{\mathrm{u}}_{\mathrm{n}}\left(\mathrm{x}^{*}, \mathrm{t}^{*}\right)=(1-\delta) \underline{\mathrm{u}}_{\mathrm{n}}\left(\mathrm{x}^{*}, \mathrm{t}^{*}\right)-\delta^{2} \mathrm{e}^{\mathrm{t}} \xi^{*}\left(\delta\left[\mathrm{x}_{\mathrm{n}}^{*}+\eta\left(\overline{\mathrm{x}}^{*}\right)\right]\right)
$$

Since $\underline{u}_{\mathrm{n}} \leq \mathrm{f}$ and $\xi^{\prime}(\mathrm{r}) \geq 0$ for $\mathrm{r} \geq 0$, we have $\bar{u}_{\mathrm{n}}\left(\mathrm{x}^{*}, \mathrm{t}^{*}\right)<\mathrm{f}\left(\mathrm{t}^{*}\right)$ if $\mathrm{x}_{\mathrm{n}}^{*} \geq-\eta\left(\overline{\mathrm{x}}^{*}\right)$. But if $\mathrm{x}_{\mathrm{n}}^{*}<-\eta\left(\mathrm{x}^{*}\right), \overline{\mathrm{u}}_{\mathrm{n}}\left(\mathrm{x}^{*}, \mathrm{t}^{*}\right)<\mathrm{f}\left(\mathrm{t}^{*}\right)$ due to (6.2) and (6.5)(i). Hence at $\left(\mathrm{x}^{*}, \mathrm{t}^{*}\right), \overline{\mathrm{u}}_{\mathrm{n}}<\mathrm{f}$ and

$$
\begin{equation*}
\bar{u}_{\mathrm{t}}\left(\mathrm{x}^{*}, \mathrm{t}^{*}\right)-\Delta \overline{\mathrm{u}}\left(\mathrm{x}^{*}, \mathrm{t}^{*}\right) \geq \mathrm{h}\left(\mathrm{x}^{*}, \mathrm{t}^{*}\right) . \tag{6.6}
\end{equation*}
$$

Then using (6.6) and the fact that $\underline{u}$ is a subsolution we obtain that

$$
\begin{align*}
0 & \leq \phi_{\mathrm{t}}^{\delta}\left(\mathrm{x}^{*}, \mathrm{t}^{*}\right)-\Delta \phi^{\delta}\left(\mathrm{x}^{*}, \mathrm{t}^{*}\right)  \tag{6.7}\\
& \leq \mathrm{e}^{-\mathrm{t}^{*}}\left[-\delta \mathrm{h}\left(\mathrm{x}^{*}, \mathrm{t}^{*}\right)-(1-\delta) \underline{u}\left(\mathrm{x}^{*}, \mathrm{t}^{*}\right)+\overline{\mathrm{u}}\left(\mathrm{x}^{*}, \mathrm{t}^{*}\right)\right]+\delta^{2} \mathrm{~K}^{\delta}\left(\mathrm{x}^{*}\right),
\end{align*}
$$

where

$$
\begin{aligned}
\mathrm{K}^{\delta}(\mathrm{x}) \triangleq \delta \sum_{\mathrm{i}=1}^{\mathrm{n}-1} \xi^{\prime \prime}\left(\delta \mathrm{x}_{\mathrm{i}}\right) & +\delta\left(1+|\nabla \eta(\overline{\mathrm{x}})|^{2}\right) \xi^{\prime \prime}\left(\delta\left[\mathrm{x}_{\mathrm{n}}+\eta(\overline{\mathrm{x}})\right]\right) \\
& +\Delta \eta(\overline{\mathrm{x}}) \xi^{\prime}\left(\delta\left[\mathrm{x}_{\mathrm{n}}+\eta(\overline{\mathrm{x}})\right]\right)
\end{aligned}
$$

Using the inequalities (6.5)(ii), $\left|\xi^{\prime}\right| \leq 5 \xi$ and $\xi^{\prime \prime} \leq 5 \xi$, we estimate $\mathrm{K}^{\delta}(\mathrm{x})$ by

$$
\mathrm{K}^{\delta}(\mathrm{x}) \leq 5 \delta \sum_{\mathrm{i}=1}^{\mathrm{n}-1} \xi\left(\delta \mathrm{x}_{\mathrm{i}}\right)+\mathrm{C} \xi\left(\delta\left[\mathrm{x}_{\mathrm{n}}+\eta(\overline{\mathrm{x}})\right]\right)
$$

for some suitable constant $C>0$. Substitute the above estimate into (6.7)
to obtain

$$
0 \leq-\phi^{\delta}\left(\mathbf{x}^{*}, \mathrm{t}^{*}\right)+\delta \sum_{\mathrm{i}=1}^{\mathrm{n}-1}\left(5 \delta^{2}-1\right) \xi\left(\delta \mathrm{x}_{\mathrm{i}}^{*}\right)+\delta(\mathrm{C} \delta-1) \xi\left(\delta\left[\mathrm{x}^{*}+\eta\left(\overline{\mathrm{x}}^{*}\right)\right]\right)
$$

Send $\delta$ to zero to arrive at (6.3).
$\square$

Corollary 6.2. The function $u$ defined by (5.5) is non-negative and polynomially growing.

Proof. The fact that $u$ is polynomially growing follows from the polynomial growth of $\mathrm{U}, \mathrm{V}$ and v . Let $\underline{\mathbf{u}} \equiv 0$, and $\overline{\mathrm{u}}=\mathrm{u}$ in the previous lemma. Since $\mathrm{g}, \mathrm{h} \geq 0, \underline{\mathbf{u}}$ is a subsolution. Also, Lemma 3.5 implies that $\overline{\mathrm{u}}=\mathrm{u}$ satisfies (6.2). Hence, $\mathrm{u}=\overline{\mathrm{u}} \geq \underline{\mathrm{u}}=0$.

Theorem 6.3. There is a unique polynomially growing, non-negative, (classical) solution of (1.1) and (1.2), and it is defined by (5.5).

Proof. In view of the previous results it suffices to show that any polynomially growing, non-negative solution of (1.1) and (1.2) satisfies (6.2). Indeed let $u(x, t)$ be such a solution and for $y \in \mathbb{R}^{n-1}, \quad t \geq 0$, define $x_{n}(y, t)$ and $p(y, t)$ by

$$
p(y, t) \triangleq \inf \left\{x_{n}: h_{n}\left(y, x_{n}, t\right)>f^{\prime}(t)\right\}, x_{n}(y, t) \triangleq \inf \left\{x_{n}: G\left(y, x_{n}, t\right)<0\right\}
$$

where $G(x, t) \triangleq u_{t}(x, t)-\Delta u(x, t)-h(x, t)$. We claim that $\chi_{n}(y, t) \geq p(y, t)$ for all $y \in \mathbb{R}^{n-1}$ and $t \geq 0$. Indeed, if this inequality does not hold for some
$\mathrm{y}^{*}, \mathrm{t}^{*}$, then there is $\delta>0$ such that $\mathrm{G}\left(\mathrm{y}^{*}, \mathrm{p}\left(\mathrm{y}^{*}, \mathrm{t}^{*}\right)-\delta, \mathrm{t}^{*}\right)<0$. By the continuity of G , there is a neighborhood of $\left(\mathrm{y}^{*}, \mathrm{p}\left(\mathrm{y}^{*}, \mathrm{t}^{*}\right)-\delta, \mathrm{t}^{*}\right)$ on which $G$ is strictly negative. Hence on this neighborhood $u_{n}=f, \Delta u_{n}=0$ and

$$
\begin{aligned}
\mathrm{G}_{\mathrm{n}}\left(\mathrm{y}^{*}, \mathrm{p}\left(\mathrm{y}^{*}, \mathrm{t}^{*}\right)-\delta, \mathrm{t}^{*}\right) & =\mathrm{f}^{\prime}\left(\mathrm{t}^{*}\right)-\mathrm{h}_{\mathrm{n}}\left(\mathrm{y}^{*}, \mathrm{p}\left(\mathrm{y}^{*}, \mathrm{t}^{*}\right)-\delta, \mathrm{t}^{*}\right) \\
& =\int_{\mathrm{p}\left(\mathrm{y}^{*}, \mathrm{t}^{*}\right)-\delta}^{\mathrm{p}\left(\mathrm{y}^{*}, \mathrm{t}^{*}\right)} \mathrm{h}_{\mathrm{nn}}\left(\mathrm{y}^{*}, \xi, \mathrm{t}^{*}\right) \mathrm{d} \xi \geq \alpha \delta .
\end{aligned}
$$

The above inequality and the argument leading to it imply that $\mathrm{G}\left(\mathrm{y}^{*}, \mathrm{x}_{\mathrm{n}}, \mathrm{t}^{*}\right)<0$ and $\mathrm{u}_{\mathrm{n}}\left(\mathrm{y}^{*}, \mathrm{x}_{\mathrm{n}}, \mathrm{t}^{*}\right)=\mathrm{f}\left(\mathrm{t}^{*}\right)$ whenever $\mathrm{x}_{\mathrm{n}} \leq \mathrm{p}\left(\mathrm{y}^{*}, \mathrm{t}^{*}\right)-\delta$. But this contradicts the positivity of $u$. Hence $\chi_{\mathrm{n}}(\mathrm{y}, \mathrm{t}) \geq \mathrm{p}(\mathrm{y}, \mathrm{t})$. The assumption (2.2) yields that $\mathrm{p}(\mathrm{y}, \mathrm{t}) \geq-\mathrm{C}(\mathrm{T})(|\mathrm{y}|+1), \forall \mathrm{y} \in \mathbb{R}^{\mathrm{n}-1}, \mathrm{t} \in[0, \mathrm{~T}]$ for a suitable constant $C(T)$. This shows the existence of a constant $C(T)$ such that $u_{t}(x, t)-\Delta u(x, t)-h(x, t)=0$ whenever $x_{n} \leq-C(T)(1+|\bar{x}|)$. But on the set where this equality holds, $u_{n n}>0$ and $u_{n} \leq f$, so in fact $u_{n}<f$ whenever $\mathrm{x}_{\mathrm{n}}<-\mathrm{C}(\mathrm{T})(1+|\overline{\mathrm{x}}|)$.

