# REGULARITY OF THE VALUE FUNCTION FOR A TWO-DIMENSIONAL SINGULAR STOCHASTIC CONTROL PROBLEM* 

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#### Abstract

It is desired to control a two-dimensional Brownian motion by adding a (possibly singularly) continuous process to it so as to minimize an expected infinite-horizon discounted running cost. The Hamilton-Jacobi-Bellman characterization of the value function $V$ is a variational inequality which has a unique twice continuously differentiable solution. The optimal control process is constructed by solving the Skorokhod problem of reflecting the two-dimensional Brownian motion along a free boundary in the $-\nabla V$ direction.


Key words. singular stochastic control, variational inequality, free boundary problem, Skorokhod problem

AMS(MOS) subject classifications. 93E20, 35R35

1. Introduction. We study regularity of the solution of the variational inequality associated with a two-dimensional singular stochastic control problem with a convex running cost. The solution $u$ of this variational inequality, which is the value function for the control problem, is shown to be of class $C^{2}$. We also study the regularity of the free boundary in $\mathbb{R}^{2}$ which divides the region where $u$ satisfies a second-order elliptic equation from the region where it does not. The free boundary is shown to be smooth, and this fact is instrumental in our construction of the optimal process for the stochastic control problem.

Previous work on the regularity of the value function in singular stochastic control has focused on one-dimensional problems. Beneš, Shepp, and Witsenhausen (1980) suggested that the value function for these problems should be of class $C^{2}$ and used this so-called "principle of smooth fit" to determine some otherwise free parameters that arose in the solution of their problems. It has been used in the same way by Harrison (1985), Harrison and Taylor (1978), Harrison and Taksar (1983), Karatzas (1981), (1983), Lehoczky and Shreve (1986), Shreve, Lehoczky, and Gaver (1984), and Taksar (1985). (But see Menaldi and Robin (1983), Chow, Menaldi, and Robin (1985), and Sun (1987) for a variational inequality approach to singular control that does not use the principle of smooth fit.) An important question is whether the principle of smooth fit can be expected to apply to multidimensional singular control problems, or is it strictly a one-dimensional phenomenon. Karatzas and Shreve (1986) suggested that it might apply in higher dimensions. These authors studied the singular control of a one-dimensional Brownian motion under a constraint on the total variation of the control process (a "finite-fuel" constraint). The fuel remaining constitutes a second state variable, and the value function for this problem was found to be of class $C^{2}$ jointly in both state variables. One should observe, however, that the second state variable in this problem is not a diffusion; indeed, the fuel remaining is constant until control is exercised, at which time it decreases an amount equal to the displacement caused by the control.

[^0]This paper concerns the control of a two-dimensional Brownian motion, and control can cause displacement in any direction. Thus, the discovery of a $C^{2}$ value function provides strong support for belief in a widely applicable principle of smooth fit. Nevertheless, the argument of this paper depends heavily on the fact that only two dimensions are involved (see Remark 6.2), and we have not found a way to obtain a similar result in higher dimensions.

This paper is organized as follows. Section 2 defines the underlying stochastic control problem, and § 3 relates it to a free boundary problem, the so-called Hamilton-Jacobi-Bellman (HJB) equation. Section 4 constructs a $C^{1,1}$, nonnegative convex solution $u$ to the HJB equation and proves its uniqueness. Sections 5-10 upgrade the regularity of $u$ to $C^{2}$. The key idea here is to use the gradient flow of $u$ to change to a more convenient pair of coordinates. This is a generalization of the device used by many authors in one-dimensional problems of differentiating the Bellman equation so as to obtain a more standard free boundary problem. In § 11 the free boundary is shown to be of class $C^{2, \alpha}$ for any $\alpha \in(0,1)$. In $\S 12$ we return to the stochastic control problem, which now reduces to the Skorokhod problem of finding a Brownian motion reflected along the free boundary in the $-\nabla u$ direction. The established regularity of $u$ and the free boundary allow us to assert the existence and uniqueness of a solution to the Skorokhod problem and finally complete the proof, begun in $\S 3$, that $u$ is the value function for the stochastic control problem of $\S 2$.
2. The singular stochastic control problem. Let $\left\{W_{t}, \mathscr{F}_{t} ; 0 \leqq t<\infty\right\}$ be a standard, two-dimensional Brownian motion defined on a complete probability space ( $\Omega, \mathscr{F}, P$ ), and let $\left\{\mathscr{F}_{t}\right\}$ be the augmentation of the filtration generated by $W$ (see Karatzas and Shreve (1987, p. 89)). The state process for our control problem is

$$
\begin{equation*}
X_{t} \Delta \underline{\Delta} x+\sqrt{2} W_{t}+\int_{0}^{t} N_{s} d \zeta_{s}, \quad 0 \leqq t<\infty \tag{2.1}
\end{equation*}
$$

where $x \in \mathbb{R}^{2}$ is the initial condition and the control process pair $\left\{\left(N_{t}, \zeta_{t}\right) ; 0 \leqq t<\infty\right\}$ is $\left\{\mathscr{F}_{t}\right\}$-adapted and satisfies the conditions:

$$
\begin{equation*}
\left|N_{t}\right|=1, \quad \forall 0 \leqq t<\infty \quad \text { a.s. }, \tag{2.2}
\end{equation*}
$$

where $|\cdot|$ denotes the Euclidean norm, and
$\zeta$ is nondecreasing, left-continuous, and $\zeta_{0}=0$ a.s..
The process $N$ gives the direction and $\zeta$ gives the intensity of the "push" applied by the controller to the state $X$.

Given control processes $N$ and $\zeta$, we define the corresponding cost

$$
\begin{equation*}
V_{N, \zeta}(x) \triangleq E^{x} \int_{0}^{\infty} e^{-t}\left[h\left(X_{t}\right) d t+d \zeta_{t}\right], \tag{2.4}
\end{equation*}
$$

where $h: R^{2} \rightarrow \mathbb{R}$ is a strictly convex function satisfying, for appropriate positive constants $C_{0}, c_{0}$, and $q$ :

$$
\begin{gather*}
h \in C_{l o c}^{2,1}\left(\mathbb{R}^{2}\right),  \tag{2.5}\\
0 \leqq h(x) \leqq C_{0}\left(1+|x|^{q}\right) \quad \forall x \in \mathbb{R}^{2},  \tag{2.6}\\
|\nabla h(x)| \leqq C_{0}(1+h(x)) \quad \forall x \in \mathbb{R}^{2},  \tag{2.7}\\
c_{0}|y|^{2} \leqq D^{2} h(x) y \cdot y \leqq C_{0}|y|^{2}(1+h(x)) \quad \forall x \in \mathbb{R}^{2}, y \in \mathbb{R}^{2} . \tag{2.8}
\end{gather*}
$$

Without loss of generality, we also assume that

$$
\begin{equation*}
0=h(0) \leqq h(x) \quad \forall x \in \mathbb{R}^{2} . \tag{2.9}
\end{equation*}
$$

For $x \in \mathbb{R}^{2}$, we define the value function

$$
\begin{equation*}
V(x) \xlongequal[N, 5]{\Delta} \inf _{N, 5} V_{N}(x) . \tag{2.10}
\end{equation*}
$$

3. The Hamilton-Jacobi-Bellman equation. We shall show that the value function $V$ of (2.10) is characterized by the Hamilton-Jacobi-Bellman (HJB) equation

$$
\begin{equation*}
\max \left\{u-\Delta u-h,|\nabla u|^{2}-1\right\}=0 \tag{3.1}
\end{equation*}
$$

The following theorem gives a partial description of the relationship between $V$ and the HJB equation. More definitive results are proved in § 12 .

Theorem 3.1. Let $u: \mathbb{R}^{2} \rightarrow \mathbb{R}$ be a convex, $C^{2}$ solution of (3.1). Then $u \leqq V$. For a given $x \in \mathbb{R}^{2}$, suppose there exists a control process pair $(N, \zeta)$ such that $V_{N, \zeta}(x)<\infty$ and the corresponding state process (2.1) satisfies

$$
\begin{gather*}
u\left(X_{t}\right)-\Delta u\left(X_{t}\right)-h\left(X_{t}\right)=0 \quad \forall t \in(0, \infty), \quad \text { a.s. }  \tag{3.2}\\
\int_{0}^{t} 1_{\left\{N_{s}=-\nabla u\left(X_{s}\right)\right\}} d \zeta_{s}=\zeta_{t} \quad \forall t \in[0, \infty), \quad \text { a.s., }  \tag{3.3}\\
u\left(X_{t}\right)-u\left(X_{t+}\right)=\zeta_{t+}-\zeta_{t} \quad \forall t \in[0, \infty), \quad \text { a.s. } \tag{3.4}
\end{gather*}
$$

Then

$$
u(x)=V(x)=V_{N, \xi}(x)
$$

i.e., $(N, \zeta)$ is optimal at $x$.

Proof. Let $x \in \mathbb{R}^{2}$ and any control process pair $(N, \zeta)$ be given. Applying Itô's rule for semimartingales (Meyer (1976, pp. 278, 301)) to $e^{-t} u\left(X_{t}\right)$, adjusting the result to account for the fact that $\zeta$ is left-continuous rather than right-continuous, and observing that $|\nabla u| \leqq 1$ so $E \int_{0}^{t} e^{-s} \nabla u\left(X_{s}\right) d W_{s}=0$, we obtain for $t \geqq 0$ :

$$
\begin{align*}
u(x)= & E e^{-t} u\left(X_{t}\right)+E \int_{0}^{t} e^{-s}\left[u\left(X_{s}\right)-\Delta u\left(X_{s}\right)-h\left(X_{s}\right)\right] d s \\
& +E \int_{0}^{t} e^{-s} h\left(X_{s}\right) d s+E \int_{0}^{t}\left[-e^{-s} \nabla u\left(X_{s}\right) \cdot N_{s}\right] d \zeta_{s}  \tag{3.5}\\
& +E \sum_{0 \leqq s<t} e^{-s}\left[u\left(X_{s}\right)-u\left(X_{s+}\right)+\nabla u\left(X_{s}\right) \cdot N_{s}\left(\zeta_{s+}-\zeta s\right)\right] .
\end{align*}
$$

The second and fifth terms on the right-hand side of (3.5) are nonpositive because of (3.1) and the convexity of $u$, respectively. Because $|\nabla u| \leqq 1$, the fourth term is dominated by $E \int_{0}^{t} e^{-s} d \zeta_{s}$, and thus we have

$$
\begin{equation*}
u(x) \leqq E e^{-t} u\left(X_{t}\right)+E \int_{0}^{t} e^{-s}\left[h\left(X_{s}\right) d s+d \zeta_{s}\right] \tag{3.6}
\end{equation*}
$$

We wish to let $t \rightarrow \infty$ in (3.6) to obtain

$$
\begin{equation*}
u(x) \leqq E \int_{0}^{\infty} e^{-s}\left[h\left(X_{s}\right) d s+d \zeta_{s}\right]=V_{N, \zeta}(x) \tag{3.7}
\end{equation*}
$$

Assume $E \int_{0}^{\infty} e^{-s} h\left(X_{s}\right)<\infty$, for otherwise (3.7) is obviously true. This implies that

$$
\varliminf_{t \rightarrow \infty} E e^{-t} h\left(X_{t}\right)=0
$$

Now (2.8), (2.9), and the inequality $|\nabla u| \leq 1$ (from (3.1)) imply that for all $y \in \mathbb{R}^{2}$,

$$
\begin{equation*}
u(y) \leqq u(0)+|y| \leqq u(0)+1+|y|^{2} \leqq u(0)+1+\frac{2}{c_{0}} h(y) \tag{3.8}
\end{equation*}
$$

so

$$
\varliminf_{t \rightarrow \infty} E e^{-t} u\left(X_{t}\right)=0
$$

We may therefore pass to the limit in (3.6) along a sequence $\left\{t_{n}\right\}_{n=1}^{\infty}$ such that $E e^{-t_{n}} u\left(X_{t_{n}}\right) \rightarrow 0$ as $t_{n} \rightarrow \infty$, and (3.7) follows. Since ( $N, \zeta$ ) is an arbitrary control process pair, we have $u(x) \leqq V(x)$.

If (3.2)-(3.4) are satisfied, then the second and fifth terms on the right-hand side of (3.5) are zero, and the fourth term is $E \int_{0}^{t} e^{-s} d \zeta_{s}$. It follows that equality holds in (3.6), and hence also in (3.7), i.e.,

$$
u(x) \leqq V(x) \leqq V_{(N, 5)}(x)=u(x) .
$$

Remark 3.2. Equation (3.1) is similar but not equivalent to a problem arising in elastic-plastic torsion (Ting (1966), (1967), Duvant and Lanchon (1967), Brezis and Sibony (1971)). The elastic-plastic problem is posed on a bounded domain $\Omega \subset \mathbb{R}^{n}$, and is to minimize

$$
J(v) \triangleq \int_{\Omega} \frac{1}{2}|\nabla v|^{2}-v h
$$

over $K \triangleq \xlongequal{\Delta}\left\{v \in H_{0}^{1}(\Omega) ;\|\nabla v\|_{\infty} \leqq 1\right\}$. Equivalently, one seeks $u \in K$ satisfying

$$
\int h(v-u)-\int \nabla u \cdot(\nabla v-\nabla u) \leqq 0 \quad \forall v \in K
$$

If $u$ solves the elastic-plastic torsion problem, then

$$
(\Delta u+h)(|\nabla u|-1)=0
$$

but $\Delta u+h$ may be negative. In the special case that $h$ is a nonnegative constant function, a solution to the elastic-plastic problem also satisfies a variational inequality like (3.1) (see Evans (1979, §6), but such an $h$ is not interesting in the control problem.
4. Solution of the Hamilton-Jacobi-Bellman equation. The existence of a $W_{l o c}^{2, \infty}$ solution to the HJB equation (3.1) follows from a modification of Evans (1979) (see also Ishii and Koike (1983)), who treated a bounded domain and general $h$ and space dimension. We need to refer to this construction in the next section, so we provide it here.

Let $\beta: \mathbb{R} \rightarrow \mathbb{R}$ be a $C^{\infty}$ function satisfying

$$
\begin{gather*}
\beta(r)=0 \quad \forall r \in(-\infty, 0]  \tag{4.1i}\\
\beta(r)>0 \tag{4.1ii}
\end{gather*} \quad \forall r \in(0, \infty),
$$

$$
\begin{equation*}
\beta(r)=r-1 \quad \forall r \in[2, \infty) \tag{4.1iii}
\end{equation*}
$$

$$
\begin{equation*}
\beta^{\prime}(r) \geqq 0, \quad \beta^{\prime \prime}(r) \geqq 0 \quad \forall r \in \mathbb{R} \tag{4.1iv}
\end{equation*}
$$

For each $\varepsilon>0$, we form the penalization function

$$
\begin{equation*}
\beta_{\varepsilon}(r) \triangleq \beta\left(\frac{r-1}{\varepsilon}\right) \quad \forall r \in \mathbb{R} \tag{4.2}
\end{equation*}
$$

and we consider the penalized equation

$$
\begin{equation*}
u^{\varepsilon}-\Delta u^{\varepsilon}+\beta_{\varepsilon}\left(\left|\nabla u^{\varepsilon}\right|^{2}\right)=h . \tag{4.3}
\end{equation*}
$$

The following lemma is proved in the appendix.
Lemma 4.1. For every $\varepsilon \in(0,1)$, there exists a nonnegative, convex, $C^{2}$ solution $u^{\varepsilon}$ to (4.3). There exist positive constants $C_{1}, C_{2}$, and $p$, indepenent of $\varepsilon$, such that for all $\varepsilon \in(0,1)$, for all $x \in \mathbb{R}^{2}$ :

$$
\begin{gather*}
0 \leqq u^{\varepsilon}(x) \leqq C_{1}\left(1+|x|^{p}\right),  \tag{4.4}\\
\left|\nabla u^{\varepsilon}(x)\right| \leqq C_{1}\left(1+|x|^{p}\right), \tag{4.5}
\end{gather*}
$$

and for every $y \in \mathbb{R}^{2}$,

$$
\begin{equation*}
0 \leqq D^{2} u^{\varepsilon}(x) y \cdot y \leqq C_{2}|y|^{2}\left(1+u^{\varepsilon}(x)\right) . \tag{4.6}
\end{equation*}
$$

Definition 4.2. We define a norm on the vector space of $2 \times 2$ matrices by

$$
\|A\| \triangleq \sqrt{\operatorname{trace}\left(A A^{T}\right)}
$$

If $A$ is symmetric with eigenvalues $\lambda_{1}$ and $\lambda_{2}$, then

$$
\begin{equation*}
\|A\|=\sqrt{\lambda_{1}^{2}+\lambda_{2}^{2}} \tag{4.7}
\end{equation*}
$$

Theorem 4.3. The HJB equation (3.1) has a nonnegative, convex solution $u \in W_{l o c}^{2, \infty}$ satisfying

$$
\begin{equation*}
\left\|D^{2} u(x)\right\| \leqq C_{3}\left(1+|x|^{m}\right), \quad \text { a.e. } x \in \mathbb{R}^{2}, \tag{4.8}
\end{equation*}
$$

for some $C_{3}>0$ and $m \in \mathbb{N}$.
Proof. Because $D^{2} u^{\varepsilon}$ is locally bounded uniformly in $\varepsilon \in(0,1)$, we may choose a decreasing sequence $\left\{\varepsilon_{n}\right\}_{n=1}^{\infty}$ with limit zero such that $\left\{u^{\varepsilon_{n}}\right\}_{n=1}^{\infty}$ and $\left\{\nabla u^{\varepsilon_{n}}\right\}_{n=1}^{\infty}$ converge uniformly on compact sets, and $\left\{D^{2} u^{\varepsilon_{n}}\right\}_{n=1}^{\infty}$ converges in the $L_{\text {loc }}^{\infty}$-weak* topology. Define $u=\lim _{n \rightarrow \infty} u^{\varepsilon_{n}}$, so that $\nabla u=\lim _{n \rightarrow \infty} \nabla u^{\varepsilon_{n}}$ and the weak* limit of $\left\{D^{2} u^{\varepsilon_{n}}\right\}_{n=1}^{\infty}$ is $D^{2} u$. Passage to the limit in (4.3) gives (3.1).

