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# GENERALIZED ONE-SIDED ESTIMATES FOR SOLUTIONS OF HAMILTON-JACOBI EQUATIONS AND APPLICATIONS

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## 1. INTRODUCTION

IN THIS paper we continue the analysis, started in [1], of the local properties of solutions of the Hamilton-Jacobi equation

$$-u_t + H(t, x, \nabla u) = 0 \tag{1.1}$$

and, more generally, of the equation

$$u_t + H(t, x, u, \nabla u) = 0, \tag{1.2}$$

where $t \in [0, T]$ and $x \in \Omega \subset \mathbf{R}^n$. Here and in the following we set

$$u_t = \frac{\partial u}{\partial t}, \qquad \nabla u = \left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}\right), \qquad Du = (u_t, \nabla u).$$

Moreover, the Hamiltonian $H$ is assumed to be strictly convex in the set of variables corresponding to $\nabla u$ and Lipschitz continuous in the remaining variables.

For various reasons, we consider solutions to equations (1.1) and (1.2) in the class of locally Lipschitz functions satisfying the equation in the viscosity sense, introduced by Crandall and Lions in [2]. Indeed, quite general existence and uniqueness theorems are now available for viscosity solutions, as the result of the work of several authors such as Lions [3], Crandall, Evans and Lions [4], Ishii [5], Souganidis [6] and Jensen [7]. Furthermore, solutions of this class can be interpreted as the value functions of certain variational problems, see e.g. [3, 8].

This paper is mainly concerned with the behaviour of a Lipschitz continuous viscosity solution, $u$, about a singular point, i.e. a point at which $u$ is not differentiable. This problem was studied in

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[1], assuming the following one-sided bound on $u$: for some $\alpha \in (0, 1]$

$$u(t+h, x+y) + u(t-h, x-y) - 2u(t, x) \leq C(|h| + |y|)^{1+\alpha}. \tag{1.3}$$

Estimate (1.3) holds for viscosity solutions of (1.1) if $H$ is sufficiently smooth in $t, x$ (see [1, 3, 5]).

One of the purposes of this paper is to extend the analysis of [1] to a more general class of solutions, including the solutions of (1.2). For this reason, in Section 3, we prove a generalized version of (1.3) which holds without extra assumptions on $H$.

In Section 4 we apply this generalized one-sided estimate to show that

$$D^+u(t, x) = \partial u(t, x), \tag{1.4}$$

where $D^+u$ and $\partial u$ denote, respectively, the superdifferential and the generalized gradient of $u$ in the sense of Clarke [9] (see Section 2 for definitions). The equality in (1.4) represents a connection between the theory of viscosity solutions and the approach to optimization problems described in [9]. As a consequence of this connection, we derive the existence of the (one-sided) directional derivatives of $u$ at any point $(t, x)$, a result known for the value function of some optimal control problems see [8]. Furthermore, we obtain that these derivatives can be expressed in terms of the support function of the set $D^+u(t, x)$.

Then, using the ideas of [1], we deduce analytical properties of $u$, in a neighbourhood of $(t, x)$, by using geometric properties of $D^+u(t, x)$. For example we show that, at any singular point $(t, x)$ of $u$, the superdifferential $D^+u(t, x)$ possesses "exposed faces" (theorem 5.7). Exposed faces of $D^+u(t, x)$ play a central role in our analysis. In fact, one can always find singular points of $u$, approaching $(t, x)$ along any "interior normal direction" to such a face (theorem 4.9). In particular, in Section 5 we conclude that any singularity of a Lipschitz continuous viscosity solution of (1.1) (resp. (1.2)) has to propagate backward (resp. forward) in time.

Conversely, $u$ is regular along all directions associated to an "exposed point" of $D^+u(t, x)$. These points can be also viewed as the limit points of $Du$, or even as the points of $D^+u(t, x)$ at which the equation is satisfied, see theorem 5.3. In Section 6 of this paper we apply these ideas to a variational context. Extending classical results [10], we establish a one-to-one correspondence between the minimizers of a problem in calculus of variations and the exposed points of the superdifferential of the corresponding value function.

## 2. NOTATION AND PRELIMINARIES

Let $N$ be a positive integer. We denote by $|P|$ the norm of $P \in \mathbf{R}^N$ and by $P \cdot Q$ the scalar product of $P$ and $Q$. We set

$$S^{N-1} = \{\theta \in \mathbf{R}^N : |\theta| = 1\}.$$

For any $X \in \mathbf{R}^N$, $r > 0$ and $\theta \in S^{N-1}$ we define

$$B_r(X) = \{Y \in \mathbf{R}^N : |Y - X| < r\}, \qquad B_r = B_r(0)$$

$$K_r(X, \theta) = \{X + \lambda Y \in \mathbf{R}^N : Y \in B_r(\theta), Y \cdot \theta = 1, 0 < \lambda < r\}.$$

The sets $K_r(X, \theta)$ are open cones with vertex at $X$ and $\theta$ as symmetry axis.

Let $Q$ be an open domain in $\mathbf{R}^N$ and $n > 0$ an integer. We denote by $W^{1,\infty}(Q, \mathbf{R}^n)$ the space of the bounded functions $u : Q \to \mathbf{R}^n$ that are *Lipschitz continuous* in $Q$, i.e.

$$|u|_{1,\infty,Q} = \sup_{X,Y \in Q, X \neq Y} \frac{|u(X) - u(Y)|}{|X - Y|} < +\infty.$$

We say that $u \in W^{1,\infty}(Q, \mathbf{R}^n)_{\text{loc}}$ if $u$ is locally Lipschitz continuous in $Q$, i.e. $u \in W^{1,\infty}(B, \mathbf{R}^n)$ for every ball $B \subset\subset Q$. We set

$$W^{2,\infty}(Q, \mathbf{R}^n) = \{u \in W^{1,\infty}(Q, \mathbf{R}^n) : Du \in W^{1,\infty}(Q, \mathbf{R}^{nN})\}$$

and we define $W^{2,\infty}(Q, \mathbf{R}^n)_{\text{loc}}$ in a similar way. Also, we abbreviate for $k = 1, 2$

$$W^{k,\infty}(Q, \mathbf{R}) = W^{k,\infty}(Q)$$
$$W^{k,\infty}(Q, \mathbf{R})_{\text{loc}} = W^{k,\infty}(Q)_{\text{loc}}.$$

For any function $u : Q \to R$ and $X \in Q$ the sets

$$\begin{aligned} D^+u(X) &= \left\{P \in \mathbf{R}^N : \limsup_{Y \to X} \frac{u(Y) - u(X) - P \cdot (Y - X)}{|X - Y|} \le 0\right\} \\ D^-u(X) &= \left\{P \in \mathbf{R}^N : \liminf_{Y \to X} \frac{u(Y) - u(X) - P \cdot (Y - X)}{|X - Y|} \ge 0\right\} \end{aligned} \tag{2.1}$$

are known, respectively, as the *superdifferential* and the *subdifferential* of $u$ at $X$. In general, $D^{\pm}u(X)$ are closed convex sets. They extend the usual notion of gradient in the sense that, if both $D^+u(X)$ and $D^-u(X)$ are nonempty, then $u$ is differentiable at $X$ and $D^{\pm}u(X) = \{Du(X)\}$.