## 7. THE SINGULAR STOCHASTIC CONTROL PROBLEM.

Consider the stochastic process $\mathrm{X}_{\mathrm{s}}=\left(\mathrm{X}_{\mathrm{s}}^{1}, \ldots, \mathrm{X}_{\mathrm{s}}^{\mathrm{n}}\right) \in \mathbb{R}^{\mathrm{n}}$ defined by

$$
\mathrm{X}_{\mathrm{s}}=\mathrm{x}+\sqrt{2} \mathrm{~W}_{\mathrm{s}}-\xi(\mathrm{s}) \mathrm{e}^{\mathrm{n}}, \quad \mathrm{~s} \geq 0
$$

where $x \in \mathbb{R}^{n}$ is the initial condition, $W_{s}=\left(W_{s}^{1}, \ldots, W_{s}^{n}\right) \in \mathbb{R}^{n}$ is an n -dimensional standard Brownian motion, $\mathrm{e}^{\mathrm{n}}=(0,0, \ldots, 1) \in \mathbb{R}^{\mathrm{n}}, \xi(\mathrm{s})$ is the control process, which is non-decreasing, left-continuous, adapted to the augmentation by null sets of the filtration generated by W , with $\xi(0)=0$.

For a given initial condition $x \in \mathbb{R}^{n}$ and horizon $t \geq 0$, the control problem is to select a control process so as to minimize the cost functional

$$
J(x, t, \xi(\cdot)) \triangleq E\left\{\int_{0}^{t}\left[h\left(X_{s}, t-s\right) d s+f(t-s) d \xi(s)\right]+g\left(X_{t}\right)\right\}
$$

Finally define the value function $\mathbf{u}^{*}$ by

$$
\begin{equation*}
u^{*}(x, t) \triangleq \inf _{\xi(\cdot)} J(x, t, \xi(\cdot)) \tag{7.1}
\end{equation*}
$$

THEOREM 7.1. The value function $\mathrm{u}^{*}(\mathrm{x}, \mathrm{t})$ is the unique, non-negative, polynomially growing solution of (1.1), (1.2). Moreover, the infimum in (7.1) is achieved by the left-continuous process $\xi^{*}$ given by

$$
\begin{aligned}
\xi^{*}\left(\mathrm{~s}^{+}\right) & \triangleq \underset{\mathrm{r} \downarrow \mathrm{~s}}{ } \lim \xi^{*}(\mathrm{r}) \\
& =\max _{0 \leq \tau \leq \mathrm{s}}\left[\sqrt{2} \mathrm{~W}_{\tau}^{\mathrm{n}}+\mathrm{x}_{\mathrm{n}}-\mathrm{q}\left(\sqrt{2} \mathrm{~W}_{\tau}^{1}+\mathrm{x}_{1}, \cdots, \sqrt{2} \mathrm{~W}_{\tau}^{\mathrm{n}-1}+\mathrm{x}_{\mathrm{n}-1}, \mathrm{t}-\tau\right)\right]
\end{aligned}
$$

where $q$ is as in Corollary 3.6.

Proof. Let $u$ be the solution to (1.1), (1.2), and let $U$ be as defined in Section 5. We develop some preparatory results. Define $F(x, t) \triangleq U(x, t)-$ $u^{*}(x, t)$, and note that for $\delta \geq 0$,

$$
\begin{equation*}
F(x, t)-F\left(x-\delta e^{n}, t\right) \tag{7.2}
\end{equation*}
$$

$$
\begin{aligned}
& \geq \inf _{\xi(\cdot)}\left[(\mathrm{U}(\mathrm{x}, \mathrm{t})-\mathrm{J}(\mathrm{x}, \mathrm{t}, \xi(\cdot)))-\left(\mathrm{U}\left(\mathrm{x}-\delta \mathrm{e}^{\mathrm{n}}, \mathrm{t}\right)-\mathrm{J}\left(\mathrm{x}-\delta \mathrm{e}^{\mathrm{n}}, \mathrm{t}, \xi(\cdot)\right)\right)\right] \\
& =\inf _{\xi(\cdot)} \mathrm{E}\left\{\int _ { 0 } ^ { \mathrm { t } } \left[\mathrm{~h}\left(\mathrm{x}+\sqrt{2} \mathrm{~W}_{\mathrm{s}}, \mathrm{t}-\mathrm{s}\right)-\mathrm{h}\left(\mathrm{x}+\sqrt{2} \mathrm{~W}_{\mathrm{s}}-\xi(\mathrm{s}) \mathrm{e}^{\mathrm{n}}, \mathrm{t}-\mathrm{s}\right)\right.\right. \\
& \left.-\mathrm{h}\left(\mathrm{x}+\sqrt{2} \mathrm{~W}_{\mathrm{s}}-\delta \mathrm{e}^{\mathrm{n}}, \mathrm{t}-\mathrm{s}\right)+\mathrm{h}\left(\mathrm{x}+\sqrt{2} \mathrm{~W}_{\mathrm{s}}-\xi(\mathrm{s}) \mathrm{e}^{\mathrm{n}}-\delta \mathrm{e}^{\mathrm{n}}, \mathrm{t}-\mathrm{s}\right)\right] \mathrm{ds} \\
& \quad+\left[\mathrm{g}\left(\mathrm{x}+\sqrt{2} \mathrm{~W}_{\mathrm{t}}\right)-\mathrm{g}\left(\mathrm{x}+\sqrt{2} \mathrm{~W}_{\mathrm{t}}-\xi(\mathrm{t}) \mathrm{e}^{\mathrm{n}}\right)-\mathrm{g}\left(\mathrm{x}+\sqrt{2} \mathrm{~W}_{\mathrm{t}}-\delta \mathrm{e}^{\mathrm{n}}\right)\right. \\
& \left.\left.\quad+\mathrm{g}\left(\mathrm{x}+\sqrt{2} \mathrm{~W}_{\mathrm{t}}-\xi(\mathrm{t}) \mathrm{e}^{\mathrm{n}}-\delta \mathrm{e}^{\mathrm{n}}\right)\right]\right\} \geq 0
\end{aligned}
$$

due to the convexity of $h$ and $g$. Thus, $F(x, t)$ is nondecreasing in the $x_{n}-$ variable. Note also, from Itô's lemma, that with $\xi$ an arbitrary control process and $\tau_{\mathrm{m}} \triangleq \inf \left\{\mathrm{s} \geq 0:\left|\mathrm{X}_{\mathrm{s}}\right|>\mathrm{m}\right\}$ we have
(7.3) $u(x, t)=E \int_{0}^{t \wedge \tau} m\left[u_{t}\left(X_{s}, t-s\right)-\Delta u\left(X_{s}, t-s\right)-h\left(X_{\mathrm{s}}, t-s\right)\right] d s$

$$
\begin{aligned}
& +\mathrm{E} \int_{0}^{\mathrm{t} \wedge \tau_{\mathrm{m}}} \mathrm{~h}\left(\mathrm{X}_{\mathrm{s}}, \mathrm{t}-\mathrm{s}\right) \mathrm{ds}+\mathrm{E} \int_{0}^{\mathrm{t} \wedge \tau_{\mathrm{m}}} \mathrm{~m}_{\mathrm{n}}\left(\mathrm{X}_{\mathrm{s}}, \mathrm{t}-\mathrm{s}\right) \mathrm{d} \xi(\mathrm{~s}) \\
& +\mathrm{E} \underset{0 \leq \mathrm{s}<\mathrm{t} \wedge \tau_{\mathrm{m}}}{\Sigma}\left(\mathrm{u}\left(\mathrm{X}_{\mathrm{s}}, \mathrm{t}-\mathrm{s}\right)-\mathrm{u}\left(\mathrm{X}_{\mathrm{s}+}, \mathrm{t}-\mathrm{s}\right)-\mathrm{u}_{\mathrm{n}}\left(\mathrm{X}_{\mathrm{s}}, \mathrm{t}-\mathrm{s}\right)\left[\xi\left(\mathrm{s}^{+}\right)-\xi(\mathrm{s})\right]\right) \\
& +\mathrm{Eu}\left(\mathrm{X}_{\mathrm{t} \Lambda \tau_{\mathrm{m}}}, \mathrm{t}-\mathrm{t} \wedge \tau_{\mathrm{m}}\right)
\end{aligned}
$$