LEMMA 4.4. Let $u \in W_{l o c}^{2, \infty}$ be a nonnegative, convex solution to the HJB equation (3.1), and define

$$
\begin{equation*}
\mathscr{C} \triangleq\left\{x \in \mathbb{R}^{2} ;|\nabla u(x)|^{2}<1\right\} . \tag{4.9}
\end{equation*}
$$

Then for every unit vector $\nu$,

$$
\begin{equation*}
u_{\nu \nu} \triangleq\left(D^{2} u\right) \nu \cdot \nu>0 \quad \text { on } \mathscr{C} . \tag{4.10}
\end{equation*}
$$

$\mathscr{C}$ is bounded, and $u$ attains its unique minimum over $\mathbb{R}^{2}$ inside $\mathscr{C}$.
Proof. We have

$$
\begin{equation*}
u-\Delta u=h \quad \text { on } \mathscr{C}, \tag{4.11}
\end{equation*}
$$

and $h \in C_{\text {loc }}^{2,1}\left(\mathbb{R}^{2}\right)$, so $u \in C^{4, \alpha}(\mathscr{C})$ for all $\alpha \in(0,1)$. Differentiating (4.11), we obtain

$$
u_{\nu \nu}-\Delta u_{\nu \nu}=h_{\nu \nu} \quad \text { on } \mathscr{C} .
$$

and since $h_{\nu \nu}>0$, relation (4.10) holds. Equation (4.11) also implies that $u \geqq h$ on $\mathscr{C}$, and since $|\nabla u| \leqq 1$ on $\mathbb{R}^{2}$ but $h$ grows at least quadratically (see (2.8)), $\mathscr{C}$ must be bounded.

Let $\delta \in\left(0, \frac{1}{2}\right)$ be given, and choose $x^{\delta} \in \mathbb{R}^{2}$ such that

$$
u\left(x^{\delta}\right) \leqq u(x)+\delta \quad \forall x \in \mathbb{R}^{2} .
$$

Define

$$
\psi_{\delta}(x) \triangleq \stackrel{\Delta}{\triangleq} u(x)+\delta\left|x-x^{\delta}\right|^{2} \quad \forall x \in \mathbb{R}^{2}
$$

and note that $\psi_{\delta}$ attains its minimum over $\mathbb{R}^{2}$ at some point $y^{\delta}$. In particular,

$$
\begin{equation*}
0=\nabla \psi_{\delta}\left(y^{\delta}\right)=\nabla u\left(y^{\delta}\right)+2 \delta\left(y^{\delta}-x^{\delta}\right) \tag{4.12}
\end{equation*}
$$

But also

$$
u\left(y^{\delta}\right)+\delta\left|y^{\delta}-x^{\delta}\right|^{2}=\psi_{\delta}\left(y^{\delta}\right) \leqq \psi_{\delta}\left(x^{\delta}\right)=u\left(x^{\delta}\right) \leqq u\left(y^{\delta}\right)+\delta .
$$

It follows that $\left|y^{\delta}-x^{\delta}\right| \leqq 1$, and returning to (4.12), we see that $\left|\nabla u\left(y^{\delta}\right)\right| \leqq 2 \delta<1$. Therefore, $y^{\delta} \in \mathscr{C}$ for all $\delta \in\left(0, \frac{1}{2}\right)$, and the sequence $\left\{y^{1 / n}\right\}_{n=3}^{\infty}$ accumulates at some $y^{0} \in \overline{\mathscr{C}}$. From (4.12) we have $\nabla u\left(y^{0}\right)=0$, so $y^{0} \in \mathscr{C}$, and the convexity of $u$ on $\mathbb{R}^{2}$ implies that $u$ attains its minimum at $y_{0}$. This minimum is unique because of (4.10).

Theorem 4.5. There is only one nonnegative, convex solution $u \in W_{l o c}^{2, \infty}$ to the HJB equation (3.1).

Proof. Let $u_{1}$ and $u_{2}$ be two nonnegative, convex solutions to (3.1), and let $y^{0}$ be the point where $u_{2}$ attains its minimum. Given $\delta>0$, define

$$
\varphi_{\delta}(x) \stackrel{\Delta}{=} u_{1}(x)-u_{2}(x)-\delta\left|x-y^{0}\right|^{2} \quad \forall x \in \mathbb{R}^{2} .
$$

The function $\varphi_{\delta}$ attains its maximum at some $x^{\delta} \in \mathbb{R}^{2}$, and $0=\nabla \varphi_{\delta}\left(x^{\delta}\right)=$ $\nabla u_{1}\left(x^{\delta}\right)-\nabla u_{2}\left(x^{\delta}\right)-2 \delta\left(x^{\delta}-y^{0}\right)$. Consequently,

$$
1 \geqq\left|\nabla u_{1}\left(x^{\delta}\right)\right|^{2}=\left|\nabla u_{2}\left(x^{\delta}\right)\right|^{2}+4 \delta^{2}\left|x^{\delta}-y^{0}\right|^{2}+4 \delta \nabla u_{2}\left(x^{\delta}\right) \cdot\left(x^{\delta}-y^{0}\right)
$$

Because $u_{2}$ is convex, $\nabla u_{2}\left(x^{\delta}\right) \cdot\left(x^{\delta}-y^{0}\right) \geqq 0$, so either $\left|\nabla u_{2}\left(x^{\delta}\right)\right|^{2}<1$ or $x^{\delta}=y^{0}$. This last equality would imply that $\nabla u_{2}\left(x^{\delta}\right)=0$, so in any event, $\left|\nabla u_{2}\left(x^{\delta}\right)\right|^{2}<1$. From (3.1) we have

$$
\Delta u_{2}\left(x^{\delta}\right)=u_{2}\left(x^{\delta}\right)-h\left(x^{\delta}\right) .
$$

Because $\varphi$ attains its maximum at $x^{\delta}$, we have from the Bony maximum principle (Bony (1967), Lions (1983))

$$
\begin{aligned}
0 & \geqq \liminf _{x \rightarrow x^{\delta}} \operatorname{ess} \Delta \varphi_{\delta}(x) \\
& =\liminf _{x \rightarrow x^{\delta}} \operatorname{ess}\left[\Delta u_{1}(x)-\Delta u_{2}(x)-4 \delta\right] \\
& \geqq u_{1}\left(x^{\delta}\right)-u_{2}\left(x^{\delta}\right)-4 \delta .
\end{aligned}
$$

It follows that for all $x \in \mathbb{R}^{2}$,

$$
\begin{aligned}
u_{1}(x)-u_{2}(x) & =\varphi_{\delta}(x)+\delta\left|x-y^{0}\right|^{2} \\
& \leqq \varphi_{\delta}\left(x^{\delta}\right)+\delta\left|x-y^{0}\right|^{2} \\
& \leqq \delta\left(4+\left|x-y^{0}\right|^{2}\right)
\end{aligned}
$$

Letting $\delta \downarrow 0$, we obtain $u_{1} \leq u_{2}$. The reverse inequality is proved by interchanging $u_{1}$ and $u_{2}$.

Remark 4.6. Throughout the remainder of the paper, $u$ will denote the unique nonnegative, convex solution in $W_{l o c}^{2, \infty}$ to (3.1). The set $\mathscr{C}$ will be given by (4.9), and $y^{0} \in \mathscr{C}$ will denote the unique minimizer of $u$. We shall prove that $u \in C_{\text {loc }}^{2, \alpha}\left(\mathbb{R}^{2}\right)$ for all $\alpha \in(0,1)$ (Theorem 10.3), $\partial \mathscr{C}$ is of class $C^{2, \alpha}$ for all $\alpha \in(0,1)$ (Corollary 11.3), and $n(x) \cdot \nabla u(x) \geqq \sigma$ for all $x \in \partial \mathscr{C}$, where $n(x)$ is the outward normal to $\mathscr{C}$ at $x$ and $\sigma$ is a positive constant (Lemma 12.2).
5. An obstacle problem. Let us return to the construction of $u$ in the proof of Theorem 4.3 as the limit of a sequence of functions $\left\{u^{\varepsilon_{n}}\right\}_{n=1}^{\infty}$, where each $u^{\varepsilon_{n}}$ satisfies (4.3). Define $\boldsymbol{w}^{\varepsilon_{n}} \underline{\underline{\Delta}}\left|\nabla \boldsymbol{u}^{\varepsilon_{n}}\right|^{2}$ and compute the product of $\nabla u^{\varepsilon_{n}}$ with the gradient of both sides of (4.3) to obtain

$$
\begin{equation*}
w^{\varepsilon_{n}}-\frac{1}{2} \Delta w^{\varepsilon_{n}}+2 \beta_{\varepsilon_{n}}^{\prime}\left(w^{\varepsilon_{n}}\right)\left(D^{2} u^{\varepsilon_{n}}\right) \nabla u^{\varepsilon_{n}} \cdot \nabla u^{\varepsilon_{n}}=H^{\varepsilon_{n}} \tag{5.1}
\end{equation*}
$$

where

$$
H^{\varepsilon_{n}} \stackrel{\Delta}{\underline{\Delta}} \nabla h \cdot \nabla u^{\varepsilon_{n}}-\left\|D^{2} u^{\varepsilon_{n}}\right\|^{2} .
$$

Along a subsequence, which we also call $\left\{\varepsilon_{n}\right\}_{n=1}^{\infty},\left\{H^{\varepsilon_{n}}\right\}_{n=1}^{\infty}$ converges to

$$
\begin{equation*}
\bar{H} \stackrel{\Delta}{\underline{ }} \nabla h \cdot \nabla u-\chi, \tag{5.2}
\end{equation*}
$$

where $\chi$ is the limit of $\left\|D^{2} u^{\varepsilon_{n}}\right\|^{2}$ in the weak* topology on $L_{\text {loc }}^{\infty}$. We will show that

$$
w=|\nabla u|^{2}=\lim _{n \rightarrow \infty}\left|\nabla u^{\varepsilon_{n}}\right|^{2}
$$

solves an obstacle problem involving $\bar{H}$, and we will then obtain $W_{l o c}^{2, p}$ regularity for $w$ by invoking the theory of variational inequalities.

For $r>0$ chosen so that $B_{r}(0) \triangleq\left\{x \in \mathbb{R}^{2} ;|x|<r\right\}$ contains $\mathscr{C}$, define

$$
K_{r} \stackrel{\Delta}{=}\left\{v \in W^{1,2}\left(B_{r}\right) ; 0 \leqq v \leqq 1 \text { on } B_{r} \text { and } v-1 \in W_{0}^{1,2}\left(B_{r}\right)\right\} .
$$

We pose the problem of finding $\varphi \in K_{r}$ such that

$$
\begin{equation*}
\frac{1}{2} \int_{B_{r}(0)} \nabla \varphi \cdot(\nabla v-\nabla \varphi) \geqq \int_{B_{r}(0)}(\bar{H}-w)(v-\varphi) \quad \forall v \in K_{r} . \tag{5.3}
\end{equation*}
$$

Lemma 5.1. The function $w=|\nabla u|^{2}$ solves (5.3).
Proof. Let $v \in K_{r}$ be given. From (5.1) we have

$$
\begin{align*}
\int_{B_{r}(0)} & \left(w^{\varepsilon_{n}}-\frac{1}{2} \Delta w^{\varepsilon_{n}}-H^{\varepsilon_{n}}\right)\left(v-w^{\varepsilon_{n}}\right) \\
& =-\int_{B_{r}(0)} 2 \beta_{\varepsilon_{n}}^{\prime}\left(w^{\varepsilon_{n}}\right)\left(D^{2} u^{\varepsilon_{n}}\right) \nabla u^{\varepsilon_{n}} \cdot \nabla u^{\varepsilon_{n}}\left(v-w^{\varepsilon_{n}}\right) \tag{5.4}
\end{align*}
$$

The function $u^{\varepsilon_{n}}$ is convex, $\boldsymbol{\beta}_{\varepsilon_{n}}^{\prime}\left(\boldsymbol{w}^{\varepsilon_{n}}\right)=0$ whenever $\boldsymbol{w}^{\varepsilon_{n}} \leqq 1$, and $v-\boldsymbol{w}^{\varepsilon_{n}}<0$ whenever $w^{\varepsilon_{n}}>1$. Therefore, the right-hand side of (5.4) is nonnegative, and integration by parts yields

$$
\begin{gather*}
-\frac{1}{2} \int_{\partial B_{r}(0)}\left(v-w^{\varepsilon_{n}}\right) \nabla w^{\varepsilon_{n}} \cdot n+\frac{1}{2} \int_{B_{r}(0)} \nabla w^{\varepsilon_{n}} \cdot\left(\nabla v-\nabla w^{\varepsilon_{n}}\right)  \tag{5.5}\\
\geqq \int_{B_{r}(0)}\left(H^{\varepsilon_{n}}-w^{\varepsilon_{n}}\right)\left(v-w^{\varepsilon_{n}}\right),
\end{gather*}
$$

where $n$ is the outward normal on $\partial B_{r}(0)$. Now $w^{\varepsilon_{n}} \rightarrow v$ uniformly on $\partial B_{r}(0), w^{\varepsilon_{n}} \rightarrow w$ uniformly on $B_{r}(0)$, and $H^{\varepsilon_{n}} \rightarrow \bar{H}, \nabla w^{\varepsilon_{n}} \rightarrow \nabla w$, both the latter convergences being weak*
in $L^{\infty}\left(B_{r}(0)\right)$. Because the weak* limit of $\left|\nabla w^{\varepsilon_{n}}\right|^{2}$ dominates $|\nabla w|^{2}$, we may pass to the limit in (5.5) to obtain

$$
\begin{equation*}
\frac{1}{2} \int_{B_{r}(0)} \nabla w \cdot(\nabla v-\nabla w) \geqq \int_{B_{r}(0)}(\bar{H}-w)(v-w) \quad \forall v \in K_{r} . \tag{5.6}
\end{equation*}
$$

Theorem 5.2. For every $p \in(1, \infty), w \triangleq|\nabla u|^{2} \in W_{l o c}^{2, p}$.
Proof. This is a classical result. See, for example, Lemma 5.1 and Theorem 3.11, p. 29 of Chipot (1984).

Corollary 5.3. We have $w \in C^{1, \alpha}\left(\mathbb{R}^{2}\right)$ for any $\alpha \in(0,1)$.
Proof. This follows from Sobolev imbedding (Gilbarg and Trudinger (1983, Thm. 7.17, p. 163)).

Remark 5.4. Integration by parts allows us to rewrite (5.6) as

$$
\int_{B_{r}(0)}\left(w-\frac{1}{2} \Delta w-\bar{H}\right)(v-w) \geqq 0 \quad \forall v \in K_{r},
$$

for all sufficiently large $r$, and so

$$
\begin{equation*}
\max \left\{w-\frac{1}{2} \Delta w-\bar{H}, w-1\right\}=0 . \tag{5.7}
\end{equation*}
$$

Now $\chi$ appearing in (5.2) dominates $\left\|D^{2} u\right\|$, and so $\bar{H}$ is dominated by

$$
\begin{equation*}
H \triangleq \nabla h \cdot \nabla u-\left\|D^{2} u\right\|^{2} . \tag{5.8}
\end{equation*}
$$

But let $x^{0} \in \mathscr{C}$ be given and choose $\varepsilon>0$ such that the closed disk $\overline{B_{2 \varepsilon}\left(x_{0}\right)}$ is contained in $\mathscr{C}$. Choose a positive integer $N$ such that

$$
\left|\nabla u^{\varepsilon_{n}}(x)\right|<1 \quad \forall n \geqq N, \quad x \in \overline{B_{2 \varepsilon}\left(x^{0}\right)} .
$$

From (4.1i), (4.2), and (4.3), we see that

$$
u^{\varepsilon_{n}}-\Delta u^{\varepsilon_{n}}=h \quad \text { on } \overline{B_{2 \varepsilon}\left(x^{0}\right)}
$$

According to Gilbarg and Trudinger (1983, Thm. 4.6, p. 6), for every $\alpha \in(0,1)$, $\left|u^{\varepsilon_{n}}\right|_{C^{2, \alpha}\left(B_{\varepsilon}\left(x^{0}\right)\right)}$ is bounded uniformly in $n \geqq N$. Thus, on $B_{\varepsilon}\left(x^{0}\right), D^{2} u^{\varepsilon_{n}}$ is continuous and converges uniformly to $D^{2} u, \chi=\left\|D^{2} u\right\|^{2}$, and $\bar{H}=H$. We conclude that (5.7) remains valid if $\bar{H}$ is replaced by $H$, i.e.,

$$
\begin{equation*}
\max \left\{w-\frac{1}{2} \Delta w-H, w-1\right\}=0 \tag{5.9}
\end{equation*}
$$

6. $\boldsymbol{D}^{2} \boldsymbol{u}$ inside $\overline{\mathscr{C}}$. Inside the set $\mathscr{C}$ defined by (4.9), $u$ satisfies the elliptic equation $u-\Delta u=h$, and is therefore smooth (at least $C^{4, \alpha}$ for all $\alpha \in(0,1)$ because $h$ is $C^{2,1}$ ). In this section, we describe the behavior of $D^{2} u$ as $\partial \mathscr{C}$ is approached from inside $\mathscr{C}$.