The generalized gradients above are used to define viscosity solutions of a first-order PDE

$$f(X, u(X), Du(X)) = 0 \qquad \text{in } Q, \tag{2.2}$$

where $f$ is a continuous function on $Q \times \mathbf{R} \times \mathbf{R}^N$ [2, 4]. A *viscosity solution* $u$ of (2.2) is a continuous function on $Q$ satisfying

$$\begin{aligned} f(X, u(X), P) &\le 0 \qquad \forall\, X \in Q,\ \forall\, P \in D^+u(X) \\ f(X, u(X), P) &\ge 0 \qquad \forall\, X \in Q,\ \forall\, P \in D^-u(X). \end{aligned} \tag{2.3}$$

Another extension of the notion of the gradient is the following, see Clarke [9]. Let $u \in W^{1,\infty}(Q)_{\text{loc}}$. For any $\theta \in \mathbf{R}^N$ we define

$$\begin{aligned} \partial^+u(X;\theta) &= \limsup_{Y \to X, \lambda \downarrow 0} \frac{u(Y + \lambda\theta) - u(Y)}{\lambda} \\ \partial^-u(X;\theta) &= \liminf_{Y \to X, \lambda \downarrow 0} \frac{u(Y + \lambda\theta) - u(Y)}{\lambda}. \end{aligned} \tag{2.4}$$

*Remark* 2.1. It is easy to show (see proposition 2.1.1 in [9]) that the function $\theta \to \partial^+u(X;\theta)$ (resp. $\theta \to \partial^-u(X;\theta)$) is Lipschitz continuous and subadditive (resp. superadditive) on $\mathbf{R}^N$, while the function $X \to \partial^+u(X;\theta)$ (resp. $X \to \partial^-u(X;\theta)$) is upper (resp. lower) semicontinuous on $Q$.

Since $\partial^+ u(X;\theta) = -\partial^- u(X;-\theta)$, we have that

$$\{P \in \mathbf{R}^N\colon \partial^+ u(X;\theta) \ge P\cdot\theta,\ \forall\, \theta \in \mathbf{R}^N\} = \{P \in \mathbf{R}^N\colon \partial^- u(X;\theta) \le P\cdot\theta,\ \forall\, \theta \in \mathbf{R}^N\}. \tag{2.5}$$

So, we define the *generalized gradient* $\partial u(X)$ to be any of the sets in (2.5). From (2.1) and (2.5) it follows that

$$D^+ u(X) \subset \partial u(X). \tag{2.6}$$

Moreover, $\partial u(X)$ is nonempty, convex, compact and

$$\partial^- u(X;\theta) = \min\{P\cdot\theta : P \in \partial u(X)\}, \qquad \forall\, \theta \in \mathbf{R}^N. \tag{2.7}$$

Also, $\partial u$ is closed on $Q$ as a set-valued function (see proposition 2.1.5 in [9]), i.e.

$$X_k \to X, \qquad P_k \in \partial u(X_k), \qquad P_k \to P \Rightarrow P \in \partial u(X). \tag{2.8}$$

We say that $P \in \mathbf{R}^N$ is *achievable for $u$ at $X$* if there exists a sequence of points $X_k \in Q$, at which $u$ is differentiable, such that

$$X = \lim_k X_k, \qquad P = \lim_k Du(X_k). \tag{2.9}$$

We denote by $Au(X)$ the set of all the achievable points $P$ for $u$ at $X$. Clearly, $Au(X)$ is closed and

$$X_k \to X, \qquad P_k \in Au(X_k), \qquad P_k \to P \Rightarrow P \in Au(X). \tag{2.10}$$

A useful characterization of $\partial u(X)$ is the following,

$$\partial u(X) = \operatorname{conv} Au(X) \tag{2.11}$$

where "conv" denotes the convex hull (see theorem 2.5.1 in [9]).

When studying the differentiability of a function $u \in W^{1,\infty}(Q)_{\rm loc}$ at a point $X \in Q$, it is useful to blow up the space at that point. This is done as follows. For $0 < \delta < d = \operatorname{dist}(X, \partial Q)$, we define

$$u_{\delta,X}(Y) = \frac{u(X+\delta Y) - u(X)}{\delta}, \qquad |Y| \le 1. \tag{2.12}$$

Then, the generalized gradients of $u$ and $u_{\delta,X}$ are related as follows:

$$D^+ u_{\delta,X}(Y) = D^+ u(X+\delta Y), \qquad \partial u_{\delta,X}(Y) = \partial u(X+\delta Y). \tag{2.13}$$

Moreover, the functions $u_{\delta,X}$ are Lipschitz continuous and bounded in $\bar{B}_1$ uniformly with respect to $\delta$.

*Definition* 2.2. We denote by $\mathbf{U}_x$ the set of all the limit points of $u_{\delta,X}$ as $\delta \downarrow 0$, in the topology of the uniform convergence in $B_1$.

The following lemma provides an upper and a lower bound for the elements of $\mathbf{U}_x$.

LEMMA 2.3. *For each $v \in \mathbf{U}_x$ and $\theta \in \bar{B}_1$*

$$\min\{P\cdot\theta\colon P \in \partial u(X)\} \le v(\theta) \le \inf\{P\cdot\theta : P \in D^+ u(X)\}. \tag{2.14}$$

Obviously, we assume the infimum in (2.14) to be $+\infty$ if $D^+ u(X) = \varnothing$.

*Proof.* Since by the definition (2.4) we have

$$\partial^- u(X; \theta) \leq v(\theta), \qquad \forall\ \theta \in \bar{B}_1$$

the first inequality in (2.14) follows from (2.7). To prove the remaining estimate, notice that

$$v(\theta) \leq \limsup_{\delta \downarrow 0} u_{\delta, X}(\theta) \leq P \cdot \theta$$

for all $\theta \in \bar{B}_1$ and $P \in D^+ u(X)$. ■

We now recall some properties of the classical *Legendre tranformation.* Let $H = H(X, p)$ be a real-valued function defined on $Q \times \mathbf{R}^n$ and assume that

$$\begin{aligned} &H(X, \lambda p^1 + (1 - \lambda)p^2) < \lambda H(X, p^1) + (1 - \lambda)H(X, p^2) \\ &\forall\ X \in Q, \qquad \forall\ p^1, p^2 \in \mathbf{R}^n, p^1 \neq p^2, \qquad \forall \lambda \in (0, 1) \end{aligned} \tag{2.15}$$

$$\lim_{|p| \to \infty} \inf_{X \in Q \cap B_r} \frac{H(X, p)}{|p|} = +\infty \qquad \text{for all } r > 0; \tag{2.16}$$

$$|H(X, p) - H(Y, p)| < C_r|X - Y| \qquad \text{for } |X|, |Y|, |p| \leq r. \tag{2.17}$$

Then, the *Legendre transform* of $H$, defined as

$$L(X, p) = \sup_{p \in R^n} [-p \cdot q - H(X, p)], X \in Q, q \in \mathbf{R}^n \tag{2.18}$$

is convex in $q$ and satisfies

$$\lim_{|q| \to \infty} \inf_{X \in Q \cap B_r} \frac{L(X, q)}{|q|} = +\infty \qquad \text{for all } r > 0; \tag{2.19}$$

$$|L(X, q) - L(Y, q)| \leq C_r|X - Y| \qquad \text{for } |X|, |Y|, |q| \leq r. \tag{2.20}$$

Moreover, from (2.15) it follows that the supremum in (2.18) is attained at a unique point $p^*(X, q)$. Consequently, $L$ is continuously differentiable with respect to $q$ and

$$p^*(X, q) = -D_q L(X, q). \tag{2.21}$$

In particular, there exists a function $\omega(r, s) \geq 0$, with $\omega(r, s) \to 0$ as $s \to 0$, such that $\forall\ r > 0$

$$|p^*(X, q) - p^*(X, q')| \leq \omega(r, |q - q'|) \tag{2.22}$$

for all $X, q, q'$ satisfying $|X|, |q|, |q'| \leq r$. We may assume that $\omega$ is nondecreasing in both $r$ and $s$.

LEMMA 2.4. Assume (2.15), (2.16) and (2.17). Then, for any $\sigma \in [0, 1]$ and $r > 0$

$$\sigma L(X, q + (1 - \sigma)q') + (1 - \sigma)L(X, q - \sigma q') - L(X, q) \leq \sigma(1 - \sigma)|q'|\omega(2r, |q'|) \tag{2.23}$$

for all $X, q, q'$ satisfying $|X|, |q|, |q'| \leq r$.