Using (1.1), (1.2) and the convexity of $u$ in the $\mathrm{x}_{\mathrm{n}}$ - variable, we obtain from (7.3)
(7.4)

$$
\begin{aligned}
u(x, t) \leq & E\left\{\int_{0}^{\mathrm{t} \wedge \tau_{m}^{m}}\left[\mathrm{~h}\left(\mathrm{X}_{\mathrm{s}}, \mathrm{t}-\mathrm{s}\right) \mathrm{ds}+\mathrm{f}(\mathrm{t}-\mathrm{s}) \mathrm{d} \xi(\mathrm{~s})\right]+\mathrm{g}\left(\mathrm{X}_{\mathrm{t}}\right)\right\} \\
& +\mathrm{E}\left\{\left(\mathrm{u}\left(\mathrm{X}_{\tau_{\mathrm{m}}}, \mathrm{t}-\tau_{\mathrm{m}}\right)-\mathrm{g}\left(\mathrm{X}_{\mathrm{t}}\right)\right) \chi_{\left\{\tau_{\mathrm{m}}<\mathrm{t}\right\}}\right\}
\end{aligned}
$$

where $\chi_{\mathrm{A}}$ denotes the indicator of the set A . If we take $\xi \equiv 0$, then $\mathrm{X}_{\mathrm{t}}=\mathrm{x}+\sqrt{2} \mathrm{~W}_{\mathrm{t}}$ and (7.4) implies

$$
\begin{align*}
u(x, t) \leq & \underset{m \uparrow \infty}{\lim E\left\{\int_{0}^{\mathrm{t} \Lambda \tau_{\mathrm{m}}} \mathrm{~m}\left(\mathrm{x}+\sqrt{2} \mathrm{~W}_{\mathrm{s}}, \mathrm{t}-\mathrm{s}\right) \mathrm{ds}+\mathrm{g}\left(\mathrm{x}+\sqrt{2} \mathrm{~W}_{\mathrm{t}}\right)\right\}}  \tag{7.5}\\
& +\underset{\mathrm{m} \uparrow \infty}{\lim } \mathrm{E}\left\{\left(\mathrm { u } \left(\mathrm{x}+\sqrt{2} \mathrm{~W}_{\tau_{\mathrm{m}}}^{\left.\left.\left., \mathrm{t}-\tau_{\mathrm{m}}\right)-\mathrm{g}\left(\mathrm{x}+\sqrt{2} \mathrm{~W}_{\mathrm{t}}\right)\right) \chi_{\left\{\tau_{\mathrm{m}}<\mathrm{t}\right\}}\right\}}\right.\right.\right. \\
= & E\left\{\int_{0}^{\mathrm{t}} \mathrm{~h}\left(\mathrm{x}+\sqrt{2} \mathrm{~W}_{\mathrm{s}}, \mathrm{t}-\mathrm{s}\right) \mathrm{ds}+\mathrm{g}\left(\mathrm{x}+\sqrt{2} \mathrm{~W}_{\mathrm{t}}\right)\right\}=\mathrm{U}(\mathrm{x}, \mathrm{t})
\end{align*}
$$

Now let an arbitrary control process $\xi$ be given. We shall show that

$$
\begin{equation*}
u(x, t) \leq J(x, t, \xi(\cdot)), \tag{7.6}
\end{equation*}
$$

and so we assume without loss of generality that $J(x, t, \xi(\cdot))<\infty$. This implies that

$$
\begin{equation*}
\lim _{\mathrm{m} \uparrow \infty} E \int_{t \wedge \tau_{\mathrm{m}}}^{\mathrm{t}}\left[\mathrm{~h}\left(\mathrm{X}_{\mathrm{s}}, \mathrm{t}-\mathrm{s}\right) \mathrm{ds}+\mathrm{f}(\mathrm{t}-\mathrm{s}) \mathrm{d} \xi(\mathrm{~s})\right]=0 \tag{7.7}
\end{equation*}
$$

from which we have

$$
\begin{align*}
& \lim _{\mathrm{m} \rightarrow \infty} \sup \mathrm{E}\left\{\left(\mathrm{u}^{*}\left(\mathrm{X}_{\tau_{\mathrm{m}}}, \mathrm{t}-\tau_{\mathrm{m}}\right)-\mathrm{g}\left(\mathrm{X}_{\mathrm{t}}\right)\right) \chi_{\left\{\tau_{\mathrm{m}}<\mathrm{t}\right\}}\right\}  \tag{7.8}\\
& \leq \lim _{\mathrm{m} \rightarrow \infty} \mathrm{E} \int_{\mathrm{t} \Lambda \tau_{\mathrm{m}}}^{\mathrm{t}}\left(\mathrm{~h}\left(\mathrm{X}_{\mathrm{s}}, \mathrm{t}-\mathrm{s}\right) \mathrm{ds}+\mathrm{f}(\mathrm{t}-\mathrm{s}) \mathrm{d} \xi(\mathrm{~s})\right)=0 .
\end{align*}
$$

From (7.4), (7.5) and the definition of $\mathrm{J}(\mathrm{x}, \mathrm{t}, \xi(\cdot))$, we have

$$
\begin{aligned}
u(x, t) \leq J(x, t, \xi(\cdot)) & -E\left\{\int_{t \wedge \tau_{m}}^{t} h\left(X_{s}, t-s\right) d s+f(t-s) d \xi(s)\right\} \\
& +E\left\{\left(U\left(X_{\tau_{m}}, t-\tau_{m}\right)-g\left(X_{t}\right)\right) \chi_{\left\{\tau_{m}<t\right\}}\right\}
\end{aligned}
$$

and (7.6) will follow from (7.7) provided we show

$$
\begin{equation*}
\underset{\mathrm{m} \rightarrow \infty}{\lim \sup } \operatorname{E}\left\{\left(\mathrm{U}\left(\mathrm{X}_{\tau_{\mathrm{m}}}, \mathrm{t}-\tau_{\mathrm{m}}\right)-\mathrm{g}\left(\mathrm{X}_{\mathrm{t}}\right)\right) \chi_{\left\{\tau_{\mathrm{m}}<\mathrm{t}\right\}}\right\}=0 \tag{7.9}
\end{equation*}
$$

But (7.9) will follow from (7.8) and

$$
\begin{equation*}
\underset{\mathrm{m} \rightarrow \infty}{\lim \sup } E\left[F\left(\mathrm{X}_{\tau_{\mathrm{m}}}, \mathrm{t}-\tau_{\mathrm{m}}\right) \chi_{\left\{\tau_{\mathrm{m}}<\mathrm{t}\right\}}\right] \leq 0 . \tag{7.10}
\end{equation*}
$$

Recalling that $F(x, t)$ is nondecreasing in the $x_{n}$-variable, we may write

$$
\begin{aligned}
\lim \sup _{\mathrm{m} \rightarrow \infty} & E\left[F\left(X_{\tau_{m}}, t-r_{m}\right) \chi_{\left\{\tau_{m}<t\right\}}\right] \\
& \leq \lim _{m \rightarrow \infty} \sup E\left[F\left(x+\sqrt{2} W_{\tau_{m}}, t-\tau_{m}\right) \chi_{\left\{\tau_{m}<t\right\}}\right]
\end{aligned}
$$

$$
\leq \lim _{\mathrm{m} \rightarrow \infty} \sup \mathrm{E}\left[\mathrm{U}\left(\mathrm{x}+\sqrt{2} \mathrm{~W}_{\tau_{\mathrm{m}}}, \mathrm{t}-\tau_{\mathrm{m}}\right) \chi_{\left\{\tau_{\mathrm{m}}<\mathrm{t}\right\}}\right]=0
$$

because U grows polynomially and $\tau_{\mathrm{m}} \uparrow \mathrm{t}$. This completes the proof of (7.6) for arbitrary $\xi$, from which we immediately conclude that $u(x, t) \leq$ $u^{*}(x, t)$.

To prove the reverse inequality, let $\xi\left({ }^{*}\right)$ be the control process defined in the statement of the theorem and let $X^{*}$. be the state process corresponding to it. The following follows from the definition of $\xi^{*}$ :

$$
\begin{equation*}
\left(\mathrm{X}_{\mathrm{s}}^{*}, \mathrm{t}-\mathrm{s}\right) \in \overline{\mathscr{B}} \quad \text { for all } \mathrm{s} \in(0, \mathrm{t}] \tag{7.11}
\end{equation*}
$$

$$
\begin{equation*}
X^{*}, \xi^{*}(\cdot) \text { are continuous on }(0, t] \tag{7.11}
\end{equation*}
$$

$$
\begin{equation*}
\xi^{*}\left(0^{+}\right)=\left[\mathrm{x}_{\mathrm{n}}-\mathrm{q}\left(\left(\mathrm{x}_{1}, \ldots, \mathrm{x}_{\mathrm{n}-1}\right), \mathrm{t}\right)\right]^{+} \tag{7.11}
\end{equation*}
$$

(7.11) (iv)

$$
\int_{0}^{\mathrm{t}} \mathrm{~d} \xi^{*}(\mathrm{~s})=\int_{0}^{\mathrm{t}} \chi_{\left\{\mathrm{s}:\left(\mathrm{X}_{\mathrm{s}}^{*}, \mathrm{t}-\mathrm{s}\right) \in \partial \not \subset\right\}} \mathrm{d} \xi^{*}(\mathrm{~s})
$$

Using (7.11) in (7.2) we arrive at $u(x, t)=J\left(x, t, \xi^{*}(\cdot)\right)$. $\quad \square$

## 8. APPENDIX

For $\varepsilon>0$, there is a smooth, polynomially growing, positive solution $\mathrm{v}^{\epsilon}$ to $(2.5)^{\epsilon}$ and (1.4). Such a solution is constructed as the limit of solutions to a sequence of boundary-value problems. See Sections 5.6 and 5.8 in Ladyzenskaya et al. [21], especially Remark 8.2 on page 496. Also, the details of such an approximation for an elliptic problem are given in the Appendix of [28].

The results of this section are either known or are obtained by slight modification of the proofs of known results. The reader may refer to [3], [4], [5], [6], [8], [9], [10], [11], [22], [23]. However, none of these references provides the results we need under precisely the conditions of our model. Our analysis is closely related to the one in Evans [8], especially the proofs of Lemma 8.3 and 8.4 below. We start with a comparison result for equation $(2.5)^{\epsilon}$.