Lemma 6.1. Let $z \in \partial \mathscr{C}$ be given. As $x \in \mathscr{C}$ approaches $z, D^{2} u(x)$ approaches the matrix

$$
A(z) \triangleq=(u(z)-h(z))\left[\begin{array}{cc}
u_{2}^{2}(z) & -u_{1}(z) u_{2}(z) \\
-u_{1}(z) u_{2}(z) & u_{1}^{2}(z)
\end{array}\right],
$$

where $u_{i}$ denotes the ith partial derivative of $u$.
Proof. Because $w=|\nabla u|^{2}=1$ on $\partial \mathscr{C}, A(z)$ can be characterized as the unique $2 \times 2$ positive semidefinite matrix with eigenvalues zero and $u(z)-h(z)$, and with $\nabla u(z)$ an eigenvector corresponding to the eigenvalue zero. Let $v$ be a unit vector orthogonal to the unit vector $\nabla u(z)$. It suffices to show that

$$
\begin{gather*}
\lim _{\substack{x \rightarrow z \\
x \in \mathscr{C}}} D^{2} u(x) \nabla u(z)=0  \tag{6.1}\\
\lim _{\substack{x \rightarrow z \\
x \in \mathscr{C}}} D^{2} u(x) \nu=(u(z)-h(z)) \nu . \tag{6.2}
\end{gather*}
$$

Because $w=|\nabla u|^{2}$ attains its maximum value of 1 at $z$, and $\nabla w$ is continuous (Corollary 5.3), we have

$$
0=\nabla w(z)=\lim _{\substack{x \rightarrow z \\ x \in \mathscr{C}}} \nabla w(x)=\lim _{\substack{x \rightarrow z \\ x \in \mathscr{C}}} D^{2} u(x) \nabla u(x) .
$$

Since $\nabla u$ is continuous and $D^{2} u \in L_{\text {loc }}^{\infty}$, (6.1) follows.
Let $0=\lambda_{1}(x) \leqq \lambda_{2}(x)$ denote the eigenvalues of $D^{2} u(x)$. Then $u(x)-h(x)=$ $\Delta u(x)=\lambda_{1}(x)+\lambda_{2}(x)$ for all $x \in \mathscr{C}$, and (6.1) shows that $\lim _{x \rightarrow z, x \in \mathscr{C}} \lambda_{1}(x)=0$. Consequently,

$$
\begin{equation*}
\lim _{\substack{x \rightarrow z \\ x \in \mathscr{C}}} \lambda_{2}(x)=u(z)-h(z), \tag{6.3}
\end{equation*}
$$

which is thus nonnegative. If $u(z)-h(z)=0$, then $D^{2} u(x)$ approaches the zero matrix and (6.2) holds. If $u(z)-h(z)>0$, then (6.1) implies that any unit eigenvector corresponding to $\lambda_{1}(x)$ must, as $x \in \mathscr{C}$ approaches $z$, approach colinearity with $\nabla u(z)$. Hence, any unit eigenvector corresponding to $\lambda_{2}(x)$ approaches colinearity with $\nu$, and (6.2) follows from (6.3).

Remark 6.2. The characterization of $A(z)$ used in the proof of Lemma 6.1 makes critical use of the fact that our problem is posed in two dimensions. The two-dimensional nature of the problem also plays a fundamental role in Lemma 8.1, and together these lemmas provide the basis for $\S 10$, where the existence of a continuous version of $D^{2} u$ on $\mathbb{R}^{2}$ is established.

Theorem 6.3. For every $\alpha \in(0,1), u \in C^{2, \alpha}(\overline{\mathscr{C}})$, i.e., $D^{2} u$ restricted to $\mathscr{C}$ has an $\alpha$-Hölder continuous extension to $\overline{\mathscr{C}}$.

Proof. Because $|\nabla u|=1$ on $\partial \mathscr{C}$, we can choose an open set $G \subset \mathscr{C}$ such that $|\nabla u|$ is bounded away from zero on $\mathscr{C} \backslash G$. Elliptic regularity implies the Hölder continuity of $D^{2} u$ on $\bar{G}$, so it suffices to prove uniform Hölder continuity of $D^{2} u$ on $\mathscr{C} \backslash G$.

Let a unit vector $\nu$ be given, and define on $\mathscr{C} \backslash G, \eta \stackrel{\Delta}{\nabla} \boldsymbol{u} /|\nabla u|, z \triangleq \nu-(\nu \cdot \eta) \eta$,

$$
\gamma \triangleq \begin{cases}\frac{z}{|z|} & \text { if } z \neq 0 \\
{\left[\begin{array}{rr}
0 & 1 \\
-1 & 0
\end{array}\right] \eta} & \text { if } z=0\end{cases}
$$

Observe that $\eta \cdot \gamma=0$ and $|\eta|=|\gamma|=1$. Therefore,

$$
\Delta u=\left(D^{2} u\right) \eta \cdot \eta+\left(D^{2} u\right) \gamma \cdot \gamma \quad \text { on } \mathscr{C} \backslash G .
$$

Direct calculation shows that on $\mathscr{C} \backslash G$,

$$
\begin{aligned}
\left(D^{2} u\right) \nu \cdot \nu= & \left(D^{2} u\right) z \cdot z+2(\nu \cdot \eta)\left(D^{2} u\right) \eta \cdot z+(\nu \cdot \eta)^{2}\left(D^{2} u\right) \eta \cdot \eta \\
= & |z|^{2}\left(\Delta u-\left(D^{2} u\right) \eta \cdot \eta\right)+2(\nu \cdot \eta)\left(D^{2} u\right) \eta \cdot(\nu-(\nu \cdot \eta) \eta) \\
& +(\nu \cdot \eta)^{2}\left(D^{2} u\right) \eta \cdot \eta .
\end{aligned}
$$

Since $\Delta u=u-h$ and $2\left(D^{2} u\right) \eta=(\nabla w /|\nabla u|)$ on $\mathscr{C} \backslash G$, we have

$$
\begin{align*}
\left(D^{2} u\right) \nu \cdot \nu= & \left|\nu-\frac{(\nu \cdot \nabla u)}{|\nabla u|^{2}} \nabla u\right|^{2}\left(u-h-\frac{1}{2} \frac{(\nabla w \cdot \nabla u)}{|\nabla u|^{2}}\right) \\
& +\frac{(\nu \cdot \nabla u)}{|\nabla u|^{2}} \nabla w \cdot \nu-\frac{(\nu \cdot \nabla u)^{2}}{2|\nabla u|^{4}} \nabla w \cdot \nabla u \quad \text { on } \mathscr{C} \backslash G . \tag{6.4}
\end{align*}
$$

All the terms appearing on the right-hand side of (6.4) are uniformly Hölder continuous in $\mathscr{C} \backslash G$ (recall Corollary 5.3).
7. The gradient flow. Recalling Remark 4.6, we let $y^{0} \in \mathscr{C}$ denote the unique minimizer of $u$. Using the strict convexity of $u$ in $\mathscr{C}$ (Lemma 4.4), we choose $\delta>0$, $\mu>0$ such that

$$
\begin{gather*}
B_{2 \delta}\left(y^{0}\right) \subset \mathscr{C},  \tag{7.1}\\
D^{2} u(x) y \cdot y \geqq \mu|y|^{2} \quad \forall x \in B_{2 \delta}\left(y^{0}\right),  \tag{7.2}\\
\mu \leqq|\nabla u(x)|^{2} \leqq \frac{1}{2} \quad \forall x \in \partial B_{\delta}\left(y^{0}\right), \tag{7.3}
\end{gather*}
$$

$$
\begin{equation*}
\nabla u\left(y^{0}+\delta \theta\right) \cdot \theta \geqq \mu \quad \forall \theta \in S_{1}, \tag{7.4}
\end{equation*}
$$

where $S_{1} \stackrel{\Delta}{\Delta} \partial B_{1}(0)$ is the set of unit vectors in $\mathbb{R}^{2}$. For $\theta \in S_{1}$, we define the gradient flow $\psi(t, \theta)$ to be the unique solution to the differential equation

$$
\begin{equation*}
\frac{d}{d t} \psi(t, \theta)=\nabla u(\psi(t, \theta)), \quad t \geqq 0 \tag{7.5}
\end{equation*}
$$

with the initial condition

$$
\begin{equation*}
\psi(0, \theta)=y^{0}+\delta \theta \tag{7.6}
\end{equation*}
$$

We will find it convenient to use $\psi$ to change coordinates in $\mathbb{R}^{2}$. The following theorem justifies this.

Theorem 7.1. The map $\psi$ is a homeomorphism from $[0, \infty] \times S_{1}$ onto $\mathbb{R}^{2} \backslash B_{\delta}\left(y^{0}\right)$.
Proof. Let us for the moment fix $\theta \in S_{1}$ and define $n(t) \triangleq \psi(t, \theta)-y^{0}$ for all $t \geqq 0$. Because $|\nabla u| \leqq 1$, we have $|n(t)| \leqq t+\delta$, and $y^{0}+(\delta \wedge t / t) n(t) \in B_{2 \delta}\left(y^{0}\right)$ for all $t>0$. We conclude from the convexity of $u$ on $\mathbb{R}^{2}$ and from (7.2) that for $t>0$ :

$$
\begin{align*}
\frac{d}{d t}|n(t)|^{2}= & 2 \nabla u\left(y^{0}+n(t)\right) \cdot n(t) \\
= & 2\left[\nabla u\left(y^{0}+n(t)\right)-\nabla u\left(y^{0}+\frac{\delta \wedge t}{t} n(t)\right)\right] \cdot n(t) \\
& +2\left[u\left(y^{0}+\frac{\delta \wedge t}{t} n(t)\right)-\nabla u\left(y^{0}\right)\right] \cdot n(t)  \tag{7.7}\\
\geqq & 2 \int_{0}^{\delta \wedge t / t} D^{2} u\left(y^{0}+\tau n(t)\right) n(t) \cdot n(t) d \tau \\
\geqq & 2 \mu\left(1 \wedge \frac{\delta}{t}\right)|n(t)|^{2} .
\end{align*}
$$

Since $|n(0)|^{2}=\delta^{2}$, we can integrate (7.7) to obtain the inequality

$$
\begin{equation*}
\left|\psi(t, \theta)-y^{0}\right|^{2} \geqq \delta^{2}\left(1 \vee \frac{t}{\delta}\right)^{2 \mu \delta} e^{2 \mu(t \wedge \delta)} \quad \forall t \geqq 0, \quad \theta \in S_{1} \tag{7.8}
\end{equation*}
$$

One consequence of (7.8) is that

$$
\begin{equation*}
|\psi(s, \theta)-\psi(0, \varphi)|>0 \quad \forall s>0, \quad \theta \in S_{1}, \quad \varphi \in S_{1} . \tag{7.9}
\end{equation*}
$$

Now let $s, t \in[0, \infty)$ and $\theta, \varphi \in S_{1}$ be given. Again using the convexity of $u$, we may write

$$
|\psi(t+s, \theta)-\psi(t, \varphi)|^{2}=|\psi(s, \theta)-\psi(0, \varphi)|^{2}
$$

$$
\begin{align*}
& +2 \int_{0}^{t}[\nabla u(\psi(\tau+s, \theta))-\nabla u(\psi(\tau, \varphi))] \cdot[\psi(\tau+s, \theta)-\psi(\tau, \varphi)] d \tau  \tag{7.10}\\
& \geqq|\psi(s, \theta)-\psi(0, \varphi)|^{2}
\end{align*}
$$

If $\theta, \varphi$ are in $S_{1}$ and $t_{1}, t_{2}$ are in [ $0, \infty$ ) and $t_{1} \neq t_{2}$, then (7.9), (7.10) imply that $\psi\left(t_{1}, \theta\right) \neq \psi\left(t_{1}, \varphi\right)$. If $t_{1}=t_{2}$ but $\theta \neq \varphi$, then the uniqueness of solutions to (7.5) implies that $\psi\left(t_{1}, \theta\right) \neq \psi\left(t_{2}, \varphi\right)$. This concludes the proof that $\psi$ is injective.

It is clear from its definition that $\psi$ is continuous. Define

$$
D \triangleq{ }^{\Delta} \psi\left([0, \infty) \times S_{1}\right) \subset \mathbb{R}^{2} \backslash B_{\delta}\left(y^{0}\right)
$$

to be the range of $\psi$. Let $x \in D$ and $\varepsilon>0$ be given. It follows from (7.8) that there exists $T>0$ such that

$$
D \cap B_{\varepsilon}(x) \subset \psi\left([0, T] \times S_{1}\right)
$$

But an injective, continuous map on a compact set has a continuous inverse, so $\psi^{-1}$ is continuous at $x$.

It remains to show that $D=\mathbb{R}^{2} \backslash B_{\delta}\left(y^{0}\right)$. There is a function $\hat{\psi}:[0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}^{2}$ such that

$$
\hat{\psi}(t, \beta)=\psi(t,(\cos \beta, \sin \beta)) \quad \forall(t, \beta) \in[0, \infty) \times \mathbb{R}
$$

and $\hat{\psi}$ is continuous and locally injective. It follows from Deimling (1985, Thm. 4.3, p. 23) that

$$
D \cap\left(\mathbb{R}^{2} \backslash \overline{\left.B_{\delta}\left(y^{0}\right)\right)}=\hat{\psi}((0, \infty) \times \mathbb{R})\right.
$$

is open. On the other hand, if $\left\{x^{n}\right\}_{n=1}^{\infty} \subset D$ is a sequence with limit $x^{0} \in \mathbb{R}^{2}$, then (7.8) shows that $\left\{\psi^{-1}\left(x^{n}\right)\right\}_{n=1}^{\infty}$ is bounded and thus has an accumulation point $\left(t^{0}, \theta^{0}\right) \in$ $[0, \infty) \times S_{1}$. The continuity of $\psi$ implies that $x^{0}=\psi\left(t^{0}, \theta^{0}\right)$, so $D$ is closed. It follows that $D=\mathbb{R}^{2} \backslash B_{\delta}\left(y^{0}\right)$.

Corollary 7.2. For $\theta \in S_{1}$ and $\gamma \in\left[\frac{1}{2}, 1\right]$, define

$$
\begin{equation*}
T_{\gamma}(\theta) \triangleq \inf \left\{t \geqq 0 ;|\nabla u(\psi(t, \theta))|^{2} \geqq \gamma\right\} . \tag{7.11}
\end{equation*}
$$

Then

$$
\sup _{\substack{1 / 2 \leq \gamma \leqq 1 \\ \theta \in S_{1}}} T_{\gamma}(\theta) \leqq \sup _{\theta \in S_{1}} T_{1}(\theta)<\infty
$$

Proof. According to Lemma $4.4, \mathscr{C}$ is bounded. We can use (7.8) to choose $t^{*} \in(0, \infty)$ such that

$$
\mathscr{C} \subset \psi\left(\left[0, t^{*}\right] \times S^{1}\right)
$$

THEOREM 7.3. The homeomorphism $\psi$ is Lipschitz continuous on compact subsets of $[0, \infty) \times S_{1}$, and $\psi^{-1}$ is Lipschitz continuous on all of $\mathbb{R}^{2} \backslash B_{\delta}\left(y^{0}\right)$.