*Proof.* Set

$$p^1 = p^*(X, q + (1 - \sigma)q'), \qquad p^2 = p^*(X, q - \sigma q').$$

Then

$$\begin{aligned}
&\sigma L(X, q + (1-\sigma)q') + (1-\sigma)L(X, q - \sigma q') - L(X, q) \\
&\quad \le \sigma\{-p^1 \cdot [q + (1-\sigma)q'] - H(X, p^1)\} + (1-\sigma)\{-p^2 \cdot (q - \sigma q') - H(X, p^2)\} \\
&\qquad - \sigma\{-p^1 \cdot q - H(X, p^1)\} - (1-\sigma)\{-p^2 \cdot q - H(X, p^2)\} \\
&\quad \le \sigma(1-\sigma)|q'|\,|p^1 - p^2|.
\end{aligned}$$

So, the conclusion follows by (2.22). ■

## 3. A GENERALIZED ONE-SIDED ESTIMATE

Consider a locally Lipschitz viscosity solution $u$ of the equation

$$-u_t + H(t, x, \nabla u) = 0 \qquad (t, x) \in (0, T) \times \Omega, \tag{3.1}$$

where $\Omega$ is an open domain in $\mathbf{R}^n$ and $T > 0$. The Hamiltonian $H(t, x, p)$ is a real-valued function defined on $(0, T) \times \Omega \times \mathbf{R}^n$. We assume that *$H$ satisfies* (2.15) *and* (2.17) *with* $X = (t, x)$, $Y = (s, y)$, $Q = (0, T) \times \Omega$.

Fix $(t_0, x_0) \in (0, T) \times \Omega$ and define $u_\delta$ according to (2.12), that is

$$u_\delta(t, x) = \delta^{-1}\{u(t_0 + \delta t, x_0 + \delta x) - u(t_0, x_0)\}, \qquad x \in B_2,\, t \in [-1, 1], \tag{3.2}$$

where

$$0 < \delta < d_0 = 2^{-1} \min\{t_0, T - t_0, \text{dist.}(x_0, \partial\Omega)\}.$$

Recalling (2.13), it is easy to see that the functions $u_\delta$ satisfy, in the viscosity sense,

$$-(u_\delta)_t + H_\delta(t, x, \nabla u_\delta) = 0, \tag{3.3}$$

where

$$H_\delta(t, x, p) = H(t_0 + \delta t, x_0 + \delta x, p).$$

Clearly, $u_\delta \in W^{1,\infty}([-1, 1] \times B_2)$, uniformly for $\delta \in (0, d_0)$. Therefore, we may assume, without loss of generality, that $H_\delta$ satisfies (2.16) uniformly for $\delta \in (0, d_0)$. In fact, equation (3.3) is not affected by modifying $H$ outside a large ball. Consequently, by dynamic programming (see e.g. [3, 8]),

$$u_\delta(t, x) = \inf\left\{\int_t^\theta L_\delta(s, \eta(s), \dot\eta(s))\,ds + u_\delta(\theta, \eta(\theta)) : \eta \in W^{1,\infty}([t, \theta], \mathbf{R}^n), \eta(t) = x\right\}, \tag{3.4}$$

where $L_\delta$ is the Legendre transform of $H_\delta$ defined according to (2.18), and

$$\theta = \inf\{\tau \in [t, 1] : (\tau, \eta(\tau)) \in (\{1\} \times B_2) \cup ([t, 1] \times \partial B_2)\}.$$

*Remark* 3.1. Using standard control arguments it follows that $\eta(s)$ in (3.4) may be assumed to satisfy

$$|\dot\eta(s)| \le C_0 \qquad \text{for a.e. } s \in [t, \theta], \tag{3.5}$$

where $C_0$ is a constant independent of $x$, $t$ and $\delta$. In fact, since $L_\delta$ satisfies (2.19) uniformly for

$\delta \in (0, d_0)$, for each $r > 0$ there exists $C(r) > 0$ such that

$$|p| \leq r \Rightarrow H_\delta(t, x, p) = \sup_{|q| \leq C(r)} [-p \cdot q - L_\delta(t, x, q)]$$

for all $x \in B_2$, $t \in [-1, 1]$ and $\delta \in (0, d_0)$. Then, since $u$ is locally Lipschitz, there exists $r_0 > 0$ such that, for each $x \in B_2$, $t \in [-1, 1]$ and $(p_t, p_x) \in D^{\pm}u_\delta(t, x)$,

$$|p_t| + |p_x| \leq r_0$$

for all $\delta \in (0, d_0)$. Therefore,

$$-(u_\delta)_t(t, x) + \sup_{|q| \leq C(r_0)} [-\nabla u_\delta(t, x) \cdot q - L_\delta(t, x, q)] = 0 \qquad \text{in } [-1, 1] \times B_2$$

in the viscosity sense and (3.4), (3.5) hold with $C_0 = C(r_0)$.

Now, let $T_1$ be such that $0 < T_1 < 1/2C_0$ and $(t, x) \in [-T_1, T_1] \times B_2$. Then $(s, \eta(s)) \in [-T_1, T_1] \times B_2$. So,

$$u_\delta(t, x) = \inf\left\{\int_t^{T_1} L_\delta(s, \eta(s), \dot\eta(s))\mathrm{d}s + u_\delta(T_1, \eta(T_1)) \colon \eta(t) = x, |\dot\eta(s)| \leq C_0\right\} \tag{3.6}$$

for all $(t, x) \in [-T_1, T_1] \times B_1$.

Using the representation formula (3.6), we now prove a one-sided estimate on $u_\delta$ which generalizes the one obtained in [1]. Let $T_0 = T_1/2$.

THEOREM 3.2. Assume that $H(t, x, p)$ satisfies (2.15) and (2.17) with $X = (t, x)$, $Y = (s, y)$, $Q = (0, T) \times \Omega$. Let $u \in W^{1,\infty}(Q)_{\mathrm{loc}}$ be a viscosity solution of (3.1) and define $u_\delta$ as in (3.2) on $[-T_0, T_0] \times B_1$. Then, there exist a constant $K > 0$ and a function $\omega(s) \geq 0$, with $\omega(s) \to 0$ as $s \to 0$, so that

$$\begin{aligned} &\lambda u_\delta(t + (1-\lambda)h, x + (1-\lambda)y) + (1-\lambda)u_\delta(t - \lambda h, x - \lambda y) - u_\delta(t, x) \\ &\quad \leq K\lambda(1-\lambda)(|h| + |y|)\{\delta + \omega(|h| + |y|)\} \end{aligned} \tag{3.7}$$

for all $\lambda \in [0, 1]$, $0 < \delta < d_0$, $(t, x) \in [-T_0/2, T_0/2] \times B_1$ and $(h, y)$ satisfying $(t, x) \pm (h, y) \in B_1$.

*Proof.* First, we prove (3.7) in the case of $y = 0$. Fix $(t, x) \in [-T_0/2, T_0/2] \times B_1$ and let $\eta_0$ be a minimizer for the expression in (3.6). Define

$$\eta_i(s) = \eta_0(\tau_i(s)), \qquad i = 1, 2,$$

where

$$\tau_1(s) = \frac{(T_1 - t)s - T_1(1-\lambda)h}{T_1 - t - (1-\lambda)h}, \qquad t + (1-\lambda)h \leq s \leq T_1$$

$$\tau_2(s) = \frac{(T_1 - t)s + \lambda h T_1}{T_1 - t + \lambda h}, \qquad t - \lambda h \leq s \leq T_1.$$