Lemma 8.1. Suppose that $\underline{\underline{v}}, \overline{\mathrm{v}} \in \mathrm{C}^{2,1}\left(\mathbb{R}^{\mathrm{n}} \times(0, \infty)\right) \cap \mathrm{C}\left(\mathbb{R}^{\mathrm{n}} \times[0, \infty)\right)$ are polynomially growing sub and supersolutions to (2.5) ${ }^{\epsilon}$, respectively. Then, for all $(\mathrm{x}, \mathrm{t}) \in \mathbb{R}^{\mathrm{n}} \times[0, \infty)$,

$$
\begin{equation*}
\underline{v}(x, t)-\bar{v}(x, t) \leq e^{t} \sup _{z \in \mathbb{R}^{n}}(\underline{v}(z, 0)-\bar{v}(z, 0))^{+} . \tag{8.1}
\end{equation*}
$$

Proof. Since $\underline{v}$ and $\overline{\mathbf{v}}$ are polynomially growing, for each $\delta>0$ there is $m \geq 1$ such that the function

$$
\phi^{\delta}(\mathrm{x}, \mathrm{t}) \triangleq \mathrm{e}^{-\mathrm{t}}[\underline{\mathrm{v}}(\mathrm{x}, \mathrm{t})-\overline{\mathrm{v}}(\mathrm{x}, \mathrm{t})]-\delta|\mathrm{x}|^{2 \mathrm{~m}}
$$

achieves its maximum on $\mathbb{R}^{n} \times[0, \infty)$ at some point $\left(x^{*}, t^{*}\right)$. If $t^{*}=0$, then (8.1) holds, so we may assume that $t^{*}>0$. Then

$$
\begin{gathered}
0 \leq \phi_{\mathrm{t}}^{\delta}\left(\mathrm{x}^{*}, \mathrm{t}^{*}\right)-\Delta \phi^{\delta}\left(\mathrm{x}^{*}, \mathrm{t}^{*}\right) \leq \mathrm{e}^{-\mathrm{t}^{*}}\left[-\mathrm{v}\left(\mathrm{x}^{*}, \mathrm{t}^{*}\right)+\overline{\mathrm{v}}\left(\mathrm{x}^{*}, \mathrm{t}^{*}\right)+\beta_{\epsilon}\left(\overline{\mathrm{v}}\left(\mathrm{x}^{*}, \mathrm{t}^{*}\right)-\mathrm{f}\left(\mathrm{t}^{*}\right)\right)\right. \\
\\
\left.-\beta_{\epsilon}\left(\underline{\mathrm{v}}\left(\mathrm{x}^{*}, \mathrm{t}^{*}\right)-\mathrm{f}\left(\mathrm{t}^{*}\right)\right)\right]+\delta 2 \mathrm{~m}(2 \mathrm{~m}+\mathrm{n}-2)\left|\mathrm{x}^{*}\right|^{2(\mathrm{~m}-1)}
\end{gathered}
$$

$$
\begin{aligned}
= & -\phi^{\delta}\left(\mathrm{x}^{*}, \mathrm{t}^{*}\right)+\mathrm{e}^{-\mathrm{t}^{*}}\left[\beta_{\epsilon}\left(\overline{\mathrm{v}}\left(\mathrm{x}^{*}, \mathrm{t}^{*}\right)-\mathrm{f}\left(\mathrm{t}^{*}\right)\right)-\beta_{\epsilon}\left(\mathrm{v}\left(\mathrm{x}^{*}, \mathrm{t}^{*}\right)-\mathrm{f}\left(\mathrm{t}^{*}\right)\right)\right] \\
& +\delta\left[2 \mathrm{~m}(2 \mathrm{~m}+\mathrm{n}-2)\left|\mathrm{x}^{*}\right|^{2(\mathrm{~m}-1)}-\left|\mathrm{x}^{*}\right|^{2 \mathrm{~m}}\right]
\end{aligned}
$$

Using the inequality (3.9) we obtain

$$
(8.2) \phi^{\delta}\left(\mathrm{x}^{*}, \mathrm{t}^{*}\right) \leq \mathrm{e}^{-\mathrm{t}^{*}}\left[\beta_{\epsilon}\left(\overline{\mathrm{v}}\left(\mathrm{x}^{*}, \mathrm{t}^{*}\right)-\mathrm{f}\left(\mathrm{t}^{*}\right)\right)-\beta_{\epsilon}\left(\mathrm{v}\left(\mathrm{x}^{*}, \mathrm{t}^{*}\right)-\mathrm{f}\left(\mathrm{t}^{*}\right)\right)\right]+\delta \mathrm{K}_{\mathrm{m}}
$$

where $K_{\mathrm{m}} \triangleq 2(2 \mathrm{~m}+\mathrm{n}-2)^{\mathrm{m}}(2 \mathrm{~m}-2)^{\mathrm{m}-1}$. We claim that $\phi^{\delta}\left(\mathrm{x}^{*}, \mathrm{t}^{*}\right) \leq \delta \mathrm{K}_{\mathrm{m}}$. Indeed, if $\phi^{\delta}\left(\mathrm{x}^{*}, \mathrm{t}^{*}\right)$ is negative this inequality follows trivially. If $\phi^{\delta}\left(\mathbf{x}^{*}, \mathrm{t}^{*}\right)$ is non-negative then $\overline{\mathrm{v}}\left(\mathrm{x}^{*}, \mathrm{t}^{*}\right) \leq \mathrm{v}\left(\mathrm{x}^{*}, \mathrm{t}^{*}\right)$ and the claimed inequality follows from (8.2) and the monotonicity of $\beta_{\epsilon}$. Hence, for every $(x, t) \in \mathbb{R}^{n} \times[0, \infty)$,

$$
\begin{aligned}
\underline{v}(x, t)-\bar{v}(x, t)=e^{t}\left[\phi^{\delta}(x, t)\right. & \left.+\delta|x|^{2 m}\right] \leq e^{t}\left[\phi^{\delta}\left(x^{*}, t^{*}\right)+\delta|x|^{2 m}\right] \\
& \leq \delta e^{t}\left[\mathrm{~K}_{\mathrm{m}}+|\mathrm{x}|^{2 \mathrm{~m}}\right]
\end{aligned}
$$

Let $\delta$ go to zero to complete the proof.
回

Lemma 8.2. There are positive constants $K, m$, independent of $\epsilon$, such that

$$
\begin{equation*}
\left|v^{\epsilon}(x, t)\right|+\left|\nabla v^{\epsilon}(x, t)\right|+\left|v_{t}^{\epsilon}(x, t)\right| \leq K\left(1+|x|^{2 m}+t^{2 m}\right) \tag{8.3}
\end{equation*}
$$

Proof. Let V be the polynomially growing solution of (5.2) and (5.4).
Then, V is a supersolution to $(2.5)^{\epsilon}$ and Lemma 8.1 yields that
$v^{\epsilon}(x, t) \leq V(x, t), \forall x \in \mathbb{R}^{n}, t \geq 0, \epsilon>0$. For the lower bound, let $\underline{V}(x, t)$ be the non-negative, polynomially growing solution to (4.3). Then, $\underline{V}$ is a subsolution to $(2.5)^{\epsilon}$ and $v^{\epsilon}(x, t) \geq \underline{V}(x, t), \forall x \in \mathbb{R}^{n}, t \geq 0, \epsilon>0$.

To obtain the spatial derivative estimate, fix a unit vector $\nu \in \mathrm{R}^{\mathrm{n}}$. Set $\mathrm{w}^{\epsilon}(\mathrm{x}, \mathrm{t})=\nabla \mathrm{v}^{\epsilon}(\mathrm{x}, \mathrm{t}) \cdot \nu$. Differentiate $(2.5)^{\epsilon}$ to obtain
$(8.4)^{\epsilon} \quad \mathrm{w}_{\mathrm{t}}^{\epsilon}(\mathrm{x}, \mathrm{t})-\Delta \mathrm{w}^{\epsilon}(\mathrm{x}, \mathrm{t})+\beta_{\epsilon}^{\prime}\left(\mathrm{v}^{\epsilon}(\mathrm{x}, \mathrm{t})-\mathrm{f}(\mathrm{t})\right) \mathrm{w}^{\epsilon}(\mathrm{x}, \mathrm{t})=\nabla \mathrm{h}_{\mathrm{n}}(\mathrm{x}, \mathrm{t}) \cdot \nu$
with initial condition $\mathrm{w}^{\epsilon}(\mathrm{x}, 0)=\nabla \mathrm{g}_{\mathrm{n}}(\mathrm{x}) \cdot \nu$. Consider the equation $\mathrm{W}_{\mathrm{t}}(\mathrm{x}, \mathrm{t})-\Delta \mathrm{W}(\mathrm{x}, \mathrm{t})=\left|\nabla \mathrm{h}_{\mathrm{n}}(\mathrm{x}, \mathrm{t})\right|$, with initial condition $\mathrm{W}(\mathrm{x}, 0)=\left|\nabla \mathrm{g}_{\mathrm{n}}(\mathrm{x})\right|$. Then $W \geq 0$, and the nonnegativity of $\beta_{\epsilon}^{\prime}$ implies that $W$ is a supersolution to the linear equation $(8.4)^{\epsilon}$. Hence, the maximum principle yields $w^{\epsilon}(x, t) \leq W(x, t)$ for all unit vectors $\nu$ and consequently $\left|\nabla v^{\epsilon}(x, t)\right| \leq$ $W(x, t), \forall x \in \mathbb{R}^{n}, t \geq 0, \epsilon>0$. Finally, set $z^{\epsilon}(x, t)=v_{t}^{\epsilon}(x, t)-f^{\prime}(t)$. Then, $(2.5)^{\epsilon}$ implies that
$(8.5)^{\epsilon} \quad \mathrm{z}_{\mathrm{t}}^{\epsilon}(\mathrm{x}, \mathrm{t})-\Delta \mathrm{z}^{\epsilon}(\mathrm{x}, \mathrm{t})+\beta_{\epsilon}^{\prime}\left(\mathrm{v}^{\epsilon}(\mathrm{x}, \mathrm{t})-\mathrm{f}(\mathrm{t})\right) \mathrm{z}^{\epsilon}(\mathrm{x}, \mathrm{t})=\mathrm{h}_{\mathrm{nt}}(\mathrm{x}, \mathrm{t})-\mathrm{f}^{\prime \prime}(\mathrm{t})$,
with initial condition $z^{\epsilon}(x, 0)=\Delta g_{n}(x)+h_{n}(x, 0)-f^{\prime}(0)$. This equality follows from (recall (2.3))

$$
v_{t}^{\epsilon}(x, 0)=\Delta v^{\epsilon}(x, 0)+h_{n}(x, 0)-\beta_{\epsilon}\left(v^{\epsilon}(x, 0)-f(0)\right)=\Delta g_{n}(x)+h_{n}(x, 0)
$$

Let $Z$ be the unique polynomially growing solution to $Z_{t}(x, t)-\Delta Z(x, t)=$ $\left|h_{n t}(x, t)-f^{\prime \prime}(t)\right| \quad$ with initial condition $Z(x, 0)=\mid \Delta g_{n}(x)+h_{n}(x, 0)-$ $f^{\prime}(0) \mid$. Then, $Z,-Z$ are super and subsolutions to $(8.5)^{\epsilon}$, respectively.