Proof. It follows immediately from (7.5) that $|(d / d t) \psi(t, \theta)| \leqq 1$ for all $(t, \theta) \in$ $[0, \infty) \times S_{1}$. Now let $T>0$ be given and use Theorem 4.3 to choose a Lipschitz constant $C$ for $\nabla u$ on $\psi\left([0, T] \times S_{1}\right)$. For $\theta, \varphi \in S_{1}$ and $t \in[0, T]$, we have

$$
\begin{aligned}
|\psi(t, \theta)-\psi(t, \varphi)| \leqq & |\psi(0, \theta)-\psi(0, \psi)| \\
& +\int_{0}^{t}|\nabla u(\psi(\tau, \theta))-\nabla u(\psi(\tau, \varphi))| d \tau \\
\leqq & \delta|\theta-\varphi|+C \int_{0}^{t}|\psi(\tau, \theta)-\psi(\tau, \varphi)| d \tau
\end{aligned}
$$

Gronwall's inequality gives

$$
|\psi(t, \theta)-\psi(t, \varphi)| \leqq \delta e^{C T}|\theta-\varphi|,
$$

and the local Lipschitz continuity of $\psi$ is proved.
To prove the global Lipschitz continuity of $\psi^{-1}$, we let $x^{1}, x^{2} \in \mathbb{R}^{2} \backslash \boldsymbol{B}_{\delta}\left(y^{0}\right)$ be given and define $\left(t_{1}, \theta_{1}\right)=\psi^{-1}\left(x^{1}\right),\left(t_{2}, \theta_{2}\right)=\psi^{-1}\left(x^{2}\right)$. Assume without loss of generality that $\left|x^{1}-x^{2}\right| \leqq 1$ and that $t_{1} \geqq t_{2}$. Set $s=t_{1}-t_{2}$. According to (7.10) and (7.8),

$$
\begin{align*}
\left|x^{1}-x^{2}\right| & \geqq\left|\psi\left(s, \theta_{1}\right)-\psi\left(0, \theta_{2}\right)\right| \\
& \geqq\left|\psi\left(s, \theta_{1}\right)-y^{0}\right|-\left|y^{0}-\psi\left(0, \theta_{2}\right)\right|  \tag{7.12}\\
& \geqq \delta\left(1 \vee \frac{s}{\delta}\right)^{\mu \delta} e^{\mu(s \wedge \delta)}-\delta \\
& \geqq \delta \mu\left(1 \vee \frac{s}{\delta}\right)^{\mu \delta}(s \wedge \delta) .
\end{align*}
$$

If $0 \leqq s \leqq \delta$, then (7.12) yields

$$
\begin{equation*}
\left|t_{1}-t_{2}\right| \leqq \frac{1}{\delta \mu}\left|x^{1}-x^{2}\right| \tag{7.13}
\end{equation*}
$$

If $s \geqq \delta$ and $\mu \delta \geqq 1$, (7.12) again yields (7.13). Finally, if $s \geqq \delta$ and $0<\mu \delta<1$, (7.12) yields $\left|x^{1}-x^{2}\right| \geqq \mu \delta^{1-\mu \delta} s^{\mu \delta}$, so

$$
\begin{equation*}
\left|t_{1}-t_{1}\right| \leqq\left(\mu \delta^{1-\mu \delta}\right)^{1 / \mu \delta}\left|x^{1}-x^{2}\right|^{1 / \mu \delta} \leqq\left(\mu \delta^{1-\mu \delta}\right)^{1 / \mu \delta}\left|x^{1}-x^{2}\right| . \tag{7.14}
\end{equation*}
$$

Relations (7.14) and (7.15) imply the global Lipschitz continuity of the first component of $\psi^{-1}$, i.e., there exists a constant $L>0$ such that

$$
\begin{equation*}
\left|t_{1}-t_{2}\right| \leqq L\left|\psi\left(t_{1}, \theta_{1}\right)-\psi\left(t_{2}, \theta_{2}\right)\right| \quad \forall\left(t_{1}, \theta_{1}\right),\left(t_{2}, \theta_{2}\right) \in[0, \infty) \times S_{1} . \tag{7.15}
\end{equation*}
$$

Now let $x^{1}, x^{2} \in \mathbb{R}^{2} \backslash B_{\delta}\left(y^{0}\right)$ be given, and define $\left(t_{1}, \theta_{1}\right),\left(t_{2}, \theta_{2}\right)$, and $s=t_{1}-t_{2} \geqq 0$ as before. From (7.10), (7.5), and (7.6), we have

$$
\begin{aligned}
\left|x^{1}-x^{2}\right| & \geqq\left|\psi\left(s, \theta_{1}\right)-\psi\left(0, \theta_{2}\right)\right| \\
& \geqq-\left|\psi\left(s, \theta_{1}\right)-\psi\left(0, \theta_{1}\right)\right|+\left|\psi\left(0, \theta_{1}\right)-\psi\left(0, \theta_{2}\right)\right| \\
& \geqq-s+\delta\left|\theta_{1}-\theta_{2}\right| .
\end{aligned}
$$

Relation (7.15) gives us

$$
\left|\theta_{1}-\theta_{2}\right| \leqq \frac{1}{\delta}\left|t_{1}-t_{2}\right|+\frac{1}{\delta}\left|x^{1}-x^{2}\right| \leqq \frac{1}{\delta}(1+L)\left|x^{1}-x^{2}\right| .
$$

Remark 7.4. In much of what follows, we will use the coordinates $(t, \theta) \in$ $[0, \infty) \times S_{1}$ rather than the coordinates $x \in \mathbb{R}^{2} \backslash B_{\delta}\left(y^{0}\right)$. We may identify $S_{1}$ with the unit circle, and let $[0, \infty) \times S_{1}$ have the product of Lebesgue measure and arc length measure. An important consequence of Theorem 7.3 is that $\psi$ maps measure zero subsets of $[0, \infty) \times S_{1}$ onto Lebesgue measure zero subsets of $\mathbb{R}^{2} \backslash B_{\delta}\left(y^{0}\right)$. Likewise, $\psi^{-1}$ preserves measure zero sets.
8. $\boldsymbol{W}^{\mathbf{2}, \infty}$ regularity for the obstacle problem. The purpose of this section is to show that the function $w=|\nabla u|^{2}$ is in $W_{l o c}^{2, \infty}$. This improves the regularity result of Theorem 5.2.

Lemma 8.1. We have

$$
\begin{equation*}
\left(D^{2} u\right) \nabla u=0, \quad\left\|D^{2} u\right\|=\Delta u \quad \text { a.e. on } \mathbb{R}^{2} \backslash \mathscr{C} . \tag{8.1}
\end{equation*}
$$

Proof. By the definition of $\mathscr{C}, w$ attains its maximum value of 1 at every point in $\mathbb{R}^{2} \backslash \mathscr{C}$, so $\nabla w=0$ everywhere on $\mathbb{R}^{2} \backslash \mathscr{C}$. But $\nabla w=2\left(D^{2} u\right) \nabla u$ almost everywhere on $\mathbb{R}^{2}$, and the first part of (8.1) follows. Since $D^{2} u$ is singular almost everywhere on $\mathbb{R}^{2} \backslash \mathscr{C}$, the second part of (8.1) also holds.

Remark 8.2. Because $D^{2} u$ is positive definite on $\mathscr{C}$ and positive semidefinite almost everywhere on $\mathbb{R}^{2}$, and since (recalling Remark 7.4)

$$
\begin{array}{r}
\frac{d}{d t} w(\psi(t, \theta))=2 D^{2} u(\psi(t, \theta)) \nabla u(\psi(t, \theta)) \cdot \nabla u(\psi(t, \theta)) \\
\quad \text { a.e. }(t, \theta) \in[0, \infty) \times S^{1}, \tag{8.2}
\end{array}
$$

the function $t \mapsto w(\psi(t, \theta))$ is nondecreasing for almost every $\theta \in S^{1}$. In particular, with $T_{1}(\theta)$ defined by (7.11), we have

$$
\begin{equation*}
w(\psi(t, \theta)) \equiv 1 \quad \forall t \geqq T_{1}(\theta), \quad \text { a.e. } \theta \in S^{1} . \tag{8.3}
\end{equation*}
$$

Theorem 8.3. The function $w=|\nabla u|^{2}$ is in $W^{2, \infty}$.
Proof. Recall that $w$ satisfies (5.9), where for all $\alpha \in(0,1), H \triangleq \nabla h \cdot \nabla u-\left\|D^{2} u\right\|^{2}$ is of class $C^{0, \alpha}$ inside $\mathscr{C}$, and $H$ is defined up to almost everywhere equivalence on $\mathbb{R}^{2} \backslash \mathscr{C}$. We define

$$
\hat{H}(x) \stackrel{\Delta}{\underline{\Delta}}\left\{\begin{array}{l}
\nabla h(x) \cdot \nabla u(x)-\left\|D^{2} u(x)\right\|^{2} \quad \forall x \in \mathscr{C}  \tag{8.4}\\
\nabla h(x) \cdot \nabla u(x)-\left[(u(x)-h(x))^{+}\right]^{2} \quad \text { if } x \in \mathbb{R}^{2} \backslash \mathscr{C} .
\end{array}\right.
$$

Now $u-h=\Delta u \geqq 0$ on $\mathscr{C}$, so $u-h \geqq 0$ on $\partial \mathscr{C}$. Theorem 6.3 and Lemma 6.1 then show that $\hat{H}$ is locally Hölder continuous with exponent $\alpha$ for any $\alpha \in(0,1)$. Because of (3.1) and Lemma 8.1,

$$
u-h \leqq \Delta u=\left\|D^{2} u\right\| \quad \text { a.e. on } \mathbb{R}^{2} \backslash \mathscr{C} .
$$

But $\Delta u \geqq 0$ almost everywhere $\mathbb{R}^{2}$, so

$$
\left[(u-h)^{+}\right]^{2} \leqq\left\|D^{2} u\right\|^{2} \quad \text { a.e. on } \mathbb{R}^{2} \backslash \mathscr{C} .
$$

Therefore $\hat{H} \geqq H$ amost everywhere $\mathbb{R}^{2} \backslash \mathscr{C}$, and $\hat{H}=H$ on $\mathscr{C}$, so (5.9) yields

$$
\begin{equation*}
\max \left\{w-\frac{1}{2} \Delta w-\hat{H}, w-1\right\}=0 . \tag{8.5}
\end{equation*}
$$

With the aid of (8.5) and the Hölder continuity of $\hat{H}$, we can obtain the $W^{2, \infty}$ regularity of $w$ from the theory of variational inequalities. More precisely, choose $r$ so that $\mathscr{C} \subset B_{r}(0)$ and observe that the Dirichlet problem

$$
\varphi-\frac{1}{2} \Delta \varphi=\hat{H} \quad \text { on } B_{r}(0), \quad \varphi=0 \quad \text { on } \partial B_{r}(0) .
$$

has a solution $\varphi$ which is in $C^{2, \alpha} \overline{\left(B_{r}(0)\right)}$ for any $\alpha \in(0,1)$ (Ladyzhenskaya and Ural'tseva (1968, Thm. 3.1.3, p. 115)). Set $\bar{w} \triangleq$| $\underline{\Delta}$ |
| :---: | , so that $\bar{w} \in W^{2, p} \overline{\left(B_{r}(0)\right)}$ for any $p \in(1, \infty)$ and

$$
\begin{gather*}
\max \left\{\bar{w}-\frac{1}{2} \Delta \bar{w}, \bar{w}-1+\varphi\right\}=0 \quad \text { in } B_{r}(0),  \tag{8.6}\\
\bar{w}=1 \quad \text { on } \partial B_{r}(0) \tag{8.7}
\end{gather*}
$$

Define

$$
L_{r} \triangleq\left\{v \in W^{1,2}\left(B_{r}(0)\right) ;-\varphi \leqq v \leqq 1-\varphi \text { on } B_{r}(0) \text { and } v-1 \in W_{0}^{1,2}\left(B_{r}\right)\right\}
$$

and note from (8.6), (8.7) that $\bar{w} \in L_{r}$ and

$$
\frac{1}{2} \int_{B_{r}(0)} \nabla \bar{w} \cdot(\nabla v-\nabla \bar{w}) \geqq-\int_{B_{r}(0)} \bar{w}(v-\bar{w}) \quad \forall v \in L_{r}
$$

It follows from Chipot (1984, Thm. 3.25, p. 49), that $\bar{w} \in W^{2, \infty}\left(B_{r}(0)\right)$, so also $w \in$ $W^{2, \infty}\left(B_{r}(0)\right)$. On $\mathbb{R}^{2} \backslash B_{r}(0), w \equiv 1$.

Corollary 8.4. We have $D^{2} u \in W^{1, \infty}(\overline{\mathscr{C}})$.
Proof. Use the $W^{1, \infty}$ regularity of $\nabla w$ in (6.4).
9. Lipschitz continuity of $\boldsymbol{T}_{\gamma}$. Recall the mappings $T_{\gamma}: S_{1} \mapsto[0, \infty)$ defined by (7.11) for each $\gamma \in\left[\frac{1}{2}, 1\right]$. The continuity of $\nabla u \circ \psi$ implies the lower semicontinuity of each $T_{\gamma}$. In this section we prove that for each $\gamma \in\left[\frac{1}{2}, 1\right], T_{\gamma}$ is, in fact, Lipschitz continuous.

Lemma 9.1. We have

$$
\begin{equation*}
K \triangleq \sup _{\nu \in S_{1}, x \in \mathscr{C}} \frac{|\nabla w(x)|}{D^{2} u(x) \nu \cdot \nu}<\infty \tag{9.1}
\end{equation*}
$$

Proof. Let $\nu, \eta \in S_{1}$ be given and set $f \stackrel{\Delta}{\Delta}\left(D^{2} u\right) \nu \cdot \nu$ and $g \stackrel{\Delta}{\Delta} \nabla w \cdot \eta$. Then in $\mathscr{C}$,

$$
f-\Delta f=\left(D^{2} h\right) \nu \cdot \nu \geqq c_{0}, \quad g-\Delta g=2 \nabla H \cdot \eta-g,
$$

where $c_{0}>0$ is the constant in (2.8), and $H$, defined by (5.8), is in $W^{1, \infty}(\overline{\mathscr{C}})$ because of Corollary 8.4. Furthermore, $g=0 \leqq f$ on $\partial \mathscr{C}$. Therefore the maximum principle implies that $g-K f \leqq 0$ in $\mathscr{C}$, where

$$
K \triangleq \stackrel{1}{c_{0}}\left(2\|\nabla H\|_{L^{\infty}(\bar{G})}+\|\nabla w\|_{L^{\infty}(\overline{\mathscr{G}})}\right)
$$

In other words, $\nabla w \cdot \eta \leqq K\left(D^{2} u\right) \nu \cdot \nu$. $\quad \square$
Theorem 9.2. For each $\gamma \in\left[\frac{1}{2}, 1\right]$, the mapping $T_{\gamma}: S_{1} \mapsto[0, \infty)$ is Lipschitz continuous with a Lipschitz constant which is independent of $\gamma$.

Proof. For each $\gamma \in\left[\frac{1}{2}, 1\right]$, define

$$
\mathscr{C}_{\gamma} \triangleq\left\{\psi(t, \theta) ; 0 \leqq t<T_{\gamma}(\theta)\right\} \cup B_{\delta}\left(y^{0}\right)
$$

(with $\psi, \delta$, and $y^{0}$ as in (7.1)-(7.6)). Each $\mathscr{C}_{\gamma}$ is open, $w<\gamma$ on $\mathscr{C}_{\gamma}$ and $w=\gamma$ on $\partial \mathscr{C}_{\gamma}$. For $\gamma \in\left[\frac{1}{2}, 1\right.$ ), we also have $\mathscr{C}_{\gamma} \subset \mathscr{C}$. Because of (4.10), $\nabla w$ does not vanish on $\mathscr{C}$, so for fixed $\gamma \in\left[\frac{1}{2}, 1\right)$ and $z \in \partial \mathscr{C}_{\gamma}$, the outward normal to $\mathscr{C}_{\gamma}$ exists and is

$$
n(z) \triangleq \frac{\Delta w(z)}{|\nabla w(z)|}=\frac{2 D^{2} u(z) \nabla u(z)}{|\nabla w(z)|} .
$$

In fact $D^{2} w$ is continuous in $\mathscr{C}$ and bounded in $\mathbb{R}^{2}$ (Theorem 8.3), so for every $\gamma \in\left[\frac{1}{2}, 1\right), \partial \mathscr{C}_{\gamma}$ has bounded curvature, i.e., there are constants $\varepsilon>0, K_{\gamma}>0$ such that for every $z \in \partial \mathscr{C}_{\gamma}$, and for every $x \in B_{\varepsilon}(z)$ :

$$
\begin{equation*}
(x-z) \cdot n(z) \geqq K_{\gamma}|x-z|^{2} \Rightarrow x \in \mathbb{R}^{2} \backslash \mathscr{C}_{\gamma} . \tag{9.2}
\end{equation*}
$$

We may use the local boundedness of $\left(d^{2} / d t^{2}\right) \psi(t, \theta)=\frac{1}{2} \nabla w(\psi(t, \theta))$ and the Lipschitz continuity of $\psi$ to choose a constant $K_{2}>0$ such that for every $\gamma \in\left[\frac{1}{2}, 1\right)$, every $\beta \in[0,1]$, and every $\theta, \varphi \in S_{1}$ :

$$
\begin{gather*}
\left|\psi\left(T_{\gamma}(\theta)+\beta, \theta\right)-\psi\left(T_{\gamma}(\theta), \theta\right)-\beta \nabla u\left(\psi\left(T_{\gamma}(\theta), \theta\right)\right)\right| \leqq K_{2} \beta^{2},  \tag{9.3}\\
\left|\psi\left(T_{\gamma}(\theta)+\beta, \theta\right)-\psi\left(T_{\gamma}(\theta)+\beta, \varphi\right)\right| \leqq K_{2}|\theta-\varphi| . \tag{9.4}
\end{gather*}
$$

With $K$ as in (9.1), choose $L>\max \left\{\frac{1}{2} K K_{2}, 1\right\}$. Let $\theta, \varphi \in S_{1}$ be given with $|\theta-\varphi| \leqq 1 / L$, and set

$$
\beta=L|\theta-\varphi|, \quad z=\psi\left(T_{\gamma}(\theta), \theta\right), \quad x=\psi\left(T_{\gamma}(\theta)+\beta, \varphi\right) .
$$

Then (9.3), (9.4) imply the existence of vectors $\nu, \eta \in B_{1}(0)$ such that

$$
x=z+\beta \nabla u(z)+K_{2} \beta^{2} \nu+K_{2}|\theta-\varphi| \eta .
$$

We calculate

$$
\begin{aligned}
(x-z) \cdot n(z) & =\frac{2 \beta D^{2} u(z) \nabla u(z) \cdot \nabla u(z)}{|\nabla w(z)|}+K_{2} \beta^{2} n(z) \cdot \nu+K_{2}|\theta-\varphi| n(z) \cdot \eta \\
& \geqq \frac{2 \beta}{K}-K_{2} \beta^{2}-K_{2}|\theta-\varphi| \\
& =\left(\frac{2 L}{K}-K_{2}\right)|\theta-\varphi|-K_{2} L^{2}|\theta-\varphi|^{2},
\end{aligned}
$$

and

$$
\begin{aligned}
K_{\gamma}|x-z|^{2} & =K_{\gamma}\left|\beta \nabla u(z)+K_{2} \beta^{2} \nu+K_{2}\right| \theta-\varphi|\eta|^{2} \\
& \leqq 9 K_{\gamma}\left(L^{2}+K_{2}^{2} L^{4}+K_{2}^{2}\right)|\theta-\varphi|^{2} .
\end{aligned}
$$

It is clear that for $|\theta-\varphi|$ sufficiently small, $x \in B_{\varepsilon}(z)$ and

$$
(x-z) \cdot n(z) \geqq K_{\gamma}|x-z|^{2}
$$

from which we conclude (see (9.2)) that $x \in \mathbb{R}^{2} \backslash \mathscr{C}_{\gamma}$, i.e.,

$$
T_{\gamma}(\varphi) \leqq T_{\gamma}(\theta)+\beta=T_{\gamma}(\theta)+L|\theta-\varphi| .
$$

Interchanging the roles of $\theta$ and $\varphi$, we obtain

$$
\left|T_{\gamma}(\theta)-T_{\gamma}(\varphi)\right| \leqq L|\theta-\varphi|
$$

for all $\theta, \varphi \in S_{1}$ such that $|\theta-\varphi|$ is sufficiently small.
For each $\theta \in S_{1}$, the mapping $t \mapsto w(\psi(t, \theta))$ is strictly increasing on [ $0, T_{1}(\theta)$ ] (see (8.2) and (4.10)). Therefore, the mapping $\gamma \mapsto T_{\gamma}(\theta)$ is continuous on $\left[\frac{1}{2}, 1\right]$. The Lipschitz continuity of $T_{1}$ follows from the uniform Lipschitz continuity of $T_{\gamma}$ for $\gamma \in\left[\frac{1}{2}, 1\right)$.