Since

$$\eta_1(t + (1 - \lambda)h) = x = \eta_2(t - \lambda h), \qquad \eta_1(T_1) = \eta_0(T_1) = \eta_2(T_1),$$

we can plug $\eta_1$, $\eta_2$ and $\eta_0$ into (3.6) to obtain

$$\begin{aligned}
&\lambda u_\delta(t + (1 - \lambda)h, x) + (1 - \lambda)u_\delta(t - \lambda h, x) - u_\delta(t, x) \\
&\le \lambda \int_{t+(1-\lambda)h}^{T_1} L_\delta(s, \eta_1(s), \dot\eta_1(s))\, ds \\
&\quad + (1 - \lambda) \int_{t-\lambda h}^{T_1} L_\delta(s, \eta_2(s), \dot\eta_2(s))\, ds - \int_t^{T_1} L_\delta(s, \eta_0(s), \dot\eta_0(s))\, ds \\
&= \lambda \int_{t+(-\lambda)h}^{T_1} \{L_\delta(s, \eta_1(s), \dot\eta_1(s)) - L_\delta(\tau_1(s), \eta_1(s), \dot\eta_1(s))\}\, ds \\
&\quad + (1 - \lambda) \int_{t-\lambda h}^{T_1} \{L_\delta(s, \eta_2(s), \dot\eta_2(s)) - L_\delta(\tau_2(s), \eta_2(s), \dot\eta_2(s))\}\, ds \\
&\quad + \lambda \int_{t+(1-\lambda)h}^{T_1} L_\delta(\tau_1(s), \eta_0(\tau_1(s)), \dot\eta_0(\tau_1(s))\dot\tau_1(s))\, ds \\
&\quad + (1 - \lambda) \int_{t-\lambda h}^{T_1} L_\delta(\tau_2(s), \eta_0(\tau_2(s)), \dot\eta_0(\tau_2(s))\dot\tau_2(s))\, ds \\
&\quad - \int_t^{T_1} L_\delta(s, \eta_0(s), \dot\eta_0(s))\, ds \\
&= A + B + C + D + E.
\end{aligned} \tag{3.8}$$

Now, from (2.20) and the definition of $H_\delta$ we have that

$$\begin{aligned}
A + B &\le K\delta\lambda \int_{t+(1-\lambda)h}^{T_1} |s - \tau_1(s)|\, ds + K\delta(1 - \lambda) \int_{t-\lambda h}^{T_1} |s - \tau_2(s)|\, ds \\
&\le K\delta\lambda(1 - \lambda)|h| T_0.
\end{aligned} \tag{3.9}$$

Also, by changing variables in the integrals $C$ and $D$,

$$C + D + E = \int_t^{T_1} \left\{\frac{\lambda}{\dot\tau_1} L_\delta(\tau, \eta_0, \dot\eta_0 \dot\tau_1) + \frac{1 - \lambda}{\dot\tau_2} L_\delta(\tau, \eta_0, \dot\eta_0 \dot\tau_2) - L_\delta(\tau, \eta_0, \dot\eta_0)\right\} d\tau.$$

Since

$$\frac{\lambda}{\dot\tau_1} + \frac{1 - \lambda}{\dot\tau_2} = 1$$

and $\eta_0$ satisfies (3.5), from (2.23) with $\sigma = \lambda/\dot\tau_1$, $q = \dot\eta_0$ and

$$q' = \lambda^{-1}\dot\tau_1(1 - \dot\tau_2)\dot\eta_0(s) = \dot\eta_0(s)(T_1 - t)h[T_1 - t + \lambda h]^{-1}[T_1 - t - (1 - \lambda)h]^{-1}$$

we conclude that

$$C + D + E \le K\frac{\lambda}{\dot\tau_1}\frac{1-\lambda}{\dot\tau_2}\int_t^{T_1} |\lambda^{-1}\dot\tau_1(1-\dot\tau_2)\dot\eta_0(s)|\omega(|\lambda^{-1}\tau_1(1-\dot\tau_2)\dot\eta_0(s)|)\,ds$$

$$\le K\lambda(1-\lambda)|h|\omega(2C_0, 2C_0|h|/T_0). \tag{3.10}$$

From (3.8), (3.9) and (3.10) we obtain (3.7) in the case of $y = 0$.

Next, to prove (3.7) for general $y$, define

$$\eta_1^*(s) = \eta_0(s-(1-\lambda)h) + \rho_1(s), \qquad \rho_1(s) = \frac{T_0 + (1-\lambda)h - s}{T_0 - t}(1-\lambda)y$$

for $t + (1-\lambda)h \le s \le T_0 + (1-\lambda)h$ and

$$\eta_2^*(s) = \eta_0(s+\lambda h) + \rho_2(s), \qquad \rho_2(s) = -\frac{T_0 - \lambda h - s}{T_0 - t}\lambda y$$

for $t - \lambda h \le s \le T_0 - \lambda h$, where $T_0 = T_1/2$. Since

$$\eta_1^*(t+(1-\lambda)h) = x + (1-\lambda)y, \qquad \eta_2^*(t-\lambda h) = x - \lambda y,$$

$$\eta_1^*(T_0 + (1-\lambda)h) = \eta_0(T_0) = \eta_2^*(T_0 - \lambda h),$$

again from (3.6) we obtain

$$\begin{aligned}
&\lambda u_\delta(t+(1-\lambda)h, x+(1-\lambda)y) + (1-\lambda)u_\delta(t-\lambda h, x-\lambda y) - u_\delta(t,x)\\
&\quad \le \{\lambda u_\delta(T_0+(1-\lambda)h, \eta_0(T_0)) + (1-\lambda)u_\delta(T_0-\lambda h, \eta_0(T_0)) - u_\delta(T_0, \eta_0(T_0))\}\\
&\qquad + \lambda\int_{t+(1-\lambda)h}^{T_0+(1-\lambda)h} l_\delta(s, \eta_1^*(s), \dot\eta_1^*(s))\,ds + (1-\lambda)\int_{t-\lambda h}^{T_0-\lambda h} L_\delta(s, \eta_2^*(s), \dot\eta_2^*(s))\,ds\\
&\qquad - \int_t^{T_0} L_\delta(s, \eta_0(s), \dot\eta_0(s))\,ds = A' + B' + C' + D'.
\end{aligned} \tag{3.11}$$

By changing variables in the integrals $B'$, $C'$ and $D'$, we have

$$\begin{aligned}
&B' + C' + D'\\
&= \lambda\int_t^{T_0}\{L_\delta(r+(1-\lambda)h, \eta_0(r) + \rho_1(r+(1-\lambda)h), \dot\eta_0 + \dot\rho_1) - L_\delta(r, \eta_0(r), \dot\eta_0 + \dot\rho_1)\}\,dr\\
&\quad + (1-\lambda)\int_t^{T_0}\{L_\delta(r-\lambda h, \eta_0(r) + \rho_2(r-\lambda h), \dot\eta_0 + \dot\rho_2) - L_\delta(r, \eta_0(r), \dot\eta_0 + \dot\rho_2)\}\,dr\\
&\quad + \int_t^{T_0}\{\lambda L_\delta(s, \eta_0, \dot\eta_0 + \dot\rho_1) + (1-\lambda)L_\delta(s, \eta_0, \dot\eta_0 + \dot\rho_2) - L_\delta(s, \eta_0, \dot\eta_0)\}\,ds.
\end{aligned}$$

Therefore, recalling (2.20) and (2.23)

$$B' + C' + D' \le K\delta\lambda(1-\lambda)(|h| + |y|) + \lambda(1-\lambda)|y|\omega(2/T_0, 2|y|/T_0). \tag{3.12}$$

Finally, using the first part of the proof to bound $A'$, we derive (3.7) from (3.11) and (3.12). ■

## 4. ANALYSIS OF THE GENERALIZED GRADIENTS

Let $Q$ be an open domain in $\mathbf{R}^N$, $u \in W^{1,\infty}(Q)_{\rm loc}$ and $X^0 \in Q$. We define $u_\delta = u_{\delta}, X^0$ on $B_1$ as in (2.12) and we assume the existence of $\omega(s) > 0$, with $\omega(s) \to 0$ as $s \to 0$, and of a constant $C > 0$ so that the *generalized one-sided estimate at* $X^0$

$$\lambda u_\delta(X + (1-\lambda)Y) + (1-\lambda)u_\delta(X - \lambda Y) - u_\delta(X) \leq C\lambda(1-\lambda)|Y|\{\delta + \omega(|Y|)\} \quad (4.1)$$

*holds for all* $\lambda \in [0, 1]$, $0 < \delta < d_0 = \operatorname{dist}(X^0, \partial Q)$, $X \in B_1$ *and* $Y$ *satisfying* $X \pm Y \in B_1$.