Hence the maximum principle yields that

$$
\left|z^{\epsilon}(x, t)\right|=\left|v_{t}^{\epsilon}(x, t)-f^{\prime}(t)\right| \leq Z(x, t), \forall x \in \mathbb{R}^{n}, t \geq 0
$$

Lemma 8.3. There are positive constants $K, m$, independent of $\epsilon$, such that

$$
\begin{equation*}
\mathrm{v}_{\nu \nu}^{\epsilon}(\mathrm{x}, \mathrm{t}) \triangleq \sum_{\mathrm{i}, \mathrm{j}=1}^{\mathrm{n}} \mathrm{v}_{\mathrm{i} \mathrm{j}}^{\epsilon}(\mathrm{x}, \mathrm{t}) \nu_{\mathrm{i}} \nu_{\mathrm{j}} \leq \mathrm{K}\left(1+|\mathrm{x}|^{2 \mathrm{~m}}+\mathrm{t}^{2 \mathrm{~m}}\right) \tag{8.6}
\end{equation*}
$$

for all $\mathrm{x} \in \mathbb{R}^{\mathrm{n}}, \mathrm{t} \geq 0$ and unit vectors $\nu \in \mathbb{R}^{\mathrm{n}}$.

Proof. Differentiate $(2.5)^{\epsilon}$ twice:
$(8.7)^{\epsilon} \nu_{\nu \nu \mathrm{t}}^{\epsilon}(\mathrm{x}, \mathrm{t})-\Delta \mathrm{v}_{\nu \nu}^{\epsilon}(\mathrm{x}, \mathrm{t})+\beta_{\epsilon}^{\prime}\left(\mathrm{v}^{\epsilon}(\mathrm{x}, \mathrm{t})-\mathrm{f}(\mathrm{t})\right) \mathrm{v}_{\nu \nu}^{\epsilon}(\mathrm{x}, \mathrm{t}) \leq \mathrm{h}_{\mathrm{n} \nu \nu}(\mathrm{x}, \mathrm{t})$.

The initial condition is

$$
\begin{equation*}
\mathrm{v}_{\nu \nu}^{\epsilon}(\mathrm{x}, 0)=\mathrm{g}_{\mathrm{n} \nu \nu}(\mathrm{x}) \tag{8.8}
\end{equation*}
$$

Let $\bar{W}(x, t)$ be the unique polynomially growing solution to $\bar{W}_{t}(x, t)-$ $\Delta \bar{W}(x, t)=\left|h_{n \nu \nu}(x, t)\right|$ with initial conditon $\bar{W}(x, 0)=\left|g_{n \nu \nu}(x)\right|$. Then $\bar{W}$ is a supersolution to

$$
\overline{\mathrm{W}}_{\mathrm{t}}(\mathrm{x}, \mathrm{t})-\Delta \overline{\mathrm{W}}(\mathrm{x}, \mathrm{t})+\beta_{\epsilon}^{\prime}\left(\mathrm{v}^{\epsilon}(\mathrm{x}, \mathrm{t})-\mathrm{f}(\mathrm{t})\right) \overline{\mathrm{W}}(\mathrm{x}, \mathrm{t}) \geq \mathrm{h}_{\mathrm{n} \nu \nu}(\mathrm{x}, \mathrm{t})
$$

and the maximum principle yields that $v_{\nu \nu}^{\epsilon}(\mathrm{x}, \mathrm{t}) \leq \overline{\mathrm{W}}(\mathrm{x}, \mathrm{t}), \forall \mathrm{x} \in \mathbb{R}^{\mathrm{n}}, \mathrm{t} \geq 0 . \quad$.

Lemma 8.4 There are positive constants $K, m$, independent of $\epsilon$, such that

$$
\begin{equation*}
\left|\Delta v^{\epsilon}(x, t)\right| \leq K\left(1+|x|^{2 m}+t^{2 m}\right) \tag{8.9}
\end{equation*}
$$

Proof. Let $\eta$ be a smooth cut-off function satisfying

$$
\begin{equation*}
0 \leq \eta(\mathrm{x}) \leq 1, \tag{8.10}
\end{equation*}
$$

$$
\begin{equation*}
\eta \in \mathrm{C}^{\infty}\left(\mathbb{R}^{\mathrm{n}}\right) \tag{8.10}
\end{equation*}
$$

$$
\begin{align*}
& \eta(\mathrm{x})=1, \forall|\mathrm{x}| \leq 1,  \tag{8.10}\\
& \eta(\mathrm{x})=0, \forall|\mathrm{x}| \geq 2 .
\end{align*}
$$

(8.10) (iv)

For $R>0$ set

$$
\phi^{\mathrm{R}}(\mathrm{x}, \mathrm{t})=\eta\left(\frac{\mathrm{x}}{\mathrm{R}}\right) \beta_{\epsilon}\left(\mathrm{v}^{\epsilon}(\mathrm{x}, \mathrm{t})-\mathrm{f}(\mathrm{t})\right)
$$

and

$$
\left(\mathrm{x}^{*}, \mathrm{t}^{*}\right)=\arg \min \left\{\phi^{\mathrm{R}}(\mathrm{x}, \mathrm{t}): \mathrm{x} \in \mathbb{R}^{\mathrm{n}}, \mathrm{t} \in[0, \mathrm{~T}]\right\}
$$

Since we are trying to establish an upper bound for $\phi^{\mathrm{R}}\left(\mathrm{x}^{*}, \mathrm{t}^{*}\right)$, we may assume that $\phi^{R}\left(x^{*}, t^{*}\right)>1$, which implies that

$$
\beta_{\epsilon}\left(\mathrm{v}^{\epsilon}\left(\mathrm{x}^{*}, \mathrm{t}^{*}\right)-\mathrm{f}\left(\mathrm{t}^{*}\right)\right)>1
$$

The construction of $\beta_{\epsilon}$ (see (2.6)) yields that

$$
\begin{equation*}
\beta_{\epsilon}\left(\mathrm{r}^{\epsilon}\right)=\frac{\mathrm{r}^{\epsilon}}{\epsilon}-1, \tag{8.11}
\end{equation*}
$$

$$
\begin{equation*}
\beta_{\epsilon}^{\prime}\left(\mathrm{r}^{\epsilon}\right)=\frac{1}{\epsilon}, \quad \beta^{\prime \prime}\left(\mathrm{r}^{\epsilon}\right)=0 \tag{8.11}
\end{equation*}
$$

where $\mathrm{r}^{\epsilon} \triangleq \mathrm{v}^{\epsilon}\left(\mathrm{x}^{*}, \mathrm{t}^{*}\right)-\mathrm{f}\left(\mathrm{t}^{*}\right)$. We have $\mathrm{t}^{*}>0$ because $\mathrm{r}^{\epsilon}>0$, and equation (2.5) ${ }^{\epsilon}$ and (8.11) yield

$$
\begin{aligned}
& 0 \leq \phi_{\mathrm{t}}^{\mathrm{R}}\left(\mathrm{x}^{*}, \mathrm{t}^{*}\right)-\Delta \phi^{\mathrm{R}}\left(\mathrm{x}^{*}, \mathrm{t}^{*}\right)=\eta\left(\frac{\mathrm{x}^{*}}{\mathrm{R}}\right) \beta_{\epsilon}^{\prime}\left(\mathrm{r}^{\epsilon}\right)\left[\mathrm{v}_{\mathrm{t}}^{\epsilon}\left(\mathrm{x}^{*}, \mathrm{t}^{*}\right)-\mathrm{f}^{\prime}\left(\mathrm{t}^{*}\right)-\Delta \mathrm{v}^{\xi}\left(\mathrm{x}^{*}, \mathrm{t}^{*}\right)\right] \\
& -\eta\left(\frac{\mathrm{x}^{*}}{\mathrm{R}}\right) \beta_{\epsilon}^{\prime \prime}\left(\mathrm{r}^{\epsilon}\right)\left|\nabla \mathrm{v}^{\epsilon}\left(\mathrm{x}^{*}, \mathrm{t}^{*}\right)\right|^{2}-\frac{1}{\mathrm{R}^{2}} \Delta \eta\left(\frac{\mathrm{x}^{*}}{\mathrm{R}}\right) \beta_{\epsilon}\left(\mathrm{r}^{\epsilon}\right) \\
& -\frac{2}{R} \beta_{\epsilon}^{\prime}\left(\mathrm{r}^{\epsilon}\right) \nabla \eta\left(\frac{\mathrm{x}^{*}}{\mathrm{R}}\right) \cdot \nabla \mathrm{v}^{\epsilon}\left(\mathrm{x}^{*}, \mathrm{t}^{*}\right) \\
& =\frac{1}{\varepsilon}\left\{-\phi^{\mathrm{R}}\left(\mathrm{x}^{*}, \mathrm{t}^{*}\right)+\eta\left(\mathrm{x}_{\mathrm{R}}^{*}\right)\left[\mathrm{h}_{\mathrm{n}}\left(\mathrm{x}^{*}, \mathrm{t}^{*}\right)-\mathrm{f}^{\prime}\left(\mathrm{t}^{*}\right)\right]\right. \\
& \left.-\frac{1}{R^{2}}\left(\mathrm{r}^{\epsilon}-\epsilon\right) \Delta \eta\left(\frac{\mathrm{x}^{*}}{\mathrm{R}}\right)-\frac{2}{\mathrm{R}} \nabla \eta\left(\mathrm{x}_{\mathrm{R}}{ }^{*}\right) \cdot \nabla \mathrm{v}^{\epsilon}\left(\mathrm{x}^{*}, \mathrm{t}^{*}\right)\right\} \text {. }
\end{aligned}
$$