Corollary 9.3. With $\psi, \delta$, and $y^{0}$ as in (7.1)-(7.6), we have

$$
\begin{equation*}
\mathscr{C}=\left\{\psi(t, \theta) ; \theta \in S^{1}, t \in\left[0, T_{1}(\theta)\right)\right\} \cup B_{\delta}\left(y^{0}\right) . \tag{9.5}
\end{equation*}
$$

Proof. Define $\tilde{\mathscr{C}}$ to be the set on the right-hand side of (9.5). It is clear that $\tilde{\mathscr{C}} \subset \mathscr{C}$, and because of (8.3) and Remark 7.4, the Lebesgue measure of $\mathscr{C} \backslash \tilde{\mathscr{C}}$ is zero. Let $x \in \mathscr{C} \backslash \tilde{\mathscr{C}}$ be given, and define $(t, \theta) \triangleq \psi^{-1}(x)$. Then $t \geqq T_{1}(\theta)$, but because $w\left(T_{1}(\theta), \theta\right)=$ 1 , we must in fact have $t>T_{1}(\theta)$. The continuity of $T_{1}$ and $w$ allows us to chose an open neighborhood of $(t, \theta)$ contained in $\mathscr{C} \backslash \tilde{\mathscr{C}}$, and this contradicts the Lebesgue negligibility of $\mathscr{C} \backslash \tilde{\mathscr{C}}$.
10. $\boldsymbol{D}^{2} \boldsymbol{u}$ outside $\mathscr{C}$. We saw in Lemma 8.1 that $D^{2} u$ is singular almost everywhere in $\mathbb{R}^{2} \backslash \mathscr{C}$. Indeed

$$
\begin{equation*}
u_{11} u_{1}+u_{12} u_{2}=0, \quad u_{12} u_{1}+u_{22} u_{2}=0 \quad \text { a.e. on } \mathbb{R}^{2} \backslash \mathscr{C}, \tag{10.1}
\end{equation*}
$$

and because $u_{1}^{2}+u_{2}^{2}=1$ on $\mathbb{R}^{2} \backslash \mathscr{C}$, we have

$$
D^{2} u=\Delta u\left[\begin{array}{cc}
u_{2}^{2} & -u_{1} u_{2}  \tag{10.2}\\
-u_{1} u_{2} & u_{1}^{2}
\end{array}\right] \text { a.e. on } \mathbb{R}^{2} \backslash \mathscr{C} \text {. }
$$

Because $u$ has continuous first partial derivatives on $\mathbb{R}^{2}$, the proof of continuity of $D^{2} u$ on $\mathbb{R}^{2} \backslash \mathscr{C}$ reduces to a search for a continuous version of $\Delta u$ on this set. In order for $D^{2} u$ to be continuous across $\partial \mathscr{C}$, we must also have $\Delta u=u-h$ on $\partial \mathscr{C}$ (see Lemma 6.1).

We shall construct the desired continuous version of $\Delta u$ in the $(t, \theta)$ variables. Indeed, if we set

$$
\lambda(t, \theta)=\Delta u(\psi(t, \theta)) \quad \forall \theta \in S^{1}, \quad t \geqq T_{1}(\theta),
$$

then a formal calculation relying on (10.2) and the constancy of $w$ on $\mathbb{R}^{2} \backslash \mathscr{C}$ leads to

$$
\begin{align*}
\frac{d}{d t} \lambda(t, \theta) & =\frac{1}{2} \Delta w(\psi(t, \theta))-\left\|D^{2} u(\psi(t, \theta))\right\|^{2}  \tag{10.3}\\
& =-\lambda^{2}(t, \theta) \quad \forall \theta \in S^{1}, \quad t \geqq T_{1}(\theta) .
\end{align*}
$$

Integrating this equation and invoking the condition $\Delta u=u-h$ on $\partial \mathscr{C}$, we obtain

$$
\begin{align*}
\lambda(t, \theta)=\frac{u\left(\psi\left(T_{1}(\theta), \theta\right)\right)-h\left(\psi\left(T_{1}(\theta), \theta\right)\right)}{1+\left(t-T_{1}(\theta)\right)\left[u\left(\psi\left(T_{1}(\theta), \theta\right)\right)-h\left(\psi\left(T_{1}(\theta), \theta\right)\right)\right]}  \tag{10.4}\\
\forall \theta \in S_{1}, \quad t \geqq T_{1}(\theta) .
\end{align*}
$$

The task before us is to show that with $\lambda$ defined by (10.4), the function $\lambda \circ \psi^{-1}$ is a version of $\Delta u$ on $\mathbb{R}^{2} \backslash \mathscr{C}$. This is essentially a justification of the formal differentiation in (10.3), which involved third-order derivatives of $u$.

Let $\rho: \mathbb{R}^{2} \rightarrow[0, \infty)$ be a $C^{\infty}$ function with support in $B_{1}(0)$ and satisfying $\int_{\mathbb{R}^{2}} \rho=1$. For $n=1,2, \cdots$, we define mollifications of $u$ by

$$
\begin{equation*}
u^{(n)}(x) \stackrel{\Delta}{\Delta} \int_{\mathbb{R}^{2}} u\left(x-\frac{1}{n} \xi\right) \rho(\xi) d \xi=n^{2} \int_{\mathbb{R}^{2}} u(\xi) \rho(n(x-\xi)) d \xi . \tag{10.5}
\end{equation*}
$$

Then $\nabla u^{(n)}$ and $D^{2} u^{(n)}$ are locally bounded, uniformly in $n$, and $u^{(n)} \rightarrow u, \nabla u^{(n)} \rightarrow \nabla u$, and $D^{2} u^{(n)} \rightarrow D^{2} u$ in $L_{l o c}^{1}$. By passing to subsequences if necessary, we assume that these convergences occur almost everywhere. We define for $(t, \theta) \in[0, \infty) \times S^{1}$ :

$$
\begin{gather*}
l^{(n)}(t, \theta) \triangleq \Delta u^{(n)}(\psi(t, \theta)), \quad n=1,2, \cdots,  \tag{10.6}\\
l(t, \theta) \triangleq \Delta u(\psi(t, \theta)), \tag{10.7}
\end{gather*}
$$

and observe that $l^{(n)}(t, \theta) \rightarrow l(t, \theta)$ for almost every $(t, \theta) \in[0, \infty) \times S^{1}$ (Remark 7.4).
Lemma 10.1. The functions

$$
\begin{equation*}
i^{(n)}(t, \theta)=\nabla \Delta u^{(n)}(\psi(t, \theta)) \cdot \nabla u(\psi(t, \theta)) \tag{10.8}
\end{equation*}
$$

are locally bounded, uniformly in $n$.
Proof. Observe first of all that

$$
\begin{aligned}
i^{(n)} & =\nabla \Delta u^{(n)} \cdot \nabla u^{(n)}+\nabla \Delta u^{(n)} \cdot\left(\nabla u-\nabla u^{(n)}\right) \\
& =\frac{1}{2} \Delta\left(\left|\nabla u^{(n)}\right|^{2}\right)-\left\|D^{2} u^{(n)}\right\|^{2}+\nabla \Delta u^{(n)} \cdot\left(\nabla u-\nabla u^{(n)}\right),
\end{aligned}
$$

where $i^{(n)}$ is evaluated at $(t, \theta)$, and the right-hand side is evaluated at $\psi(t, \theta)$. It suffices to obtain uniform local bounds on $\Delta\left(\left|\nabla u^{(n)}\right|^{2}\right)$ and $\nabla \Delta u^{(n)} \cdot\left(\nabla u-\nabla u^{(n)}\right)$.

Define for $i \in\{1,2\}$ the functions

$$
\begin{aligned}
F_{i i}^{(n)}(x) & \triangleq \int_{\mathbb{R}^{2}}\left(\left|\nabla u\left(x-\frac{1}{n} \xi\right)\right|^{2}\right)_{i, i} \rho(\xi) d \xi \\
& =n \int_{\mathbb{R}^{2}}\left(\left|\nabla u\left(x-\frac{1}{n} \xi\right)\right|^{2}\right)_{i} \rho_{i}(\xi) d \xi \\
& =2 n \int_{\mathbb{R}^{2}} \nabla u_{i}\left(x-\frac{1}{n} \xi\right) \cdot \nabla u\left(x-\frac{1}{n} \xi\right) \rho_{i}(\xi) d \xi
\end{aligned}
$$

$$
n=1,2, \cdots,
$$

and note that these functions are uniformly bounded in $n$ (Theorem 8.3). Then

$$
\begin{aligned}
&\left(\left|\nabla u^{(n)}(x)\right|^{2}\right)_{i i}= 2 n^{6} \int_{\mathbb{R}^{2}} \int_{\mathbb{R}^{2}} \nabla u(\xi) \cdot \nabla u(\eta)\left[\rho_{i i}(n(x-\xi)) \rho(n(x-\eta))\right. \\
&\left.+\rho_{i}(n(x-\xi)) \rho_{i}(n(x-\eta))\right] d \xi d \eta \\
&= 2 n^{2} \int_{\mathbb{R}^{2}} \int_{\mathbb{R}^{2}} \nabla u\left(x-\frac{1}{n} \xi\right) \cdot \nabla u\left(x-\frac{1}{n} \eta\right)\left[\rho_{i i}(\xi) \rho(\eta)\right. \\
&\left.+\rho_{i}(\xi) \rho_{i}(\eta)\right] d \xi d \eta \\
&= 2 n \int_{\mathbb{R}^{2}} \int_{\mathbb{R}^{2}} \nabla u_{i}\left(x-\frac{1}{n} \xi\right) \cdot \nabla u\left(x-\frac{1}{n} \eta\right) \rho_{i}(\xi) \rho(\eta) d \xi d \eta \\
&+2 \int_{\mathbb{R}^{2}} \int_{\mathbb{R}^{2}} \nabla u_{i}\left(x-\frac{1}{n} \xi\right) \nabla u_{i}\left(x-\frac{1}{n} \eta\right) \rho(\xi) \rho(\eta) d \xi d \eta .
\end{aligned}
$$

The last term is locally bounded in $x$, uniformly in $n$. The next-to-last term is

$$
F_{i i}^{(n)}(x)+2 n \int_{\mathbb{R}^{2}} \int_{\mathbb{R}^{2}} \nabla u_{i}\left(x-\frac{1}{n} \xi\right) \cdot\left[\nabla u\left(x-\frac{1}{n} \eta\right)-\nabla u\left(x-\frac{1}{n} \xi\right)\right]
$$

$$
\rho_{i}(\xi) \rho(\eta) d \xi d \eta
$$

which is also locally bounded in $x$, uniformly in $n$, because for all $\xi, \eta \in B_{1}(0)$,

$$
\left|\nabla u\left(x-\frac{1}{n} \eta\right)-\nabla u\left(x-\frac{1}{n} \xi\right)\right| \leqq \frac{2}{n} \sup _{B_{1}(x)}\left\|D^{2} u\right\| .
$$

This provides a uniform local bound on $\Delta\left(\left|\nabla u^{(n)}\right|^{2}\right)$.
On the other hand,

$$
\begin{aligned}
\nabla \Delta u^{(n)}(x) \cdot & \left(\nabla u(x)-\nabla u^{(n)}(x)\right) \\
= & n^{5} \int_{\mathbb{R}^{2}} \int_{\mathbb{R}^{2}} \Delta u(\xi) \\
& \times[\nabla u(x)-\nabla u(\eta)] \cdot \nabla \rho(n(x-\xi)) \rho(n(x-\eta)) d \xi d \eta \\
= & n \int_{\mathbb{R}^{2}} \int_{\mathbb{R}^{2}} \Delta u\left(x-\frac{1}{n} \xi\right) \\
& \times\left[\nabla u(x)-\nabla u\left(x-\frac{1}{n} \eta\right)\right] \cdot \nabla \rho(\xi) \rho(\eta) d \xi d \eta
\end{aligned}
$$

and the boundedness of this expression follows from the local Lipschitz continuity of $\nabla u$.

Because of Lemma 10.1, a subsequence of $\left\{i^{(n)}\right\}_{n=1}^{\infty}$ converges in the $L_{\text {loc }}^{\infty}$-weak* topology to a function $\zeta \in L_{\text {loc }}^{\infty}\left([0, \infty) \times S_{1}\right)$. We assume without loss of generality that the full sequence converges. For each nonnegative integer $k$, choose a number $t_{k}>k$ such that $\left\{l^{(n)}\left(t_{k}, \theta\right)\right\}_{n=1}^{\infty}$ converges for almost every $\theta \in S_{1}$, and define $\lambda_{k}\left(t_{k}, \theta\right)$ to be this limit. (Whereas $l(\cdot, \cdot)$ is defined up to almost everywhere equivalence on $[0, \infty) \times$ $S_{1}$, the functions $\lambda_{k}\left(t_{k}, \cdot\right)$ are defined up to almost everywhere equivalence on $S_{1}$.) We insist furthermore that $t_{0}$ be chosen so that $\psi\left(t_{0}, \theta\right) \in \mathscr{C}$ for all $\theta \in S_{1}$. Then $\Delta u\left(\psi\left(t_{0}, \cdot\right)\right)$ is defined pointwise on $S_{1}$ because $\Delta u$ is continuous on $\mathscr{C}$, and so we may require that

$$
\lambda_{0}\left(t_{0}, \theta\right)=\Delta u\left(\psi\left(t_{0}, \theta\right)\right) \quad \forall \theta \in S_{1}
$$

For each $k=0,1, \cdots$, define $\lambda_{k}:[0, \infty) \times S^{1} \mapsto \mathbb{R}$ by

$$
\lambda_{k}(t, \theta) \triangleq \lambda_{k}\left(t_{k}, \theta\right)+\int_{t_{k}}^{t} \zeta(s, \theta) d s,
$$

so that any two versions $\hat{\lambda}_{k}$ and $\tilde{\lambda}_{k}$ of this function have the property that the set $\left\{\theta \in S_{1} \mid\right.$ there exists $t \in[0, \infty]$ with $\left.\hat{\lambda}_{k}(t, \theta) \neq \tilde{\lambda}_{k}(t, \theta)\right\}$ has measure zero.

We now relate the functions $\lambda_{k}, k=0,1, \cdots$, to the function $l$ of (10.7). Let $\varphi$ be a continuous, real-valued function on $[0, \infty) \times S_{1}$, and define

$$
\Phi(t, \theta) \stackrel{\Delta}{\underline{~}} \int_{0}^{t} \varphi(s, \theta) d s \quad \forall(t, \theta) \in[0, \infty) \times S_{1}
$$

For $k=0,1, \cdots$,

$$
\begin{aligned}
\int_{S_{1}} \int_{0}^{t_{k}} \lambda_{k}( & s, \theta) \varphi(s, \theta) d s d \theta \\
& =\int_{S_{1}}\left[\lambda_{k}\left(t_{k}, \theta\right) \Phi\left(t_{k}, \theta\right)-\int_{0}^{t_{k}} \zeta(s, \theta) \varphi(s, \theta) d s\right] d \theta \\
& =\lim _{n \rightarrow \infty} \int_{S_{1}}\left[l^{(n)}\left(t_{k}, \theta\right) \Phi\left(t_{k}, \theta\right)-\int_{0}^{t_{k}} l^{(n)}(s, \theta) \varphi(s, \theta) d s\right] d \theta \\
& =\lim _{n \rightarrow \infty} \int_{S_{1}} \int_{0}^{t_{k}} l^{(n)}(s, \theta) \varphi(s, \theta) d s d \theta \\
& =\int_{S_{1}} \int_{0}^{t_{k}} l(s, \theta) \varphi(s, \theta) d s d \theta .
\end{aligned}
$$

It follows that $\lambda_{k}=l$ almost everywhere on $\left[0, t_{k}\right] \times S_{1}$. In particular, for any two nonnegative integers $k$ and $m, \lambda_{k}$ and $\lambda_{m}$ agree almost everywhere on $\left[0, t_{k} \wedge t_{m}\right] \times S_{1}$, and hence almost everywhere on $[0, \infty) \times S_{1}$. In particular,

$$
\begin{equation*}
\lambda_{0}(t, \theta)=\Delta u(\psi(t, \theta)), \quad \text { a.e. }(t, \theta) \in[0, \infty) \times S^{1} \tag{10.9}
\end{equation*}
$$

and for almost every $\theta \in S_{1}$,

$$
\begin{equation*}
\lambda_{0}(t, \theta)=\Delta u\left(\psi\left(t_{0}, \theta\right)\right)+\int_{t_{0}}^{t} \zeta(s, \theta) d s \quad \forall t \in[0, \infty) \tag{10.10}
\end{equation*}
$$

Lemma 10.2. Almost everywhere on the set

$$
\psi^{-1}\left(\mathbb{R}^{2} \backslash \mathscr{C}\right)=\left\{(t, \theta) \in[0, \infty) \times S_{1} ; t \geqq T_{1}(\theta)\right\}
$$

the function $\zeta$ appearing in (10.10) is equal to $-\lambda_{0}^{2}$.