First, we give an equivalent form of (4.1) expressed in terms of generalized gradients.

LEMMA 4.1. Let $u \in W^{1,\infty}(Q)_{\rm loc}$ and assume (4.1). Then

$$u_\delta(Y) - u_\delta(X) - P\cdot(Y - X) \leq C|Y - X|\{\delta + \omega(|Y - X|)\} \quad (4.2)$$

for all $X, Y \in B_1$, $P \in \partial u_\delta(X)$ and $0 < \delta < d_0$.

*Proof.* Let $u_\delta$ be differentiable at $X$. Then, by (4.1),

$$\begin{aligned} & u_\delta(Y) - u_\delta(X) - Du_\delta(X)\cdot(Y - X) \\ &= \lim_{\lambda \downarrow 0} \left\{ u_\delta(X + (1-\lambda)(Y - X)) - u_\delta(X - \lambda(Y - X)) + \frac{u_\delta(X - \lambda(Y - X)) - u_\delta(X)}{\lambda} \right\} \\ &\leq C|Y - X|\{\delta + \omega(|Y - X|)\}. \end{aligned}$$

The general case follows from the previous one in view of (2.11). ■

A major step in our analysis is the following.

THEOREM 4.2. Let $u \in W^{1,\infty}(Q)_{\rm loc}$ and assume the generalized estimate (4.1) at $X^0$. Then

$$D^+u(X^0) = \partial u(X^0). \quad (4.3)$$

*Proof.* In view of (2.6), we only need to prove that

$$D^+u(X^0) \supset \partial u(X^0). \quad (4.4)$$

So, assume the contrary and let $P$ be an element of $\partial u(X^0)$ which is not in $D^+u(X^0)$. Then, there exists a sequence $X_k \to 0$ such that

$$\lim_k \frac{u(X^0 + X_k) - u(X^0) - P\cdot X_k}{|X_k|} = \alpha > 0.$$

Fix $\varepsilon > 0$ and set

$$X_{k,\varepsilon} = \varepsilon X_k/|X_k|, \qquad \delta_{k,\varepsilon} = |X_k|/\varepsilon.$$

The above limit reads as follows

$$\lim_k \frac{1}{\varepsilon}\{u_{\delta_{k,\varepsilon}}(X_{k,\varepsilon}) - P\cdot X_{k,\varepsilon}\} = \alpha > 0. \quad (4.5)$$

On the other hand, recalling (2.13), by the generalized one-sided estimate (4.2) we conclude that, for $k \geq k_\varepsilon$,

$$u_{\delta_{k,\varepsilon}}(X_{k,\varepsilon}) - P \cdot X_{k,\varepsilon} \leq C\varepsilon\{\delta_{k,\varepsilon} + \omega(\varepsilon)\}$$

Thus, by (4.5), $0 < \alpha \leq C\omega(\varepsilon)$, which is false for $\varepsilon$ sufficiently small. ∎

We state below an immediate consequence of theorem 4.2 and of lemma 2.3, namely the fact that $u$ possesses (one-sided) directional derivatives at $X^0$ in all directions $\theta \in \mathbf{R}^N$, denoted by

$$\frac{\partial u}{\partial \theta}(X^0) = \lim_{\delta \downarrow 0} u_{\delta,X^0}(\theta).$$

Corollary 4.3. Let $u \in W^{1,\infty}(Q)_{\mathrm{loc}}$ and assume the generalized estimate (4.1) at $X^0$. Then

$$\frac{\partial u}{\partial \theta}(X^0) = \min\{P \cdot \theta : P \in D^+u(X^0)\} = \partial^- u(X^0, \theta) \tag{4.6}$$

for all $\theta \in \mathbf{R}^N$. Furthermore, the convergence of $u_{\delta,X^0}$ to $\partial^- u(X^0, \cdot)$, as $\delta \downarrow 0$, is uniform in $B_1$.

Corollary 4.4. Let $u \in W^{1,\infty}(Q)_{\mathrm{loc}}$ and assume the generalized estimate (4.1) at $X^0$. Then

$$P \cdot (X - Y) \leq \partial^- u(X^0, X) - \partial^- u(X^0, Y) + C\delta \tag{4.7}$$

for all $X, Y \in B_1$, $P \in D^+u(X^0 + \delta X)$, $0 < \delta < d_0$.

*Proof.* From lemma 4.1 we derive

$$P \cdot (Y - X) \leq C\{\delta + \omega(\lambda)\} + \lambda^{-1}\{u_\delta(\lambda X) - u_\delta(\lambda Y)\} = C\{\delta + \omega(\lambda)\} + u_{\delta\lambda}(X) - u_{\delta\lambda}(Y)$$

for all $\lambda \in (0, 1)$, $X, Y \in B_1$, $P \in D^+u(X^0 + \delta X)$, $0 < \delta < d_0$. Then the conclusion follows as $\lambda \to 0$, in view of corollary 4.3. ∎

Motivated by corollary 4.3, we now focus our attention on the exposed faces (in the sense of convex analysis, see Rockafellar [11]) of $D^+u$. We recall that, given any convex set $D \subset \mathbf{R}^N$, a point of $D$ is exposed if there exists a supporting hyperplane $\Pi$ to $D$, which contains no other point of $D$. To any exposed point $P \in D$ one can associate at least a vector $\theta \in \mathbf{R}^N \setminus \{0\}$ (normal to $\Pi$) such that

$$\theta \cdot P' < \theta \cdot P \qquad \text{for all } P' \in D,\ P' \neq P.$$

We call such a vector $\theta$ an *exposed vector* for $D$.

*Definition* 4.5. Let $X \in Q$. We denote by $Eu(X)$ the set of all the exposed points of $D^+u(X)$. For $\theta \in \mathbf{R}^N \setminus \{0\}$ we set

$$D^+u(X, \theta) = \{P \in \mathbf{R}^N : P \cdot \theta = \min[P' \cdot \theta : P' \in D^+u(X)]\}.$$

Clearly, $\theta$ is an exposed vector for $D^+u(X)$ if and only if $D^+u(X, \theta)$ is a singleton.

*Remark* 4.6. By Straszewicz's theorem (see e.g. [11, theorem 18.6]), $Eu(X)$ is a (dense) subset of the set of the extreme points of $D^+u(X)$. Now, by (2.11) and (4.3), $D^+u(X^0) = \operatorname{conv} Au(X^0)$. So, every extreme point of $D^+u(X^0)$ is a point of $Au(X^0)$ (see [11, corollary 18.3.1]). In particular, $Eu(X^0) \subset Au(X^0)$.

The following proposition is a refinement of (2.8).

PROPOSITION 4.7. Let $u \in W^{1,\infty}(Q)_{\mathrm{loc}}$ and assume the generalized estimate (4.1) at $X^0$. If

$$X_k \to X^0,\ P_k \in D^+u(X_k),\ P_k \to P$$
$$(X_k - X^0)/|X_k - X^0| \to \theta$$

then $P \in D^+u(X^0, \theta)$.

*Proof.* Let $\theta_k = (X_k - X^0)/|X_k - X^0|$. Applying corollary 4.4, we obtain

$$P_k \cdot \theta_k \leq \partial^- u(X^0, \theta_k) + C|X_k - X^0|.$$

By remark 2.1, (2.8) and (4.3), in the limit we have that $P \in D^+u(X^0)$ and $P \cdot \theta \leq \partial^- u(X^0; \theta)$, which is equivalent to our conclusion.

*Definition* 4.8. We say that $u$ is regular at $X^0$ along $\theta \in \mathbf{R}^n \backslash \{0\}$, if $\theta$ is an exposed vector for $D^+u(X^0)$.

The meaning of the above notion is explained by the following result, which generalizes theorem 4.14 of [1]. Let $K_r(X^0, \theta)$ denote the open cone defined at the beginning of Section 2.