Since $\left|\mathrm{x}^{*}\right| \leq 2 \mathrm{R}$ and $\mathrm{t}^{*} \leq \mathrm{T}$, assumption (2.2) and the estimate (8.3) imply the existence of positive constants $K, m$ such that

$$
\begin{aligned}
& \phi^{\mathrm{R}}\left(\mathrm{x}^{*}, \mathrm{t}^{*}\right) \leq \eta\left(\frac{\mathrm{x}_{\mathrm{R}}}{\mathrm{R}}\right)\left[\mathrm{h}_{\mathrm{n}}\left(\mathrm{x}^{*}, \mathrm{t}^{*}\right)-\mathrm{f}^{\prime}\left(\mathrm{t}^{*}\right)\right] \\
& -\frac{1}{R^{2}}\left(\mathrm{r}^{\epsilon}-\epsilon\right) \Delta \eta\left(\frac{\mathrm{x}^{*}}{\mathrm{R}}\right)-\frac{2}{\mathrm{R}} \nabla \eta\left(\mathrm{X}_{\mathrm{R}}^{*}\right) \cdot \nabla \mathrm{v}^{\epsilon}\left(\mathrm{x}^{*}, \mathrm{t}^{*}\right) \\
& \leq \mathrm{K}\left(1+\mathrm{R}^{2 \mathrm{~m}}+\mathrm{T}^{2 \mathrm{~m}}\right) .
\end{aligned}
$$

Hence for any $\mathbf{x} \in \mathbb{R}^{\mathrm{n}}, \mathrm{T}>0$,

$$
\beta_{\epsilon}\left(\mathrm{v}^{\epsilon}(\mathrm{x}, \mathrm{~T})-\mathrm{f}(\mathrm{~T})\right)=\phi^{|\mathrm{x}|}(\mathrm{x}, \mathrm{~T}) \leq \phi^{|\mathrm{x}|}\left(\mathrm{x}^{*}, \mathrm{t}^{*}\right) \leq \mathrm{K}\left(1+|\mathrm{x}|^{2 \mathrm{~m}}+\mathrm{T}^{2 \mathrm{~m}}\right)
$$

We obtain (8.9) after observing that $\Delta v^{\epsilon}=v^{\epsilon}+h_{n}+\beta_{\epsilon}\left(v^{\epsilon}(x, t)-f(t)\right)$.

We conclude by proving a comparison result for equation (1.3). A direct consequence of it, with $\Omega=\mathbb{R}^{n}$, is the uniqueness of the polynomially growing solution to (1.3) and (1.4). The following generality is needed in Section 4.

Let $\Omega$ be a (possibly unbounded) nonempty domain in $\mathbb{R}^{n}$.

Lemma 8.5 Suppose that $\mathbf{v}, \overline{\mathbf{v}}$ satisfy the estimate (2.4) and are almost everywhere sub and supersolutions to (1.9) and (1.4) on $\Omega \times[0, \infty)$. Moreover assume that $\underline{\mathrm{y}} \leq \overline{\mathrm{v}}$ on $\partial \Omega \times[0, \infty)$. Then $\underline{\mathrm{v}} \leq \overline{\mathrm{v}}$ on $\Omega \times[0, \infty)$.

Proof. We regularize $\underline{v}$ and $\bar{v}$, first. Let $\xi$ be a $C^{\infty}$, non-negative function with the properties,

$$
\xi(x, t)=0 \text { if }|x|+|t|>\epsilon, \int_{\mathbb{R}^{n+1}} \xi(x, t) \mathrm{d} x d t=1
$$

For $\epsilon>0$, set $\Sigma_{\epsilon} \triangleq\{(x, t) \in \Omega \times[\epsilon, \infty)$ : distance $(x, \partial \Omega)>\epsilon\}$, and for $(x, t) \in \Sigma_{\epsilon}$, define

$$
\begin{gathered}
\underline{\mathbf{v}}^{\epsilon}(x, t) \triangleq \int_{|s|+|y| \leq 1} \underline{v}(x-\epsilon y, t-\epsilon s) \xi(y, s) d y d s \\
\bar{v}^{\epsilon}(x, t) \triangleq \int_{|s|+|y| \leq 1} \overline{\mathrm{v}}(\mathrm{x}-\epsilon \mathrm{y}, \mathrm{t}-\epsilon \mathrm{s}) \xi(\mathrm{y}, \mathrm{~s}) \mathrm{dy} \mathrm{ds}
\end{gathered}
$$

$$
\begin{gathered}
f^{\epsilon}(t) \triangleq \int_{|s|+|y| \leq 1} f(t-\epsilon s) d y d s \\
h_{n}^{\epsilon}(x, t) \triangleq \int_{|s|+|y| \leq 1} h_{n}(x-\epsilon y, t-\epsilon s) \xi(y, s) d y d s
\end{gathered}
$$

It is well known that $\underline{\underline{v}}^{\epsilon}$ and $\overline{\mathbf{v}}^{\epsilon}$ are infinitely differentiable, and converge to $\underline{\mathbf{v}}$ and $\overline{\mathbf{v}}$, respectively, as $\epsilon$ tends to zero. Moreover, $\underline{\mathbf{v}}^{\epsilon}$ is a subsolution to

$$
\begin{equation*}
\max \left\{\frac{\partial}{\partial t} w-\Delta w-h_{n}^{\epsilon}, w-f^{\epsilon}\right\}=0 \text { on } \Sigma_{\epsilon}, \tag{8.12}
\end{equation*}
$$

and $\overline{\mathbf{v}}^{\boldsymbol{\epsilon}}$ is a supersolution to a related equation (8.15), which we now derive.

Let $G$ be a compact subset of $\Omega$ such that $G \times[\epsilon, \infty) \subset \Sigma_{\epsilon}$. For $T>0$, set

$$
\alpha(G, \epsilon, T) \triangleq 3 \sup _{\substack{x \in G \\ 0 \leq t \leq T+\epsilon}}\left(|\overline{\mathrm{v}}(x, t)|+\left|\frac{\partial}{\partial \mathrm{t}} \overline{\mathrm{v}}(\mathrm{x}, \mathrm{t})\right|\right) .
$$

Now suppose that

$$
\begin{equation*}
\overline{\mathrm{v}}^{\epsilon}\left(\mathrm{x}_{0}, \mathrm{t}_{0}\right)<\mathrm{f}\left(\mathrm{t}_{0}\right)-\epsilon \alpha(\mathrm{G}, \epsilon, \mathrm{~T}) \tag{8.13}
\end{equation*}
$$

for some $\left(\mathrm{x}_{0}, \mathrm{t}_{0}\right) \in \mathrm{G} \times[\epsilon, \mathrm{T}]$. Then the definitions of $\overline{\mathrm{v}}^{\boldsymbol{\epsilon}}$ and $\alpha(\mathrm{G}, \epsilon, \mathrm{T})$ imply that $\overline{\mathrm{v}}(\mathrm{x}, \mathrm{t})<\mathrm{f}(\mathrm{t})$ whenever $\left|\mathrm{x}-\mathrm{x}_{0}\right|+\left|\mathrm{t}-\mathrm{t}_{0}\right| \leq \epsilon$, and consequently

$$
\frac{\partial}{\partial t} \bar{v}(x, t)-\Delta \bar{v}(x, t) \geq h_{n}(x, t) \quad \text { a.e. }\left|x-x_{0}\right|+\left|t-t_{0}\right| \leq \epsilon .
$$

Multiplying the above inequality by $\xi$ and integrating, we obtain

$$
\begin{equation*}
\frac{\partial}{\partial t} \bar{v}^{\epsilon}\left(\mathrm{x}_{0}, \mathrm{t}_{0}\right)-\Delta \overline{\mathrm{v}}^{\epsilon}\left(\mathrm{x}_{0}, \mathrm{t}_{0}\right) \geq \mathrm{h}_{\mathrm{n}}^{\epsilon}\left(\mathrm{x}_{0}, \mathrm{t}_{0}\right) . \tag{8.14}
\end{equation*}
$$