Proof. From (10.8) we have

$$
\begin{aligned}
\dot{l}^{(n)} \circ \psi^{-1}+ & \left(l \circ \psi^{-1}\right)\left(l^{(n)} \circ \psi^{-1}\right) \\
= & \nabla \Delta u^{(n)} \cdot \nabla u+\Delta u \Delta u^{(n)} \\
= & \left(u_{12}^{(n)} u_{2}+u_{11}^{(n)} u_{1}\right)_{1}+\left(u_{12}^{(n)} u_{1}+u_{22}^{(n)} u_{2}\right)_{2} \\
& +u_{11}^{(n)} u_{22}+u_{11} u_{22}^{(n)}-2 u_{12}^{(n)} u_{12} .
\end{aligned}
$$

Now $u_{11}^{(n)} u_{22}+u_{11} u_{22}^{(n)}-2 u_{12}^{(n)} u_{12}$ is locally bounded, uniformly in $n$, and converges almost everywhere to 2 det $D^{2} u$, which is zero on $\mathbb{R}^{2} \backslash \mathscr{C}$. It follows from (10.1) that for any function $\varphi \in C_{0}^{1}\left(\mathbb{R}^{2} \backslash \overline{\mathscr{C}}\right)$,

$$
\begin{aligned}
& \lim _{n \rightarrow \infty} \int_{\mathbb{R}^{2} \backslash \bar{G}}\left[i^{(n)} \circ \psi^{-1}+\left(l \circ \psi^{-1}\right)\left(l^{(n)} \circ \psi^{-1}\right)\right] \varphi \\
& \quad=\lim _{n \rightarrow \infty} \int_{R^{2} \backslash \bar{G}}\left[\left(u_{12}^{(n)} u_{2}+u_{11}^{(n)} u_{1}\right)_{1}+\left(u_{12}^{(n)} u_{1}+u_{22}^{(n)} u_{2}\right)_{2}\right] \varphi \\
& \quad=-\lim _{n \rightarrow \infty} \int_{\mathbb{R}^{2} \backslash \bar{G}}\left(u_{12}^{(n)} u_{2}+u_{11}^{(n)} u_{1}\right) \varphi_{1}+\left(u_{12}^{(n)} u_{1}+u_{22}^{(n)} u_{2}\right) \varphi_{2} \\
& \quad=0 .
\end{aligned}
$$

Because the functions $i^{(n)} \circ \psi^{-1}+\left(l \circ \psi^{-1}\right)\left(l^{(n)} \circ \psi^{-1}\right)$ are locally bounded, uniformly in $n$, we can show that for every $\varphi \in L^{1}\left(\mathbb{R}^{2} \backslash \overline{\mathscr{C}}\right)$,

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \int_{\mathbb{R}^{2} \backslash \bar{G}}\left[i^{(n)} \circ \psi^{-1}+\left(l \circ \psi^{-1}\right)\left(l^{(n)} \circ \psi^{-1}\right)\right] \varphi=0 . \tag{10.11}
\end{equation*}
$$

Now let $\gamma \in L^{1}\left(\psi^{-1}\left(\mathbb{R}^{2} \backslash \overline{\mathscr{C}}\right)\right)$ be given so that $\left(\gamma \circ \psi^{-1}\right)\left|J \Psi^{-1}\right| \in L^{1}\left(\mathbb{R}^{2} \backslash \overline{\mathscr{C}}\right)$, where $\left|J \psi^{-1}\right|$ is the bounded (Theorem 7.3) determinant of the Jacobian of $\psi^{-1}$. From (10.11) it follows that

$$
\begin{aligned}
& \lim _{n \rightarrow \infty} \int_{\psi^{-1}\left(\mathbb{R}^{2} \backslash \overline{\mathscr{G}}\right)}\left(i^{(n)}+l l^{(n)}\right) \gamma \\
& \quad=\lim _{n \rightarrow \infty} \int_{\mathbb{R}^{2} \backslash \overline{\mathscr{G}}}\left[i^{(n)} \circ \psi^{-1}+\left(l \circ \psi^{-1}\right)\left(l^{(n)} \circ \psi^{-1}\right)\right]\left(\gamma \circ \psi^{-1}\right)\left|J \psi^{-1}\right| \\
& \quad=0 .
\end{aligned}
$$

On the other hand, $i^{(n)}+l l^{(n)}$ converges in the $L_{\text {loc }}^{\infty}$-weak* topology on $[0, \infty) \times S_{1}$ to $\zeta+l^{2}=\zeta+\lambda_{0}^{2}$ almost everywhere, and the lemma follows.

Theorem 10.3. There is a Lipschitz continuous version of $D^{2} u$ on $\mathbb{R}^{2}$.
Proof. For $\theta \in S_{1}$ and $0 \leqq t<T_{1}(\theta)$, define

$$
\begin{equation*}
\lambda(t, \theta) \triangleq \Delta u(\psi(t, \theta)), \tag{10.12}
\end{equation*}
$$

where, of course, we mean the Lipschitz continuous version of $\Delta u$ inside $\mathscr{C}$ (Corollary 8.4). For $\theta \in S_{1}$ and $t \geqq T_{1}(\theta)$, define $\lambda(t, \theta)$ by (10.4), which gives us a Lipschitz function. At $t=T_{1}(\theta)$, the Lipschitz continuity of $\lambda$ follows from (10.4), Lemma 6.1, and the equality $|\nabla u|^{2}=1$ on $\partial \mathscr{C}$. The Lipschitz continuity of $\psi^{-1}$ implies the Lipschitz continuity of $\lambda \circ \psi^{-1}$.

It remains to show that $\lambda \circ \psi^{-1}$ is a version of $\Delta u$, or equivalently,

$$
\begin{equation*}
\lambda(t, \theta)=\Delta u(\psi(t, \theta)), \quad \text { a.e. }(t, \theta) \in[0, \infty) \times S_{1} . \tag{10.13}
\end{equation*}
$$

In light of (10.9) and (10.12), we need only show that for almost every $\theta \in S_{1}$,

$$
\begin{equation*}
\lambda(t, \theta)=\lambda_{0}(t, \theta) \quad \forall t \geqq T_{1}(\theta) . \tag{10.14}
\end{equation*}
$$

But (10.10) shows that for almost every $\theta \in S_{1}$, the function $t \mapsto \lambda_{0}(t, \theta)$ is absolutely continuous on $[0, \infty)$; in particular,

$$
\begin{align*}
\lambda_{0}\left(T_{1}(\theta), \theta\right) & =\lim _{\uparrow \uparrow T_{1}(\theta)} \lambda_{0}(t, \theta) \\
& =\lim _{\uparrow \uparrow T_{1}(\theta)} \Delta u(\psi(t, \theta)) \\
& =\lim _{\uparrow \uparrow T_{1}(\theta)}[u(\psi(t, \theta))-h(\psi(t, \theta))]  \tag{10.15}\\
& =u\left(\psi\left(T_{1}(\theta), \theta\right)\right)-h\left(\psi\left(T_{1}(\theta), \theta\right)\right) .
\end{align*}
$$

Equation (10.10) and Lemma 10.2 imply that for almost every $\theta \in S_{1}$,

$$
\begin{equation*}
\dot{\lambda_{0}}(t, \theta)=-\lambda_{0}^{2}(t, \theta), \quad \text { a.e. } t \geqq T_{1}(\theta) . \tag{10.16}
\end{equation*}
$$

Equations (10.15) and (10.16) imply (10.14).
11. Regularity of the free boundary. In this section we apply known regularity results for free boundaries to show that the boundary of $\mathscr{C}$ is of class $C^{2, \alpha}$ for all $\alpha \in(0,1)$. In order to apply these results, we recall that $w=|\nabla u|^{2}$ is a $W^{2, \infty}$ function (Theorem 8.3) which satisfies (see (5.9)) $1-w \geqq 0$ on $\mathbb{R}^{2}$ and

$$
\begin{equation*}
\frac{1}{2} \Delta(1-w)=H-w \quad \text { on } \mathscr{C}, \tag{11.1}
\end{equation*}
$$

where we recall that $H \triangleq \stackrel{\Delta}{\triangleq} \nabla h \cdot \nabla u-\left\|D^{2} u\right\|^{2}$. We shall establish the strict positivity of the forcing term $H-w$ on $\partial \mathscr{C}$. Recall that

$$
w-\frac{1}{2} \Delta w-H \leqq 0 \quad \text { on } \mathbb{R}^{2},
$$

and $w=1, \Delta w=0$ on $\mathbb{R}^{2} \backslash \overline{\mathscr{C}}$, so

$$
\begin{equation*}
H-w=H-1 \geqq 0 \quad \text { on } \mathbb{R}^{2} \backslash \mathscr{C} . \tag{11.2}
\end{equation*}
$$

Lemma 11.1. The function $H$ is locally Lipschitz continuous, and $H>1$ on $\partial \mathscr{C}$.
Proof. The local Lipschitz continuity of $H$ follows from Theorem 10.3. To prove that $H>1$ on $\partial \mathscr{C}$, we assume that there exists a point on $\partial \mathscr{C}$ where $H=1$. Without loss of generality, we take this point to be the origin $(0,0)$, and we take $\nabla u(0,0)=$ $(-1,0)$.

We first obtain an upper bound on $H$ near $(0,0)$. Inside $\mathscr{C}, H$ is differentiable and

$$
\begin{equation*}
\nabla H \cdot \nabla u=\left(D^{2} h\right) \nabla u \cdot \nabla u+\left(D^{2} u\right) \nabla u \cdot \nabla h-\nabla\left(\left\|D^{2} u\right\|^{2}\right) \cdot \nabla u . \tag{11.3}
\end{equation*}
$$

Let $\nu^{1}$ and $\nu^{2}$ be unit eigenvectors for $D^{2} u$, and let $\lambda_{1}$ and $\lambda_{2}$ denote their respective (nonnegative) eigenvalues. Then

$$
\begin{align*}
\nabla\left(\left\|D^{2} u\right\|^{2}\right) \cdot \nabla u & =\operatorname{tr}\left(D^{2} w D^{2} u\right)-2 \operatorname{tr}\left[\left(D^{2} u\right)^{3}\right] \\
& =\lambda_{1}\left(D^{2} w\right) \nu^{1} \cdot \nu^{1}+\lambda_{2}\left(D^{2} w\right) \nu^{2} \cdot \nu^{2}-2\left(\lambda_{1}^{3}+\lambda_{2}^{3}\right)  \tag{11.4}\\
& \leqq 2\left\|D^{2} u\right\|_{L^{\infty}(\mathscr{C})} \sup _{\nu \in S_{1}}\left(D^{2} w\right) \nu \cdot \nu .
\end{align*}
$$

Applying Theorem 1 and the remark following it from Caffarelli (1977) to the function $1-w$, we have that for some positive constants $C$ and $\varepsilon$,

$$
\begin{equation*}
\sup _{\nu \in S_{1}} D^{2} w(x, y) \nu \cdot \nu \leqq C|\log (\operatorname{dist}((x, y), \partial \mathscr{C}))|^{-\varepsilon} \quad \forall(x, y) \in \mathscr{C} . \tag{11.5}
\end{equation*}
$$

Combining (11.3)-(11.5), we conclude that

$$
\begin{gather*}
\nabla H(x, y) \cdot \nabla u(x, y) \geqq D^{2} h(x, y) \nabla u(x, y) \cdot \nabla u(x, y)+\frac{1}{2} \nabla w(x, y) \cdot \nabla h(x, y)  \tag{11.6}\\
-2\left\|D^{2} u\right\|_{L^{\infty}(\mathscr{C})} C|\log (\operatorname{dist}((x, y), \partial \mathscr{C}))|^{-\varepsilon} \quad \forall(x, y) \in \mathscr{C} .
\end{gather*}
$$

As $(x, y)$ approaches $(0,0) \in \partial \mathscr{C},|\nabla u(x, y)|$ approaches 1 and $\nabla w(x, y)$ approaches zero. Using (2.8) and (11.6), we can choose $\tilde{\varepsilon}>0$ such that

$$
\begin{equation*}
\nabla H(x, y) \cdot \nabla u(x, y) \geqq \frac{c_{0}}{2} \quad \forall(x, y) \in[-\tilde{\varepsilon}, \tilde{\varepsilon}]^{2} \cap \mathscr{C} . \tag{11.7}
\end{equation*}
$$

Let $\theta_{0} \in S_{1}$ be such that $\psi\left(T_{1}\left(\theta_{0}\right), \theta_{0}\right)=(0,0)$. For $t \in\left(0, T_{1}\left(\theta_{0}\right)\right)$ chosen so that $\psi\left(t, \theta_{0}\right) \in[-\tilde{\varepsilon}, \varepsilon]^{2}$,

$$
\frac{d}{d t} H\left(\psi\left(t, \theta_{0}\right)\right)=\nabla H\left(\psi\left(t, \theta_{0}\right)\right) \cdot \nabla u\left(\psi\left(t, \theta_{0}\right)\right) \geqq \frac{c_{0}}{2} .
$$

It follows that for some $\tau>0$,

$$
\begin{align*}
H\left(\psi\left(T_{1}\left(\theta_{0}\right)-t, \theta_{0}\right)\right) & \leqq H\left(\psi\left(T_{1}\left(\theta_{0}\right), \theta_{0}\right)\right)-\frac{1}{2} c_{0} t \\
& =1-\frac{1}{2} c_{0} t \quad \forall t \in(0, \tau) . \tag{11.8}
\end{align*}
$$

But also

$$
\begin{align*}
&\left|\psi\left(T_{1}\left(\theta_{0}\right)-t, \theta_{0}\right)-(t, 0)\right|=\mid \psi\left(T_{1}\left(\theta_{0}\right)-t, \theta_{0}\right)-\psi\left(T_{1}\left(\theta_{0}\right), \theta_{0}\right) \\
&+t \nabla u\left(\psi\left(T_{1}\left(\theta_{0}\right), \theta_{0}\right)\right) \mid  \tag{11.9}\\
& \leqq t^{2}\left\|D^{2} u\right\|_{L^{\infty}(\boldsymbol{\varepsilon})} \quad \forall t \in\left(0, T_{1}\left(\theta_{0}\right)\right) .
\end{align*}
$$

Let $\beta>0$ be a Lipschitz constant for $H$ in a sufficiently large neighborhood of ( 0,0 ). From (11.8), (11.9), we have for all $t \in(0, \tau)$ :

$$
\begin{aligned}
H(t, 0) & \leqq H\left(\psi\left(T_{1}(\theta)-t, \theta_{0}\right)\right)+\left|H(t, 0)-H\left(\psi\left(T_{1}\left(\theta_{0}\right)-t, \theta_{0}\right)\right)\right| \\
& \leqq 1-\frac{1}{2} c_{0} t+\beta t^{2}\left\|D^{2} u\right\|_{L^{\infty}(\mathscr{C})} .
\end{aligned}
$$

Choosing $\tau$ smaller, if necesary, we have $H(t, 0) \leqq 1-\frac{1}{3} c_{0} t$ for all $t \in(0, \tau)$. Again using the Lipschitz continuity of $H$, we obtain the desired upper bound

$$
\begin{equation*}
H(x, y) \leqq 1-\frac{1}{3} c_{0} x+\beta|y| \quad \forall(x, y) \in[0, \tau] \times[-\tau, \tau] . \tag{11.10}
\end{equation*}
$$

We next construct a function $\varphi: \mathbb{R}^{2} \rightarrow \mathbb{R}$ such that for appropriate $\rho, \sigma \in(0, \tau)$,

$$
\begin{gather*}
\varphi-\frac{1}{2} \Delta \varphi \geqq H \quad \text { on }[0, \rho] \times[-\sigma, \sigma],  \tag{11.11}\\
\varphi \geqq 1 \quad \text { on } \partial([0, \rho] \times[\sigma, \sigma]),  \tag{11.12}\\
\varphi(0,0)=1 . \tag{11.13}
\end{gather*}
$$

For this purpose, choose $0<\rho<\min \left\{\tau,\left(c_{0} / 6 \sqrt{2} \beta\right)\right\}$ such that

$$
\begin{equation*}
\left(1-\frac{\rho^{2}}{4}\right) \sinh \sqrt{2} \rho \geqq \sqrt{2} \rho . \tag{11.14}
\end{equation*}
$$