THEOREM 4.9. Let $u \in W^{1,\infty}(Q)_{\mathrm{loc}}$ and assume the generalized estimate (4.1) at $X^0$. Suppose also that $Au(X^0)$ contains no line segments. If $u$ is continuously differentiable on $K_r(X^0, \theta))$ for some $r > 0$, then $u$ is regular at $X^0$ along $\theta$.

*Proof.* Since the proof has much in common with the one given in [1], we only sketch it below, focussing on the main differences.

We argue by contradiction. Suppose $\theta$ is not exposed and let $d$ be the dimension of the convex set $D^+u(X^0, \theta)$. Then, $1 \leq d \leq N$. We write vectors $P \in \mathbf{R}^N$ in the form

$$P = (p', p'', p^N),\ p' \in \mathbf{R}^d,\ p'' \in \mathbf{R}^{N-1-d},\ p^N \in \mathbf{R}.$$

Similarly,

$$Du = (D'u, D''u, D_N u).$$

Arguing as in the proof of theorem 4.14 of [1] we may assume that $X^0 = 0$, $\theta = (0, 0, -1)$ and

(i) $D^+u(0) \subset \{P \in \mathbf{R}^N : p^N \leq 0\}$;

(ii) $D^+(0,(0, 0, -1)) \subset \{P \in \mathbf{R}^N : p^N \leq 0, p'' = 0\}$;

(iii) $D^+u(0,(0, 0, -1)) \supset \{(p', 0, 0) : |p'| \leq \rho\}$ for some $\rho > 0$. (4.8)

Since $u$ is continuously differentiable on $K_r(X^0, \theta))$, the mappings

$$\Phi^\delta : \bar{B}_r \subset \mathbf{R}^d \to \mathbf{R}^d, \qquad \Phi^\delta(x') = D'u(\delta x', 0, -\delta)$$

are well defined and continuous for $0 < \delta < r$. Also, by corollary 4.4

$$\Phi^\delta(x') \cdot x' \leq \partial^- u(0,(x', 0, -1)) - \partial^- u(0,(0, 0, -1)) + C\delta.$$

Now, by (4.4) and (4.8)(iii)(i),

$$\partial^- u(0,(x',0,-1)) \leq -r\rho, \qquad \partial^- u(0,(0,0,-1)) \geq 0.$$

So, if $0 < \delta < r\rho/2C$, then

$$\Phi^\delta(x') \cdot x' \leq -r\rho/2, \qquad \forall\, x' \in \partial B_r. \tag{4.9}$$

Therefore, by an easy corollary of Brouwer's fixed point theorem, (4.9) implies that for each $\bar{p}' \in B_{\rho/2} \subset \mathbf{R}^d$ there exists $x'(\delta) = x'(\delta, \bar{p}')$ such that $\Phi^\delta(x'(\delta)) = \bar{p}'$. Next set

$$P(\delta) = Du(\delta x'(\delta), 0, -\delta) = (\bar{p}', D''u(\delta x'(\delta), 0, -\delta), D_N u(\delta x'(\delta), 0, -\delta)). \tag{4.10}$$

There exists a sequence $\delta_m \to 0$ with $P(\delta_m) \to P_0$ as $m \to \infty$. Also, recalling (4.10),

$$P_0 = (\bar{p}', p_0'', p_0^N) \in Au(0,(0,0,-1))$$

by construction. Then, by (4.7),

$$\bar{p}' \cdot x'(\delta_m) - p^N(\delta_m) \leq \partial^- u(0,(x'(\delta_m),0,-1)) + C\delta_m \leq \bar{p}' x'(\delta_m) + C\delta_m$$

since $(\bar{p}', 0, 0) \in D^+u(0)$. So, recalling (4.8)(i), we have in the limit

$$p_0^N = 0.$$

Thus, $P_0 \cdot (0,0,-1) = 0$ and $P_0 \in D^+u(0,(0,0,-1))$. In particular, $p_0'' = 0$ by (4.8)(ii) and we have obtained that any point $(\bar{p}', 0, 0)$, with $\bar{p}' \in B_{\rho/2}$, is achievable. But this fact contradicts with our assumption on $Au(X^0)$. ■

## 5. APPLICATION TO HAMILTON-JACOBI EQUATIONS

The results of Sections 3 and 4 will now be combined to study the first order singularities of the locally Lipschitz viscosity solutions of the equation

$$-u_t + H(t,x,\nabla u) = 0, \qquad (t,x) \in (0,T) \times \Omega. \tag{5.1}$$

Here $\Omega$ is an open domain in $\mathbf{R}^n$ and $H = H(t,x,p)$ is a real-valued function with the properties below:

$H$ is strictly convex in $p$, i.e.

$$H(t,x,\lambda p^1 + (1-\lambda)p^2) < \lambda H(t,x,p^1) + (1-\lambda)H(t,x,p^2) \tag{5.2}$$

for all $(x,t) \in (0,T) \times \Omega$, $\lambda \in (0,1)$, $p^1, p^2 \in \mathbf{R}^n$, $p^1 \neq p^2$; for each $r > 0$ there exists $C_r > 0$ such that

$$|H(t,x,p) - H(s,y,p)| \leq C_r(|t-s| + |x-y|) \tag{5.3}$$

for all $s, t \in [0,t]$ and $x, y, p$ satisfying $|x|, |y|, |p| \leq r$.

From theorem 3.2 it follows that the generalized one-sided estimate (3.7) holds at every point $(t_0, x_0) \in (0,T) \times \Omega$. Therefore, recalling theorem 4.2, we have the following.

Corollary 5.1. Assume (5.2) and (5.3) and let $u \in W^{1,\infty}((0,T) \times \Omega)_{\mathrm{loc}}$ be a viscosity solution of (5.1). Then

$$\partial u(t,x) = D^+u(t,x)$$

for any $(t,x) \in (0,T) \times \Omega$.

*Remark* 5.2. An interesting by-product of corollary 5.1 is that viscosity solutions of (5.1) satisfy the equation in the sense of Clarke [9]. A related result is proved in [12].

New connections among the geometric objects defined in Sections 2 and 4 may be established for solutions of (5.1). We give the proposition below as an example. In the following, we decompose any vector $P \in D^+u(t,x)$ as $P = (p_t, p_x)$ where $p_t \in \mathbf{R}$ (resp. $p_x \in \mathbf{R}^n$) corresponds to the time (resp. space) derivative.

THEOREM 5.3. Assume (5.2) and (5.3) and let $u \in W^{1,\infty}((0,T) \times \Omega)_{\mathrm{loc}}$ be a viscosity solution of (5.1). Then, for any $(t,x) \in (0,T) \times \Omega$ and $\bar{P} = (\bar{p}_t, \bar{p}_x) \in D^+u(t,x)$ the following statements are equivalent:

(a) $\bar{P} \in Eu(t,x)$;
(b) $\bar{P} \in Au(t,x)$;
(c) $\bar{P}_t = H(t,x,\bar{p}_x)$.

*Proof.* From remark 4.6 it follows that (a) $\Rightarrow$ (b). Also, from definition (2.9) and the fact that (5.1) holds at each point at which u is differentiable, we conclude that (b) $\Rightarrow$ (c). So, we now proceed to show that (c) $\Rightarrow$ (a). Let $h \in \partial_p H(t,x,\bar{p}_x)$*. By (5.2)

$$H(t,x,\bar{p}_x) + h \cdot (p_x - \bar{p}_x) \le H(t,x,p_x), \qquad \forall\, p_x \in \mathbf{R}^n \tag{5.4}$$

the strict inequality being true whenever $p_x \neq \bar{p}_x$. Also, since $u$ is a viscosity solution of (5.1)

$$p_t \ge H(t,x,p_x), \qquad \forall\, P = (p_t, p_x) \in D^+u(t,x). \tag{5.5}$$

Now, from (5.4), (5.5) and (c) we conclude that

$$\bar{p}_t - h \cdot \bar{p}_x \le p_t - h \cdot p_x, \qquad \forall\, P \in D^+u(t,x)$$

the strict inequality being true whenever $p_x \neq \bar{p}_x$. But then

$$p_t - h \cdot \bar{p}_x < p_t - h \cdot p_x, \qquad \forall\, P \in D^+u(t,x), P \neq \bar{P}$$

and $\bar{P}$ is exposed with respect to $(1, -h)$. ∎

*Remark* 5.4. An immediate consequence of the strict convexity of $H$ with respect to $p$ is that $Au(t,x)$ contains no line segments. Indeed, if $P$, $Q$ and $P(\lambda)$ are points of $Au(t,x)$ and $p_x(\lambda) = \lambda p_x + (1-\lambda) q_x$, $\lambda \in (0,1)$, then by (c) above

$$p_t(\lambda) < \lambda p_t + (1-\lambda) q_t.$$

We now turn to the analysis of singularities of a viscosity solution $u \in W^{1,\infty}((0,T) \times \Omega)_{\mathrm{loc}}$ of (5.1). Let $(t,x) \in (0,T) \times \Omega$.