Recall that we assumed (8.13) to arrive at (8.14). Hence,
(8.15) $\max \left\{\frac{\partial}{\partial t} \overline{\mathrm{v}}^{\epsilon}(\mathrm{x}, \mathrm{t})-\Delta \overline{\mathrm{v}}^{\epsilon}(\mathrm{x}, \mathrm{t})-\mathrm{h}_{\mathrm{n}}^{\epsilon}(\mathrm{x}, \mathrm{t}), \overline{\mathrm{v}}^{\epsilon}(\mathrm{x}, \mathrm{t})-(\mathrm{f}(\mathrm{t})-\epsilon \alpha(\mathrm{G}, \epsilon, \mathrm{T}))\right\} \geq 0$

$$
\forall(x, t) \in G \times[\epsilon, T] .
$$

It is not difficult to construct a $\mathrm{C}^{\infty}$ function $\eta$ satisfying
(8.16) (i)

$$
0<\eta(\mathrm{x}) \leq \mathrm{e}^{-|\mathrm{x}|} \quad \forall \mathrm{x} \in \mathbb{R}^{\mathrm{n}}
$$

(8.16) (ii)

$$
|\nabla \eta(\mathbf{x})|+|\Delta \eta(\mathrm{x})| \leq 3 \eta(\mathrm{x}) \quad \forall \mathrm{x} \in \mathbb{R}^{\mathrm{n}}
$$

In fact, a mollification of $\psi(|x|)$ will work, where

$$
\psi(\xi) \triangleq \begin{cases}\left(1+\frac{n}{4}\right) \mathrm{e}^{-\mathrm{n} / 2}-\frac{\xi^{2}}{\mathrm{n}} \mathrm{e}^{-\mathrm{n} / 2} & \text { if } 0 \leq \xi \leq \frac{\mathrm{n}}{2}, \\ \mathrm{e}^{-\xi} & \text { if } \xi>\frac{\mathrm{n}}{2}\end{cases}
$$

Consider the auxiliary function

$$
\varphi(x, t) \triangleq e^{-20 t} \eta(x)(\underline{v}(x, t)-\bar{v}(x, t))
$$

Since $\underline{\mathbf{v}}$ and $\overline{\mathbf{v}}$ are polynomially growing, $\varphi$ achieves its maximum over $\bar{\Omega} \times[0, \infty)$ at some $\left(\mathrm{x}_{0}, \mathrm{t}_{0}\right)$. If $\mathrm{t}_{0}=0$ or $\mathrm{x}_{0} \in \partial \Omega$, then

$$
\begin{equation*}
\varphi\left(\mathrm{x}_{0}, \mathrm{t}_{0}\right) \leq 0 \tag{8.17}
\end{equation*}
$$

from which follows $\underline{v} \leq \overline{\mathrm{v}}$. We assume therefore that $\mathrm{x}_{0} \in \Omega$ and $\mathrm{t}_{0}>0$. For $\epsilon>0, \delta \geq 0$, define the smooth function

$$
\varphi^{\epsilon, \delta}(x, t)=e^{-20 t} \eta(x)\left(\underline{v}^{\epsilon}(x, t)-\bar{v}^{\epsilon}(x, t)\right)-\delta\left(\left|x-x_{0}\right|^{2}+\left|t-t_{0}\right|^{2}\right)
$$

Set $\epsilon_{0}=\frac{1}{2} \min \left\{\right.$ distance $\left.\left(\mathrm{x}_{0}, \partial \Omega\right), \mathrm{t}_{0}\right\}$. Then for fixed $\delta>0$, there is an $\epsilon^{*} \in\left(0, \epsilon_{0}\right)$ such that $\forall \epsilon \in\left(0, \epsilon^{*}\right)$, there exist ( $\mathrm{x}_{\epsilon}, \mathrm{t}_{\epsilon}$ ), depending on $\delta$, and satisfying $\left|x_{\epsilon}-x_{0}\right|+\left|t_{\epsilon}-t_{0}\right|<\epsilon_{0}$,

$$
\varphi^{\epsilon, \delta}\left(\mathrm{x}_{\epsilon}, \mathrm{t}_{\epsilon}\right)=\max \left\{\varphi^{\epsilon, \delta}(\mathrm{x}, \mathrm{t}):\left|\mathrm{x}-\mathrm{x}_{0}\right|+\left|\mathrm{t}-\mathrm{t}_{0}\right|<\varepsilon_{0}\right\}
$$

Note that $\left(x_{\epsilon}, \mathrm{t}_{\epsilon}\right) \in \Sigma_{\epsilon_{0}}\left[\Sigma_{\epsilon}\right.$ for all $\epsilon \in\left(0, \varepsilon_{0}\right)$. Hence, for all $\epsilon \in\left(0, \epsilon^{*}\right)$,

$$
\begin{gather*}
\max \left\{\frac{\partial}{\partial \mathrm{t}} \mathbf{y}^{\epsilon}\left(\mathrm{x}_{\epsilon}, \mathrm{t}_{\epsilon}\right)-\Delta \mathbf{y}^{\epsilon}\left(\mathrm{x}_{\epsilon}, \mathrm{t}_{\epsilon}\right)-\mathrm{h}_{\mathrm{n}}^{\epsilon}\left(\mathrm{x}_{\epsilon}, \mathrm{t}_{\epsilon}\right), \mathbf{y}^{\epsilon}\left(\mathrm{x}_{\epsilon}, \mathrm{t}_{\epsilon}\right)-\mathrm{f}^{\epsilon}\left(\mathrm{t}_{\epsilon}\right)\right\} \leq 0  \tag{8.18}\\
\max \left\{\frac{\partial}{\partial \mathrm{t}} \overline{\mathrm{v}}^{\epsilon}\left(\mathrm{x}_{\epsilon}, \mathrm{t}_{\epsilon}\right)-\Delta \overline{\mathrm{v}}^{\epsilon}\left(\mathrm{x}_{\epsilon}, \mathrm{t}_{\epsilon}\right)-\mathrm{h}_{\mathrm{n}}^{\epsilon}\left(\mathrm{x}_{\epsilon}, \mathrm{t}_{\epsilon}\right)\right. \tag{8.19}
\end{gather*}
$$

$$
\begin{gathered}
\max \left\{\frac{\partial}{\partial \mathrm{t}} \overline{\mathrm{v}}^{\epsilon}\left(\mathrm{x}_{\epsilon}, \mathrm{t}_{\epsilon}\right)-\Delta \overline{\mathrm{v}}^{\epsilon}\left(\mathrm{x}_{\epsilon}, \mathrm{t}_{\epsilon}\right)-\mathrm{h}_{\mathrm{n}}^{\epsilon}\left(\mathrm{x}_{\epsilon}, \mathrm{t} \epsilon\right)\right. \\
\left.\left.\overline{\mathrm{v}}^{\epsilon}\left(\mathrm{x}_{\epsilon}, \mathrm{t} \epsilon\right)-\mathrm{f}^{\epsilon}\left(\mathrm{t}_{\epsilon}\right)-\epsilon \alpha(\mathrm{G}, \epsilon, \mathrm{~T})\right)\right\} \leq 0
\end{gathered}
$$

where $G=\left\{x:\left|x-x_{0}\right| \leq \epsilon_{0}\right\}$ and $T=t_{0}+\epsilon_{0}$.
Continuing to hold $\delta$ fixed, we let $\varepsilon \downarrow 0$ along a subsequence so that ( $\mathrm{x}_{\epsilon}, \mathrm{t}_{\epsilon}$ ) converges to a limit $\left(\mathrm{x}^{*}, \mathrm{t}^{*}\right)$ satisfying $\left|\mathrm{x}^{*}-\mathrm{x}_{0}\right|+\left|\mathrm{t}^{*}-\mathrm{t}_{0}\right| \leq \epsilon_{0}$. But

$$
\begin{gathered}
\varphi\left(\mathrm{x}^{*}, \mathrm{t}^{*}\right)-\delta\left(\left|\mathrm{x}^{*}-\mathrm{x}_{0}\right|^{2}+\left|\mathrm{t}^{*}-\mathrm{t}_{0}\right|^{2}\right)=\lim _{\epsilon \downarrow 0} \varphi^{\epsilon, \delta}\left(\mathrm{x}_{\epsilon}, \mathrm{t} \epsilon\right) \\
\geq \lim _{\epsilon \downarrow 0} \varphi^{\epsilon, \delta}\left(\mathrm{x}_{0}, \mathrm{t}_{0}\right)=\varphi\left(\mathrm{x}_{0}, \mathrm{t}_{0}\right) \geq \varphi\left(\mathrm{x}^{*}, t^{*}\right),
\end{gathered}
$$

so $\mathrm{x}^{*}=\mathrm{x}_{0}, \mathrm{t}^{*}=\mathrm{t}_{0}$. It follows that $\lim _{\epsilon \backslash 0}\left(\mathrm{x}_{\epsilon}, \mathrm{t}_{\epsilon}\right)=\left(\mathrm{x}_{0}, \mathrm{t}_{0}\right)$, where the limit is taken over all $\epsilon \in\left(0, \epsilon^{*}\right)$.

If $\overline{\mathrm{v}}^{\epsilon}\left(\mathrm{x}_{\epsilon}, \mathrm{t}{ }_{\epsilon}\right) \geq \mathrm{f}\left(\mathrm{t}{ }_{\epsilon}\right)-\epsilon \alpha(\mathrm{G}, \epsilon, \mathrm{T})$, then (8.18) implies that $\varphi^{\epsilon, \delta}\left(\mathrm{x}_{\epsilon}, \mathrm{t}_{\epsilon}\right) \leq \epsilon \alpha(\mathrm{G}, \epsilon, \mathrm{T}) \eta\left(\mathrm{x}_{\epsilon}\right)+\mathrm{f}^{\epsilon}\left(\mathrm{t}_{\epsilon}\right)-\mathrm{f}\left(\mathrm{t} \mathrm{t}_{\epsilon}\right)$. Letting first $\mathrm{e} \downarrow 0$ and then $\delta \downarrow 0$, we obtain (8.17) and conclude as before that $\mathrm{x} \leq \overline{\mathrm{v}}$.