Then define

$$
\begin{gather*}
\sigma \triangleq \underline{\triangleq} \min \left\{\tau, \frac{\rho^{2}}{4 \sqrt{2}}\right\}  \tag{11.15}\\
A \triangleq \frac{c_{0}}{3}\left(1-\frac{\sqrt{2} \rho}{\sinh \sqrt{2} \rho \cosh \sqrt{2} \sigma}\right)^{-1},  \tag{11.16}\\
\varphi(x, y) \triangleq 1+\beta \sigma\left(2-\frac{\cosh \sqrt{2} y}{\cosh \sqrt{2} \sigma}\right)\left(1-\frac{\sinh \sqrt{2} x+\sinh \sqrt{2}(\rho-x)}{\sinh \sqrt{2} \rho}\right) \\
+A \rho\left(1-\frac{\cosh \sqrt{2} y}{\cosh \sqrt{2} \sigma}\right)\left(-\frac{x}{\rho}+\frac{\sinh \sqrt{2} x}{\sinh \sqrt{2} \rho}\right) \quad \forall(x, y) \in \mathbb{R}^{2} .
\end{gather*}
$$

Then

$$
\begin{gathered}
\varphi(0, y)=\varphi(\rho, y)=1 \quad \forall y \in[-\sigma, \sigma] \\
\varphi(x, \pm \sigma)=1+\beta \sigma\left[1-\frac{\sinh \sqrt{2} x+\sinh \sqrt{2}(\rho-x)}{\sinh \sqrt{2} \rho}\right] \geqq 1 \quad \forall x \in[0, \rho]
\end{gathered}
$$

because

$$
\begin{align*}
\sinh a+\sinh b & \leqq \sinh a \cosh b+\sinh b \cosh a \\
& =\sinh (a+b) \quad \forall a, b \in \mathbb{R} . \tag{11.17}
\end{align*}
$$

It remains to verify (11.11). Direct computation reveals

$$
\begin{aligned}
& \varphi(x, y)-\frac{1}{2} \Delta \varphi(x, y)= 1+2 \beta \sigma-A x+A \rho \frac{\sinh \sqrt{2} x \cosh \sqrt{2} y}{\sinh \sqrt{2} \rho \cosh \sqrt{2} \sigma} \\
&-\beta \sigma \frac{\cosh \sqrt{2} y}{\cosh \sqrt{2} \sigma}\left(\frac{\sinh \sqrt{2} x+\sinh \sqrt{2}(\rho-x)}{\sinh \sqrt{2} \rho}\right) \\
& \geqq 1+\beta \sigma-A x+A \rho \frac{\sqrt{2} x}{\sinh \sqrt{2} \rho \cosh \sqrt{2} \sigma} \\
& \geqq 1-\left(1-\frac{\sqrt{2} \rho}{\sinh \sqrt{2} \rho \cosh \sqrt{2} \sigma}\right) A x+\beta \sigma \\
& \geqq 1-\frac{1}{3} c_{0} x+\beta|y| \\
& \geqq H(x, y) \quad \forall(x, y) \in[0, \rho] \times[-\sigma, \sigma],
\end{aligned}
$$

where we have used (11.17), the inequality $a \leqq \sinh a$ for all $a \geqq 0$, (11.16), and (11.10).
On the other hand, (5.9) implies that

$$
\begin{gathered}
w-\frac{1}{2} \Delta w \leqq H \text { on }[0, \rho] \times[-\sigma, \sigma] \\
w \leqq 1 \text { on } \partial([0, \rho] \times[-\sigma, \sigma]) .
\end{gathered}
$$

The maximum principle implies that $w \leqq \varphi$ on $[0, \rho] \times[-\sigma, \sigma]$. In particular, for all $x \in[0, \rho]$,

$$
w(x, 0)-w(0,0)=w(x, 0)-1 \leqq \sigma(x, 0)-1=\varphi(x, 0)-\varphi(0,0),
$$

and thus

$$
\begin{equation*}
0=\frac{\partial}{\partial x} w(0,0) \leqq \frac{\partial}{\partial x} \varphi(0,0) . \tag{11.18}
\end{equation*}
$$

The final step in the proof is to show that $(\partial / \partial x) \varphi(0,0)<0$, so (11.18) is contradicted, as well as the assumption that $H=1$ at some point on $\partial \mathscr{C}$. We compute

$$
\begin{align*}
\frac{\partial}{\partial x} \varphi(0,0)= & \sqrt{2} \beta \sigma\left(2-\frac{1}{\cosh \sqrt{2} \sigma}\right)\left(\frac{\cosh \sqrt{2} \rho-1}{\sinh \sqrt{2} \rho}\right) \\
& -A\left(1-\frac{1}{\cosh \sqrt{2} \sigma}\right)\left(1-\frac{\sqrt{2} \rho}{\sinh \sqrt{2} \rho}\right) . \tag{11.19}
\end{align*}
$$

The first term on the right-hand side of (11.19) is bounded above by

$$
2 \sqrt{2} \beta \sigma\left(\frac{\cosh \sqrt{2} \rho-1}{\sinh \sqrt{2} \rho}\right) \leqq 2 \beta \sigma \rho .
$$

As for the second term, (11.14) and the inequality $\cosh \sqrt{2} \sigma-1 \geqq \sqrt{2} \sigma$ imply that

$$
\begin{aligned}
A(1 & \left.-\frac{1}{\cosh \sqrt{2} \sigma}\right)\left(1-\frac{\sqrt{2} \rho}{\sinh \sqrt{2} \rho}\right) \\
& =\frac{c_{0}}{3}\left[\left(1-\frac{\sqrt{2} \rho}{\sinh \sqrt{2} \rho}\right)^{-1}+(\cosh \sqrt{2} \sigma-1)^{-1}\right]^{-1} \\
& \geqq \frac{c_{0}}{3}\left[\frac{4}{\rho^{2}}+\frac{1}{\sqrt{2} \sigma}\right]^{-1} \\
& =\frac{c_{0}}{3}\left(\frac{\sqrt{2} \rho^{2} \sigma}{\rho^{2}+4 \sqrt{2} \sigma}\right)
\end{aligned}
$$

Therefore,

$$
\frac{\partial}{\partial x} \varphi(0,0) \leqq \sigma\left[2 \beta \rho-\frac{c_{0}}{3}\left(\frac{\sqrt{2} \rho^{2}}{\rho^{2}+4 \sqrt{2} \sigma}\right)\right],
$$

and (11.15) and the choice of $\rho$ show that

$$
\frac{\partial}{\partial x} \varphi(0,0) \leqq \sigma\left[2 \beta \rho-\frac{c_{0}}{3 \sqrt{2}}\right]<0 .
$$

Theorem 11.2. The free boundary $\partial \mathscr{C}$ is of class $C^{1}$, and $w$ has continuous second partial derivatives inside $\mathscr{C}$ up to $\partial \mathscr{C}$.

Proof. Because $T_{1}$ is Lipschitz (Theorem 9.2), for every $\bar{\theta} \in S_{1}$, the point ( $T_{1}(\bar{\theta}), \theta$ ) is a point of positive density with respect to the measure of Remark 7.4 for the set $\left\{(t, \theta) \mid \theta \in S^{1}, t \in\left(T_{1}(\theta), \infty\right)\right\}=\psi\left(\mathbb{R}^{2} \backslash \overline{\mathscr{C}}\right)$. But $\psi$ and $\psi^{-1}$ are locally Lipschitz, so every point of $\partial \mathscr{C}$ is a point of positive Lebesgue density for $\mathbb{R}^{2} \backslash \mathscr{C}$. It follows from Theorem 2 of Caffarelli (1977) that $\partial \mathscr{C}$ is Lipschitz. Caffarelli's Theorem 3 can now be applied (with $v$ in Caffarelli's Assumption (H1) equal to our $1-w$ ), and it yields the desired results.

Corollary 11.3. The boundary $\partial \mathscr{C}$ is of class $C^{2, \alpha}$ for every $\alpha \in(0,1)$.
Proof. In light of Theorems 6.3 and 11.2 and equation (6.4), $D^{2} u$ has a $C^{1}$ extension from $\mathscr{C}$ to $\overline{\mathscr{C}}$. Therefore, $H-w$ appearing on the right-hand side of (11.1) has a $C^{1}$ extension from $\mathscr{C}$ to $\overline{\mathscr{C}}$, and because $\partial \mathscr{C}$ is of class $C^{1}, H-w$ has a $C^{1}$ extension to an open set containing $\overline{\mathscr{C}}$. (In Lemma 12.4, we explain in some detail how to construct a similar extension.) Lemma 11.1 and Theorem 11.2 permit us to apply a theorem of Kinderlehrer \& Nirenberg (1977) (see also Friedman (1982, Thm. 1.1(i), p. 129)), to conclude that $\partial \mathscr{C}$ is of class $C^{1, \alpha}$ for every $\alpha \in(0,1)$.

Now observe that $\nabla w$ solves the problem

$$
\begin{gathered}
\nabla w-\frac{1}{2} \Delta \nabla w=\nabla H \quad \text { in } \mathscr{C}, \\
\nabla w=0 \quad \text { on } \partial \mathscr{C} .
\end{gathered}
$$

Since $\nabla H$ is continuous up to $\partial \mathscr{C}$ and $\partial \mathscr{C}$ is $C^{1, \alpha}$, Theorem 8.34 of Gilbarg and Trudinger (1983, p. 211), implies that $\nabla w$ is of class $C^{1, \alpha}$ on $\mathscr{C}$ up to $\partial \mathscr{C}$. Inserting this regularity into (6.4), we conclude that $D^{2} u$, and hence $H-w$, are of class $C^{1, \alpha}$ on $\mathscr{C}$ up to $\partial \mathscr{C}$. We may again appeal to Friedman (1982, Thm. 1.1) to conclude that $\partial \mathscr{C}$ is of class $C^{2, \alpha}$ for every $\alpha \in(0,1)$.

Remark 11.4. The bootstrapping in Corollary 11.3 can be continued until the regularity of $h$ is exhausted. If, in place of assumption (2.5), we assume that $h \in C_{l o c}^{k, \alpha}$ for some $k \geqq 3$ and $\alpha \in(0,1)$, then the free boundary is of class $C^{k, \alpha}, w$ is of class $C^{k, \alpha}$
inside $\mathscr{C}$ up to $\partial \mathscr{C}$, and $u$ is of class $C^{k+1, \alpha}$ inside $\mathscr{C}$ up to $\partial \mathscr{C}$. This argument uses Ladyzhenskaya and Ural'tseva (1968, Thm. 1.1, p. 107), to wit, if $\nabla H$ is of class $C^{k-3, \alpha}$ up to $\partial \mathscr{C}$ and $\partial \mathscr{C}$ is $C^{k-1, \alpha}$, then $\nabla w$ is of class $C^{k-1, \alpha}$ up to $\partial \mathscr{C}$.

## 12. Construction of the optimal control process.

Definition 12.1. Let $x \in \overline{\mathscr{C}}$ be given. A control process pair $\left\{\left(N_{t}, \zeta_{t}\right) ; 0 \leqq t<\infty\right\}$ as in $\S 2$ is called a solution to the Skorokhod problem for reflected Brownian motion in $\overline{\mathscr{C}}$ starting at $x$ and with reflection direction $-\nabla u$ along $\partial \mathscr{C}$ provided that:
(a) $\zeta$ is continuous,
(b) the process $X$ defined by (2.1) satisfies $X_{t} \in \overline{\mathscr{C}}, 0 \leqq t<\infty$, almost surely and
(c) for all $0 \leqq t<\infty$,

$$
\begin{equation*}
\zeta_{t}=\int_{0}^{t} 1_{\left\{X_{s} \in \partial \mathscr{Q}, N_{s}=-\nabla u\left(X_{s}\right)\right\}} d \zeta_{s} \tag{12.1}
\end{equation*}
$$

For every $x \in \overline{\mathscr{C}}$, the Skorokhod problem of Definition 12.1 has a solution starting at $x$. This follows from Lions and Sznitman (1984, Thm. 4.3), provided that the following three conditions are satisfied:
$(\mathrm{C} 1) \mathscr{C}$ has a $C^{1}$ boundary and satisfies a uniform exterior sphere condition,
(C2) There exists $\sigma>0$ such that $\nabla u(x) \cdot n(x) \geqq \sigma$ for all $x \in \partial \mathscr{C}$, where $n(x)$ is the outward normal vector for $\mathscr{C}$ at $x$,
(C3) $\nabla u$ on $\mathscr{C}$ has an extension to a $C^{2}$ function on an open set containing $\overline{\mathscr{C}}$. Condition (C1) is implied by Corollary 11.3. We establish (C2) and (C3).

Lemma 12.2. Condition (C2) is satisfied.
Proof. Let $x \in \partial \mathscr{C}$ be given. We construct a sequence $\left\{x_{k}\right\}_{k=2}^{\infty}$ in $\mathscr{C}$ such that $x_{k} \rightarrow x$ and $\left(\nabla w\left(x_{k}\right) /\left|\nabla w\left(x_{k}\right)\right|\right) \rightarrow n(x)$. With $K$ as in Lemma 9.1, we have

$$
\frac{\nabla w\left(x_{k}\right) \cdot \nabla u\left(x_{k}\right)}{\left|\nabla w\left(x_{k}\right)\right|} \geqq \frac{2}{K},
$$

and (C2) follows.
As for the construction of $\left\{x_{k}\right\}_{k=2}^{\infty}$, we choose $r>0$ such that $B_{r}(x+r n(x)) \cap \mathscr{C}=\phi$. Define $\bar{x}=x+\frac{1}{2} r n(x)$, so $B_{r / 2}(\bar{x}) \cap \mathscr{C}=\phi$ and $x \in \partial B_{r / 2}(\bar{x})$. Given $k \geqq 2$, we define $\mathscr{C}_{k} \triangleq\left\{x \in \mathbb{R}^{2} ; w(x)<1-(1 / k)\right\}$. We then translate $B_{r / 2}(\bar{x})$ in the $-n(x)$ direction until it touches $\partial \mathscr{C}_{k}$, i.e., we define

$$
\rho_{k}=\sup \left\{\rho>0 ; B_{r / 2}(\bar{x}-\rho n(x)) \cap \mathscr{C}_{k}=\phi\right\}
$$

and we choose $x_{k} \in \overline{B_{r / 2}\left(\bar{x}-\rho_{k} n(x)\right)} \cap \partial \mathscr{C}_{k}$. Then $B_{r / 2}\left(\bar{x}-\rho_{k} n(x)\right)$ is an exterior sphere for $\partial \mathscr{C}_{k}$ at $x_{k}$, so the outward normal to $\mathscr{C}_{k}$ at $x_{k}$ is

$$
\frac{\nabla w\left(x_{k}\right)}{\left|\nabla w\left(x_{k}\right)\right|}=\frac{\bar{x}-\rho_{k} n(x)-x_{k}}{\left|\bar{x}-\rho_{k} n(x)-x_{k}\right|} .
$$

As $k \rightarrow \infty$, we have $x_{k} \rightarrow x$ and $\rho_{k} \rightarrow 0$, so $\left(\nabla w\left(x_{k}\right) /\left|\nabla w\left(x_{k}\right)\right|\right) \rightarrow n(x)$.
Lemma 12.3. Condition (C3) is satisfied.
Proof. Given $\varepsilon>0$, we can find a finite set of open discs $\left\{B_{k}\right\}_{k=1}^{n}$, each with radius $\varepsilon$, such that $\overline{\mathscr{C}} \subset \bigcup_{k=1}^{n} B_{k}$, and we can find $C^{\infty}$ functions $\gamma_{k}: \mathbb{R}^{2} \rightarrow[0,1]$ such that $\overline{\operatorname{supp} \gamma_{k}} \subset B_{k}$ for every $k$ and $\sum_{k=1}^{n} \gamma_{k}=1$ on $\mathscr{C}$. We can decompose $\nabla u$ on $\mathscr{C}$ as $\sum_{k=1}^{n} \gamma_{k} \nabla u$, so it suffices to show that each $v_{k} \triangleq \gamma_{k} \nabla u$ has a $C^{2}$ extension from $B_{k} \cap \mathscr{C}$ to $B_{k}$. For sufficiently small $\varepsilon>0$, in each $B_{k}$ there is a $C^{2}$ change of coordinates which results in $B_{k} \cap \overline{\mathscr{C}} \subset\{(x, y) \mid x \leqq 0\}$ and $B_{k} \backslash \overline{\mathscr{C}} \subset\{(x, y) \mid x>0\}$. Now $v_{k}$ has a $C^{2}$
extension from $B_{k} \cap \mathscr{C}$ to $B_{k} \cap \overline{\mathscr{C}}$ (proof of Corollary 11.3), and taking $v_{k}$ to be zero on $\{(x, y) \mid x \leqq 0\} \backslash\left(B_{k} \cap \overline{\mathscr{C}}\right)$, we have a $C^{2}$ function on the closed left half-plane. For $x>0, y \in \mathbb{R}$, define

$$
v_{k}(x, y)=3 v_{k}(0, y)-3 v_{k}(-x, y)+v_{k}(-2 x, y) .
$$

It is easy to check that this extended $v_{k}$ is $C^{2}$ on all of $\mathbb{R}^{2}$.
Theorem 12.4. Let $x \in \mathbb{R}^{2}$ be given. If $x \in \overline{\mathscr{C}}$, then the solution to the Skorokhod problem of Definition 12.1 is an optimal control process pair for the singular stochastic control problem with initial condition $x$ posed in § 2 . If $x \notin \mathscr{C}$, then there exists a unique pair $(t, \theta) \in[0, \infty) \times S_{1}$ such that $x=\psi(t, \theta)$. Define $\hat{x} \triangleq \psi\left(T_{1}(\theta), \theta\right)$ and let $(\hat{N}, \hat{\zeta})$ be a solution to the Skorokhod problem starting at $\hat{x}$. Then $(N, \zeta)$ is optimal for the control problem with initial condition $x$, where

$$
\begin{align*}
& N_{t} \stackrel{\Delta}{=} \begin{cases}-\nabla u(\bar{x}) & \text { if } t=0, \\
\hat{N}_{t} & \text { if } t>0,\end{cases}  \tag{12.2}\\
& \zeta_{t} \stackrel{\Delta}{=} \begin{cases}0 & \text { if } t=0 \\
\hat{\zeta}_{t}+|x-\hat{x}| & \text { if } t>0\end{cases} \tag{12.3}
\end{align*}
$$

In either case, we have that $u(x)=V(x)$, where $u$ is the solution to the HJB equation (3.1) (see Theorem 4.6), and $V$ is the value function for the control problem defined by (2.10).