*Definition* 5.5. We say that $(t,x)$ is a regular point of $u$ if $D^+u(t,x)$ is a singleton. We say that $(t,x)$ is a singular point of $u$ otherwise, i.e. if $D^+u(t,x)$ has strictly positive dimension.

* Here $\partial_p H$ denotes the subgradient of $H$ in the $p$ variables. $\partial_p H$ coincides with the generalized gradient in the sense of Section 2, see [9, proposition 2.2.7].

*Remark* 5.6. Notice that $(t, x)$ is a regular point of $u$ if and only if $u$ is (strictly) differentiable at $(t, x)$ (see [9, proposition 2.2.4]).

Obviously, if $(t_0, x_0)$ is a regular point of $u$, then $u$ is regular at $(t_0, x_0)$ along any $\theta \in \mathbf{R}^{n+1}$ (in the sense of definition 4.8). The result below states that, in some sense, the converse is true for the viscosity solutions of (5.1).

THEOREM 5.7. Assume (5.2) and (5.3). Let $u \in W^{1,\infty}((0, T) \times \Omega)_{\mathrm{loc}}$ be a viscosity solution of (5.1) and $(t_0, x_0) \in (0, T) \times \Omega$. If $u$ is regular at $(t_0, x_0)$ along any $\theta = (\theta_t, \theta_x) \in \mathbf{R}^{n+1}$ with $\theta_t < 0$, then $(t_0, x_0)$ is a regular point of $u$.

In particular, in view of theorem 4.9 and remark 5.4, theorem 5.7 yields the following.

COROLLARY 5.8. Assume (5.2) and (5.3). Let $u \in W^{1,\infty}((0, T) \times \Omega)_{\mathrm{loc}}$ be a viscosity solution of (5.1) and $(t_0, x_0) \in (0, T) \times \Omega$. If $u$ is continuously differentiable on $(t_0 - \varepsilon, t_0) \times B_\varepsilon(x_0)$ for some $\varepsilon > 0$, then $(t_0, x_0)$ is a regular point of $u$.

We note that corollary 5.8 was proved in [1] in a more restrictive form. Actually, the method of [1] contains all the ideas needed for the proof of theorem 5.7, that we give below for completeness. Besides, in doing so, we want to distinguish between the use of equation (5.1) and the contribution of convex analysis, represented by the following lemma.

LEMMA 5.9. Let $D$ be a compact convex set in $\mathbf{R}^{n+1}$, with $\dim D = d + 1$, $0 \le d \le n$. For any $P = (s, p) \in D$, $s \in \mathbf{R}$, $p \in \mathbf{R}^n$, define $\bar{P} = (\bar{s}, \bar{p}) \in D$ by

$$\bar{p} = p, \qquad \bar{s} = \max\{s \in \mathbf{R} : (s, p) \in D\}. \tag{5.6}$$

Suppose that all the vectors $\theta \in \mathbf{R}^{n+1}$ with $\theta_1 < 0$ are exposed vectors for $D$ and let $P_0$ be a point in the relative interior of $D$. Then $\bar{P}_0$ is an exposed point of $D$.

*Proof.* First, we prove the conclusion in the case of $d = n$. Then, $P_0 \in D^\circ$ by hypothesis, where $D^\circ$ denotes the interior of $D$. Also, $\bar{P}_0 \in \partial D$ by construction. Therefore, as $D$ is convex, there is a vector $\tilde{\theta} \in \mathbf{R}^{d+1}$ such that

$$(P - \bar{P}_0) \cdot \tilde{\theta} \ge 0, \qquad \forall\, P \in D. \tag{5.7}$$

Taking $P = P_0$ in (5.7), we have $(s_0 - \bar{s}_0)\tilde{\theta}_1 \ge 0$. But $s_0 - \bar{s}_0 < 0$ as $P_0 \in D^\circ$, so $\tilde{\theta}_1 \le 0$. Now, suppose that $\tilde{\theta}_1 = 0$. Then, from (5.6) and (5.7) we obtain $(P - P_0) \cdot \tilde{\theta} \ge 0$, $\forall\, P \in D$, which contradicts with $P_0 \in D^\circ$. Thus, $\tilde{\theta}_1 < 0$ and the conclusion follows.

Next, we prove our lemma for $0 \le d < n$. We claim that, after possibly changing coordinates in $\mathbf{R}^n$,

$$D \subset \{(s, p', p'') \in \mathbf{R} \times \mathbf{R}^d \times \mathbf{R}^{n-d} : p'' = 0\}. \tag{5.8}$$

Indeed, since $\dim D = d + 1$, there are $n - d$ vectors $\theta^i \in S^n$, $i = d + 2, \ldots, n + 1$, such that

$$\theta^i (p - q) = 0 \qquad \text{for all } p, q \in D \tag{5.9}$$

for $i = d + 2, \ldots, n + 1$. Since all the vectors $\theta \in \mathbf{R}^{n+1}$ with $\theta_1 < 0$ are exposed vectors for $D$, (5.9) yields that $\theta_1^i = 0$, $i = d + 2, \ldots, n + 1$. This implies our claim (5.5). Now, applying the first part of the reasoning to the projection of $D$ onto $\mathbf{R}^{d+1}$, one can easily complete the proof. ■

*Proof of theorem* 5.7. If we show that

$$\dim D^+u(t_0, x_0) = 0, \tag{5.10}$$

the conclusion will follow from remark 5.6. Thus, suppose that

$$\dim D^+u(t_0, x_0) = d + 1, \qquad 0 \leq d \leq n.$$

Then, there is a segment $P(\lambda) = \lambda P + (1 - \lambda)Q$, $\lambda \in [0, 1]$, $P \neq Q$, contained in the relative interior of $D^+u(t_0, x_0)$. So, by lemma 5.9, $\bar{P}(\lambda) \in Eu(t_0, x_0)$ for all $\lambda \in [0, 1]$. Consequently, recalling remark 5.4,

$$\bar{p}_t(\lambda) < \lambda p_t + (1 - \lambda)q_t \tag{5.11}$$

for all $\lambda \in [0, 1]$. On the other hand, from the definition (5.6) it follows that

$$\bar{p}_t(\lambda) \geq \lambda p_0 + (1 - \lambda)q_t.$$

Since the above inequality contradicts with (5.11), we obtain (5.10). ■

Let now $(t_0, x_0)$ be a singular point of $u$. Then, from theorem 5.7 it follows that there is at least one direction $\theta \in S^n$, with $\theta_t < 0$, such that

$$\dim D^+u((t_0, x_0), \theta) \geq 1.$$

So, applying theorem 4.9, we obtain that the singularity at $(t_0, x_0)$ has to propagate along $\theta$ in the following sense: *there exists a sequence of singular points $(t_k, x_k)$, with $t_k < t_0$, such that $(t_k, x_k) \to (t_0, x_0)$ and*

$$\frac{(t_k - t_0, x_k - x_0)}{[(t_k - t_0)^2 + (x_k - x_0)^2]^{1/2}} \to \theta \tag{5.12}$$

*as $k \to \infty$.*

*Remark* 5.10. We note that all the results of this section have been obtained under the assumption that $H$ be locally Lipschitz in $(t, x)$ (condition (5.3)). Therefore, they can be easily extended to solutions of (1.2), assuming a Lipschitz continuous dependence of $H$ on $u$. Obviously, due to the change of sign in front of $u_t$, singularities will now propagate forward in time.