It remains to examine the case

$$
\begin{equation*}
\overline{\mathrm{v}}^{\epsilon}\left(\mathrm{x}_{\epsilon}, \mathrm{t}{ }_{\epsilon}\right)<\mathrm{f}\left(\mathrm{t}_{\epsilon}\right)-\epsilon \alpha(\mathrm{G}, \epsilon, \mathrm{~T}) . \tag{8.20}
\end{equation*}
$$

Because $\varphi^{\epsilon, \delta}$ has a local maximum at ( $\mathrm{x}_{\epsilon}, \mathrm{t}_{\epsilon}$ ), we have

$$
\begin{align*}
& 0 \leq \frac{\partial}{\partial \mathrm{t}} \varphi^{\epsilon, \delta}\left(\mathrm{x}_{\epsilon}, \mathrm{t}_{\epsilon}\right)-\Delta \varphi^{\epsilon, \delta}\left(\mathrm{x}_{\epsilon}, \mathrm{t}_{\epsilon}\right)  \tag{8.21}\\
& =\mathrm{e}^{-20 \mathrm{t}} \epsilon_{\eta\left(\mathrm{x}_{\epsilon}\right)\left\{-20\left[\underline{\mathbf{v}}^{\epsilon}\left(\mathrm{x}_{\epsilon}, \mathrm{t}_{\epsilon}\right)-\overline{\mathrm{v}}^{\epsilon}\left(\mathrm{x}_{\epsilon}, \mathrm{t}_{\epsilon}\right)\right]+\underline{\mathbf{v}}_{\mathrm{t}}{ }^{\epsilon}\left(\mathrm{x}_{\epsilon}, \mathrm{t}_{\epsilon}\right)-\overline{\mathrm{v}}_{\mathrm{t}}^{\epsilon}\left(\mathrm{x}_{\epsilon}, \mathrm{t}_{\epsilon}\right)\right.} \\
& -\Delta \underline{\mathbf{v}}^{\epsilon}\left(\mathrm{x}_{\epsilon} \mathrm{t}_{\epsilon}\right)+\Delta \overline{\mathrm{v}}^{\epsilon}\left(\mathrm{x}_{\epsilon}, \mathrm{t}_{\epsilon}\right)-\left[\underline{\mathbf{v}}^{\epsilon}\left(\mathrm{x}_{\epsilon}, \mathrm{t}_{\epsilon}\right)-\overline{\mathrm{v}}^{\epsilon}\left(\mathrm{t}_{\epsilon}, \mathrm{t}_{\epsilon}\right)\right] \frac{\Delta \eta\left(\mathrm{x}_{\epsilon}\right)}{\eta\left(\mathrm{x}_{\epsilon}\right)} \\
& \left.-2\left[\nabla \underline{\underline{v}}^{\epsilon}\left(\mathbf{x}_{\epsilon}, \hat{t}_{\epsilon}\right)-\nabla \bar{v}^{\epsilon}\left(\mathbf{x}_{\epsilon}, \mathrm{t}_{\epsilon}\right)\right] \cdot \frac{\nabla \eta\left(\mathrm{x}_{\epsilon}\right)}{\eta\left(\mathbf{x}_{\epsilon}\right)}\right\}+2 \delta\left(\mathrm{n}+\mathrm{t}_{0}-\mathrm{t}_{\epsilon}\right) \\
& \leq e^{-20 \mathrm{t}} \epsilon \eta\left(\mathrm{x}_{\epsilon}\right)\left\{-\left(20+\frac{\Delta \eta\left(\mathrm{x}_{\epsilon}\right)}{\eta\left(\mathrm{x}_{\epsilon}\right)}\left[\mathrm{v}^{\epsilon}\left(\mathrm{x}_{\epsilon}, \mathrm{t}_{\epsilon}\right)-\overline{\mathrm{v}}^{\epsilon}\left(\mathrm{x}_{\epsilon}, \mathrm{t}_{\epsilon}\right)\right]\right.\right.
\end{align*}
$$

$$
\left.-2\left[\nabla \underline{v}^{\epsilon}\left(\mathrm{x}_{\epsilon}, \mathrm{t}_{\epsilon}\right)-\nabla \bar{v}^{\epsilon}\left(\mathrm{x}_{\epsilon}, \mathrm{t}_{\epsilon}\right)\right] \cdot \frac{\nabla \eta\left(\mathrm{x}_{\epsilon}\right)}{\eta\left(\mathrm{x}_{\epsilon}\right)}\right\}+2 \delta\left(\mathrm{n}+\mathrm{t}_{0}-\mathrm{t}_{\epsilon}\right),
$$

where we have used $(8.18)-(8.20)$ to obtain the last inequality. Also,

$$
\begin{align*}
0= & \nabla \varphi^{\epsilon, \delta^{\prime}}\left(\mathrm{x}_{\epsilon}, \mathrm{t}_{\epsilon}\right) \cdot \nabla \eta\left(\mathrm{x}_{\epsilon}\right)  \tag{8.22}\\
= & \mathrm{e}^{-20 \mathrm{t}} \epsilon^{\eta}\left(\mathrm{x}_{\epsilon}\right)\left\{\left[\nabla \underline{\mathrm{v}}^{\epsilon}\left(\mathrm{x}_{\epsilon}, \mathrm{t}_{\epsilon}\right)-\nabla \bar{v}^{\epsilon}\left(\mathrm{x}_{\epsilon}, \mathrm{t}_{\epsilon}\right)\right] \cdot \cdot \nabla \eta\left(\mathrm{x}_{\epsilon}\right)\right. \\
& \left.+\left(\underline{\mathrm{v}}^{\epsilon}\left(\mathrm{x}_{\epsilon}, \mathrm{t}_{\epsilon}\right)-\overline{\mathrm{v}}^{\epsilon}\left(\mathrm{x}_{\epsilon}, \mathrm{t}_{\epsilon}\right)\right) \frac{\left|\nabla \eta\left(\mathrm{x}_{\epsilon}\right)\right|^{2}}{\eta\left(\mathrm{x}_{\epsilon}\right)}\right\}-2 \delta\left(\mathrm{x}_{\epsilon}-\mathrm{x}_{0}\right) \cdot \nabla \eta\left(\mathrm{x}_{\epsilon}\right)
\end{align*}
$$

Substitution of (8.22) into (8.21) allows us to eliminate the $\nabla \underline{v}^{\epsilon}-\nabla \overline{\mathrm{v}}{ }^{\epsilon}$ term in the latter equation. We may then invoke the bound (8.16)(ii) to obtain

$$
\begin{aligned}
& \begin{array}{l}
0 \leq \mathrm{e}^{-20 \mathrm{t}} \epsilon\left(\underline{v}^{\epsilon}\left(\mathrm{x}_{\epsilon}, \mathrm{t}_{\epsilon}\right)-\overline{\mathrm{v}}{ }^{\epsilon}\left(\mathrm{x}_{\epsilon}, \mathrm{t}_{\epsilon}\right)\right)\left(-10 \eta\left(\mathrm{x}_{\epsilon}\right)-\Delta \eta\left(\mathrm{x}_{\epsilon}\right)+2 \frac{\left|\nabla \eta\left(\mathrm{x}_{\epsilon}\right)\right|^{2}}{\eta\left(\mathrm{x}_{\epsilon}\right)}\right) \\
\quad+2 \delta\left(\mathrm{n}+\mathrm{t}_{0}-\mathrm{t}_{\epsilon}-2\left(\mathrm{x}_{\epsilon}-\mathrm{x}_{0}\right) \cdot \frac{\nabla \eta\left(\mathrm{x}_{\epsilon}\right)}{\eta\left(\mathrm{x}_{\epsilon}\right)}\right) \\
\leq-2 \mathrm{e}^{-20 \mathrm{t}} \epsilon \quad \eta\left(\mathrm{x}_{\epsilon}\right)\left(\underline{v}^{\epsilon}\left(\mathrm{x}_{\epsilon}, \mathrm{t}_{\epsilon}\right)-\bar{v}^{\epsilon}\left(\mathrm{x}_{\epsilon}, \mathrm{t}_{\epsilon}\right)\right)+2 \delta\left(\mathrm{n}+\mathrm{t}_{0}-\mathrm{t}_{\epsilon}-6\left|\mathrm{x}_{\epsilon}-\mathrm{x}_{0}\right|\right)
\end{array}
\end{aligned}
$$

Letting first $\epsilon \downharpoonright 0$ and then $\delta \downarrow 0$ again yields (8.17).

## Proofs of Theorem 2.1 and 2.2.

Estimates (8.6) and (8.9) imply the second-derivative estimate (2.4) for $\mathbf{v}^{\epsilon}$, and this estimate is uniform in $\epsilon$. This, together with (8.3), gives (2.4). Using this estimate we choose a subsequence, denoted by $\epsilon$ again, such that $\mathrm{v}^{\epsilon}, \nabla \mathrm{v}^{\epsilon}$ converge uniformly on bounded subsets. Let v be the limit; then v also satisfies (2.4). Moreover, using the weak formulation of $(2.5)^{\epsilon}$, we conclude that the limit v is a solution to (1.3), and trivially to (1.4). Suppose $\overline{\mathrm{v}}$ is another point of the sequence $\mathrm{v}^{\epsilon}$ as $\epsilon$ tends to zero. Then $\overline{\mathrm{v}}$ satisfies (1.3), (1.4) and (2.4). Since there is only one function (recall Lemma 8.5 with $\Omega=\mathbb{R}^{\mathrm{n}}$ ) satisfying (1.3), (1.4) and (2.4), $\overline{\mathrm{v}}=\mathrm{v}$. Hence $\mathrm{v}^{\epsilon}$ converges to the unique solution v on the whole sequence. $\quad$ व

## Footnotes

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