## 6. APPLICATION TO A VARIATIONAL PROBLEM

Consider a variational integrand $L(x, q)$ defined on $\mathbf{R}^n \times \mathbf{R}^n = \mathbf{R}^{2n}$, having the following properties:

$$L \in W^{2,\infty}(\mathbf{R}^{2n})_{\text{loc}} \tag{6.1}$$

*for all* $(x, q) \in \mathbf{R}^{2n}$

$$L(x, q) \geq \Lambda(|q|) \tag{6.2}$$

*where $\Lambda(r)$ is some real function, bounded below and such that $\Lambda(r)/r \to +\infty$ as $r \to +\infty$; there is a nonincreasing function $\lambda(r) > 0$, $r \in [0, +\infty)$ such that*

$$L_{qq}(x, q)\xi \cdot \xi \geq \lambda(|q|)|\xi|^2, \qquad \forall\, x, q, \xi \in \mathbf{R}^n \tag{6.3}$$

*where $L_{qq}$ denotes the matrix $\{\partial^2 L/\partial q_i \partial q_j\}$.*

Also, let $\varphi$ be such that

$$\varphi \in W^{1,\infty}(\mathbf{R}^{2n})_{\mathrm{loc}}, \qquad \varphi(x) \geq C, \qquad \forall\, x \in \mathbf{R}^n \tag{6.4}$$

for some real constant $C$.

Then, from standard results in calculus of variations it follows that for every $(t, x) \in [0, T] \times \mathbf{R}^n$ the functional

$$x(\cdot) \to \int_t^T L(x(s), \dot{x}(s))\, \mathrm{d}s + \varphi(x(T)), \tag{6.5}$$

under the condition $x(t) = x$, attains a minimum at an absolutely continuous curve $x(s)$, $s \in [t, T]$ (see e.g. [13, 14]). Moreover, condition (6.2) implies that any minimizer of (6.5) is Lipschitz continuous [15].

Now, the *value function* $u$ defined as

$$u(t, x) = \inf \left\{ \int_t^T L(x(s), \dot{x}(s))\, \mathrm{d}s + \varphi(x(T)) : x(\cdot) \in W^{1,\infty}([t\ T], \mathbf{R}^n), x(t) = x \right\} \tag{6.6}$$

is also locally Lipschitz on $[0, T] \times \mathbf{R}^n$ (see e.g. [10, 14]). Furthermore, $u$ is the unique viscosity solution of the Hamilton–Jacobi equation*

$$-u_t + H(x, \nabla u) = 0 \qquad \text{in } [0, T] \times \mathbf{R}^n \tag{6.7}$$

with initial condition $u(0, x) = \varphi(x)$, $x \in \mathbf{R}^n$, where $H(x, p)$ is the Legendre transform of $L$, that is

$$H(x, p) = \sup_{q \in \mathbf{R}^n} [-p \cdot q - L(x, q)].$$

In particular, from (6.1), (6.2) and (6.3) it follows that $H(x, p)$ is strictly convex in $p$ and $H \in W^{2,\infty}(\mathbf{R}^{2n})_{\mathrm{loc}}$.

Next, we summarize some known properties of minimizers (see [10]).

Proposition 6.1. Assume (6.1), ..., (6.4) and let $(t, x) \in (0, T) \times \mathbf{R}^n$. Let $x(s)$, $s \in [t, T]$, minimize the functional in (6.5) under the condition $x(t) = x$. Then, for all $s \in (t, T)$ the value function $u$ is differentiable at $(s, x(s))$ and

$$\begin{aligned} \dot{x}(s) &= -D_p H(x(s), p(s)) \\ \dot{p}(s) &= D_x H(x(s), p(s)) \end{aligned} \tag{6.8}$$

where $p(s) = \nabla u(s, x(s))$. Moreover, $u$ is differentiable at $(t, x)$ if and only if there is a unique minimizer $x(\cdot)$ of (6.5) with initial point $x(t) = x$.

Now, we give a criterion to select, among all the solutions $(x(\cdot), p(\cdot))$ of the Hamiltonian system (6.8), those which correspond to minimizers. Let $x(\cdot)$ be a minimizer of (6.5) at $(t, x)$

* This property of the value function is well known, see e.g. [8] and Chapter 1 in [3]; see also [1] for the case at hand.

and let $p(s) = \nabla u(s, x(s))$. Since $H \in W^{2,\infty}(\mathbf{R}^{2n})_{\text{loc}}$ by (6.8) we have that $(x(\cdot), p(\cdot)) \in W^{1,\infty}([t, T], \mathbf{R}^{2n})$. In particular, $p(s)$ converges to a limit $p(t)$ as $s \downarrow t$ and, recalling equation (6.7),

$$(H(x, p(t)), p(t)) \in Au(t, x). \tag{6.9}$$

Notice that, in view of theorem 5.3, condition (6.9) is equivalent to $p(t) \in \pi_x Au(t, x)$, where $\pi_x$ denotes the projection $\pi_x : \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^n$. Furthermore, the above conditions characterize the minimizers of (6.5), as we show below.

THEOREM 6.2. Assume (6.1), ..., (6.4) and let $(x(\cdot), p(\cdot)) \in W^{1,\infty}([t, T], \mathbf{R}^{2n})$ be a solution of system (6.8) with initial conditions $x(t) = x, p(t) = p_x$. Then $x(\cdot)$ minimizes the variational problem in (6.6) if and only if

$$p_x \in \pi_x Au(t, x). \tag{6.10}$$

*Proof.* Assume (6.10). By the definition of $Au(t, x)$ there is a sequence $(t_k, x_k) \to (t, x)$, as $k \to \infty$, such that $u$ is differentiable at $(t_k, x_k)$ and $\nabla u(t_k, x_k) \to p_x$. Now, let $(x^k(\cdot), p^k(\cdot))$ be the solution of (6.8) with initial conditions $x^k(t_k) = x_k, p^k(t_k) = \nabla u(t_k, x_k)$. Then, $x^k(\cdot)$ converges* to $x(\cdot)$ in $W^{1,\infty}$ and consequently

$$\int_{t_k}^{T} L(x^k(s), \dot{x}^k(s))\, ds + \varphi(x^k(T)) \to \int_{t}^{T} L(x(s), \dot{x}(s))\, ds + \varphi(x(T)). \tag{6.11}$$

Also, from proposition 6.1 and the successive remarks we conclude that

$$u(t_k, x_k) = \int_{t_k}^{T} L(x^k(s), \dot{x}^k(s))\, ds + \varphi(x^k(T)).$$

Thus, the above equality, (6.11) and the continuity of $u$ yield that $x(\cdot)$ is a minimizer at $(t, x)$.

Conversely, assume that $x(\cdot)$ minimizes the expression in (6.6). The conclusion will follow from (6.9), provided we show that

$$p(s) = \nabla u(s, x(s)), \tag{6.12}$$

for all $s \in (t, T)$. Now, since $(x(\cdot), p(\cdot))$ and $(x(\cdot), \nabla u(\cdot, x(\cdot)))$ are both solutions of (6.8), we obtain

$$0 = (D_p H(x(s), p(s)) - D_p H(x(s), \nabla u(s, x(s)))) \cdot (p(s) - \nabla u(s, x(s)))$$

for all $s \in (t, T)$. But $H$ is strictly convex in $p$ and so the above equality implies (6.12). ∎

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* Obviously, we can assume $x(\cdot)$ and $x^k(\cdot)$ to be defined on some interval $[t', T]$, with $t' \le t, t' \le t_k$. So, the convergence takes place on this interval.

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