Proof. The theorem follows immediately from Theorem 3.1 once we observe that in the case $x \notin \overline{\mathscr{C}}$, Lemma 8.1 implies that for all $s \geqq T_{1}(\theta)$,

$$
\begin{aligned}
\nabla u(\psi(s, \theta)) & =\nabla u(\hat{x})+\int_{T_{1}(\theta)}^{s} \frac{d}{d \tau} \nabla u(\psi(\tau, \theta)) d \tau \\
& =\nabla u(\hat{x})+\int_{T_{1}(\theta)}^{s} D^{2} u(\psi(\tau, \theta)) \nabla u(\psi(\tau, \theta)) d \tau \\
& =\nabla u(\hat{x}) .
\end{aligned}
$$

Thus, when $x \notin \mathscr{C}$, the control process pair $(N, \zeta)$ of (12.2), (12.3) causes the state to jump from $X_{0}=x$ to $X_{0^{+}}=\hat{x}$ and $u(x)-u(\hat{x})=|x-\hat{x}|$. After this initial jump, the state is kept inside $\overline{\mathscr{C}}$ by reflection in the $-\nabla u$ direction along $\partial \mathscr{C}$.
13. Appendix. Proof of Lemma 4.1. For $\varepsilon \in(0,1), R>0$, denote by $u^{\varepsilon, R}$ the solution to

$$
\begin{gather*}
u^{\varepsilon, R}-\Delta u^{\varepsilon, R}+\beta_{\varepsilon}\left(\left|\nabla u^{\varepsilon, R}\right|^{2}\right)=h \quad \text { on } B_{R}(0),  \tag{13.1}\\
u^{\varepsilon, R}=0 \quad \text { on } \partial B_{R}(0) . \tag{13.2}
\end{gather*}
$$

The existence of $u^{\varepsilon, R} \in C^{2}\left(\overline{\left.B_{R}(0)\right)}\right.$ follows from Ladyzhenskaya and Ural'tseva (1968, Thm. 4.8.3, p. 301); uniqueness follows from the following lemma.

Lemma 13.1. Suppose that $\varphi$ is a subsolution and $\psi$ is a supersolution to (13.1). Then for all $x \in R_{\mathbb{K}}(0)$ :

$$
\begin{equation*}
\varphi(x)-\psi(x) \leqq \sup _{y \in \partial B_{R}(0)}[\varphi(y)-\psi(y)]^{+} . \tag{13.3}
\end{equation*}
$$

Proof. If $\varphi-\psi$ attains its maximum over $\overline{B_{R}(0)}$ at an interior point $x^{*}$, then $\nabla \varphi\left(x^{*}\right)=\nabla \psi\left(x^{*}\right)$ and $0 \geqq \Delta \varphi\left(x^{*}\right)-\Delta \psi\left(x^{*}\right)=\varphi\left(x^{*}\right)-\psi\left(x^{*}\right)$.

Lemma 13.2. Let $q>0$ be as in (2.6). There exists a constant $C_{1}>0$, independent of $\varepsilon$ and $R$, such that

$$
\begin{equation*}
0 \leqq u^{\varepsilon, R}(x) \leqq C_{1}\left(1+|x|^{q}\right) \quad \forall x \in B_{R}(0) . \tag{13.4}
\end{equation*}
$$

Proof. To prove the nonnegativity of $u^{\varepsilon, R}$, take $\varphi \equiv 0$ and $\psi=u^{\varepsilon, R}$ in Lemma 13.1. To obtain the upper bound on $u^{\varepsilon, R}$, take $\varphi=u^{\varepsilon, R}$ and

$$
\psi(x)=E \int_{0}^{\tau_{x}} e^{-t} h\left(x+\sqrt{2} W_{t}\right) d t
$$

where $\tau_{x} \triangleq \inf \left\{t \geqq 0 ;\left|x+\sqrt{2} W_{t}\right| \geqq R\right\}$. Then $\psi-\Delta \psi=h$ on $B_{R}(0), \psi=0$ on $\partial B_{R}(0)$, and Lemma 13.1 and (2.6) imply that

$$
\begin{aligned}
u^{\varepsilon, R}(x) & \leqq E \int_{0}^{\tau_{x}} e^{-t} h\left(x+\sqrt{2} W_{t}\right) d t \\
& \leqq E \int_{0}^{\infty} e^{-t} h\left(x+\sqrt{2} W_{t}\right) d t \\
& \leqq 2^{q} C_{0} E \int_{0}^{\infty} e^{-t}\left(|x|^{q}+\left|\sqrt{2} W_{t}\right|^{q}\right) d t \\
& \leqq C_{1}\left(1+|x|^{q}\right)
\end{aligned}
$$

Lemma 13.3. There exist constants $C>0$ and $p>0$, independent of $\varepsilon$ and $R$, such that

$$
\begin{equation*}
\max _{x \in \partial B_{R}(0)}\left|\nabla u^{\varepsilon, R}(x)\right| \leqq C\left(1+R^{p}\right) \quad \forall \varepsilon \in(0,1), \quad R>0 \tag{13.5}
\end{equation*}
$$

Proof. Let $N$ be a positive integer greater than $q / 2$, and define $g, B:[0, \infty) \rightarrow \mathbb{R}$ by

$$
g(r)=\sum_{k=0}^{N} \frac{r^{2 k}}{4^{k}(k!)^{2}}, \quad B(r)=\sum_{k=0}^{\infty} \frac{r^{2 k}}{4^{k}(k!)^{2}}
$$

Then

$$
g(r)-\frac{1}{r} g^{\prime}(r)-g^{\prime \prime}(r)=\frac{r^{2 N}}{4^{N}(N!)^{2}}
$$

and

$$
\begin{equation*}
B(r)-\frac{1}{r} B^{\prime}(r)-B^{\prime \prime}(r)=0 \tag{13.6}
\end{equation*}
$$

For $R>0$, define

$$
\begin{aligned}
\psi_{R}(x)= & 2 C_{0}+C_{0} 4^{N}(N!)^{2} g(|x|) \\
& -\left[2 C_{0}+C_{0} 4^{N}(N!)^{2} g(R)\right] \frac{B(|x|)}{B(R)} \quad \forall x \in \mathbb{R}
\end{aligned}
$$

so

$$
\begin{gathered}
\psi_{R}(x)-\Delta \psi_{R}(x)=C_{0}\left(2+|x|^{2 N}\right) \geqq h(x) \quad \forall x \in B_{R}(0) \\
\psi_{R}(x)=0 \quad \forall x \in \partial B_{R}(0)
\end{gathered}
$$

It follows from Lemma 13.1 that $u^{\varepsilon, R} \leqq \psi_{R}$ on $B_{R}(0)$, and because these functions agree on $B_{R}(0)$ and because $\nabla u^{\varepsilon, R}$ on $\partial B_{R}(0)$ must point inward, where $u^{\varepsilon, R}$ is nonnegative, we have

$$
\left|\nabla u^{\varepsilon, R}(x)\right| \leqq\left|\nabla \psi_{R}(x)\right| \quad \forall x \in \partial B_{R}(0)
$$

But on $\partial B_{R}(0)$,

$$
\left|\nabla \psi_{R}(x)\right|=\left|C_{0} 4^{N}(N!)^{2} g^{\prime}(R)-\left[2 C_{0}+C_{0} 4^{N}(N!)^{2} g(R)\right] \frac{B^{\prime}(R)}{B(R)}\right| .
$$

Equation (13.6) and the nonnegativity of $B^{\prime \prime}$ show that

$$
0 \leqq B^{\prime}(r) \leqq r B(r) \quad \forall r>0,
$$

so we may bound the growth of $\max _{x \in \partial B_{R}(0)}\left|\nabla \psi_{R}(x)\right|$ by a constant times ( $1+$ $R^{2 N+1}$ ).

Lemma 13.4. There exist constants $C>0, p>0, \lambda>0$, independent of $\varepsilon$ and $R$, such that

$$
\begin{equation*}
\left|\nabla u^{\varepsilon, R}(x)\right| \leqq \lambda u^{\varepsilon, R}(x)+C|x|^{p}+C \quad \forall x \in B_{R}(0), \quad \varepsilon \in(0,1), \quad R>0 . \tag{13.7}
\end{equation*}
$$

Proof. With $C \geqq 1$ and $p \geqq 2$ satisfying (13.5), and $C_{0}$ as in (2.7), define $\lambda \triangleq$ $\max \left\{2, C_{0}\right\}, B \triangleq=C p^{p}+C_{0}$, and consider the auxiliary function

$$
\varphi(x) \stackrel{\Delta}{\Delta} \nabla u^{\varepsilon, R}(x) \cdot \nu-\lambda u^{\varepsilon, R}(x)-C|x|^{p}-B,
$$

where $\varepsilon \in(0,1), R>0$ are fixed, and $\nu$ is a fixed unit vector. It suffices to show that $\varphi(x) \leqq 0$ for all $x \in B_{R}(0)$, so let $x^{*}$ be a point at which $\varphi$ attains its maximum over $\overline{B_{R}(0)}$. If $x^{*} \in \partial B_{R}(0)$, then (13.5) implies that $\varphi\left(x^{*}\right) \leqq 0$. Thus, we need only consider the case that $x^{*} \in B_{R}(0)$, for which we have

$$
0 \geqq \Delta \varphi\left(x^{*}\right)=\Delta \nabla u^{\varepsilon, R}\left(x^{*}\right) \cdot \nu-\lambda \Delta u^{\varepsilon, R}\left(x^{*}\right)-C p^{2}\left|x^{*}\right|^{p-2} .
$$

Using (13.1), we may rewrite this as

$$
\begin{align*}
0 \geqq & \nabla u^{\varepsilon, R}\left(x^{*}\right) \cdot \nu+2 \beta_{\varepsilon}^{\prime}\left(r^{*}\right) \nabla\left[\nabla u^{\varepsilon, R}\left(x^{*}\right) \cdot \nu\right] \cdot \nabla u^{\varepsilon, R}\left(x^{*}\right) \\
& -\nabla h\left(x^{*}\right) \cdot \nu-\lambda u^{\varepsilon, R}\left(x^{*}\right)-\lambda \beta_{\varepsilon}\left(r^{*}\right)+\lambda h\left(x^{*}\right)-C p^{2}\left|x^{*}\right|^{p-2}, \tag{13.8}
\end{align*}
$$

where $r^{*}$ denotes $\left|\nabla u^{\varepsilon, R}\left(x^{*}\right)\right|^{2}$. Because of (2.7),

$$
|\nabla h(x)| \leqq C_{0}+\lambda h(x) \quad \forall x \in \mathbb{R} .
$$

Furthermore,

$$
\begin{aligned}
C p^{2}|x|^{p-2} & \leqq C p^{p}\left|\frac{x}{p}\right|^{p-2} \\
& \leqq C|x|^{p}+C p^{p} \quad \forall x \in \mathbb{R} .
\end{aligned}
$$

Adding these two inequalities, we see that

$$
\left|\nabla h\left(x^{*}\right)\right|+C p^{2}\left|x^{*}\right|^{p-2} \leqq \lambda h\left(x^{*}\right)+C\left|x^{*}\right|^{p}+B .
$$

Substitution into (13.8) yields

$$
\begin{equation*}
0 \geqq \varphi\left(x^{*}\right)+2 \beta_{\varepsilon}^{\prime}\left(r^{*}\right) \nabla\left[\nabla u^{\varepsilon, R}\left(x^{*}\right) \cdot \nu\right] \cdot \nabla u^{\varepsilon, R}\left(x^{*}\right)-\lambda \beta_{\varepsilon}\left(r^{*}\right) . \tag{13.9}
\end{equation*}
$$

Because $\nabla \varphi\left(x^{*}\right)=0$, we also have

$$
\begin{align*}
0= & \nabla \varphi\left(x^{*}\right) \cdot \nabla u^{\varepsilon, R}\left(x^{*}\right) \\
= & \nabla\left[\nabla u^{\varepsilon, R}\left(x^{*}\right) \cdot \nu\right] \cdot \nabla \boldsymbol{u}^{\varepsilon, R}\left(x^{*}\right)-\lambda r^{*}  \tag{13.10}\\
& -C p\left|x^{*}\right|^{p-2} x^{*} \cdot \nabla u^{\varepsilon, R}\left(x^{*}\right) .
\end{align*}
$$

Substitution of (13.10) into (13.9) results in the inequality

$$
\varphi\left(x^{*}\right) \leqq \lambda\left[\beta_{\varepsilon}\left(r^{*}\right)-2 \beta_{\varepsilon}^{\prime}\left(r^{*}\right) r^{*}\right]-2 C p\left|x^{*}\right|^{p-2} \beta_{\varepsilon}^{\prime}\left(r^{*}\right) x^{*} \cdot \nabla u^{\varepsilon, R}\left(x^{*}\right) .
$$

Let us assume that $\varphi\left(x^{*}\right)>0$. Then

$$
\sqrt{r^{*}} \geqq \nabla u^{\varepsilon, R}\left(x^{*}\right) \cdot \nu \geqq B \geqq 2,
$$

so $r^{*} \geqq 4$ and for all $\varepsilon \in(0,1)$,

$$
\beta_{\varepsilon}\left(r^{*}\right)=\frac{r^{*}-1}{\varepsilon}-1, \quad \beta_{\varepsilon}^{\prime}\left(r^{*}\right)=\frac{1}{\varepsilon} .
$$

Consequently,

$$
\begin{aligned}
0<\varphi\left(x^{*}\right) & \leqq-\frac{\lambda}{\varepsilon}\left(\left|\nabla u^{\varepsilon, R}\left(x^{*}\right)\right|^{2}+1+\varepsilon\right)-\frac{2 C p}{\varepsilon}\left|x^{*}\right|^{p-2} x^{*} \cdot \nabla u^{\varepsilon, R}\left(x^{*}\right) \\
& \leqq-\frac{\lambda}{\varepsilon}\left(\left|\nabla u^{\varepsilon, R}\left(x^{*}\right)\right|^{2}+1+\varepsilon\right)+\frac{2 C p}{\varepsilon}\left|x^{*}\right|^{p-1}\left|\nabla u^{\varepsilon, R}\left(x^{*}\right)\right|,
\end{aligned}
$$

which implies that

$$
\left|\nabla u^{\varepsilon, R}\left(x^{*}\right)\right| \leqq \frac{2 C p}{\lambda}\left|x^{*}\right|^{p-1} \leqq C p^{p}\left|\frac{x^{*}}{p}\right|^{p-1} \leqq C\left|x^{*}\right|^{p}+B .
$$

This inequality contradicts the assumption that $\varphi\left(x^{*}\right)>0$.
Lemma 13.5. For each $\varepsilon \in(0,1)$, there is an increasing sequence $\left\{R_{n}\right\}_{n=1}^{\infty}$ of positive numbers converging to infinity and a function $u^{\varepsilon} \in C^{2}\left(\mathbb{R}^{2}\right)$ such that $\left\{u^{\varepsilon, R_{n}}\right\}_{n=1}^{\infty}$ and $\left\{\nabla u^{\varepsilon, R_{n}}\right\}_{n=1}^{\infty}$ converge uniformly to $u^{\varepsilon}$ and $\nabla u^{\varepsilon}$, respectively, on compact sets. Furthermore, $u^{\varepsilon}$ is a solution to (4.3) and satisfies (4.4), (4.5), with $C_{1}$ and $p$ independent of $\varepsilon$.

Proof. Let $\varepsilon \in(0,1)$ be fixed and let $r>0$ be given. Then $u^{\varepsilon, R}$ and $\nabla u^{\varepsilon, R}$ are bounded on $B_{2 r}(0)$, uniformly in $R$ and $\varepsilon$ (Lemmas 13.2, 13.4). Elliptic regularity implies Hölder continuity of $\nabla u^{\varepsilon, R}$ on $B_{r}(0)$, uniformly in $R \in[2 r, \infty)$ (Gilbarg and Trudinger, Thm. 3.9, p. 41), and by the Arzela-Ascoli Theorem, we can find a sequence $\left\{R_{n}\right\}_{n=1}^{\infty}$ along which $\left\{u^{\varepsilon, R_{n}}\right\}_{n=1}^{\infty}$ and $\left\{\nabla u^{\varepsilon, R_{n}}\right\}_{n=1}^{\infty}$ converge uniformly on $B_{r}(0)$. Indeed, by diagonalization we can select $\left\{R_{n}\right\}_{n=1}^{\infty}$ so that $\left\{u^{\varepsilon, R_{n}}\right\}_{n=1}^{\infty}$ and $\left\{\nabla u^{\varepsilon, R_{n}}\right\}_{n=1}^{\infty}$ converge uniformly on compact sets to limits $u^{\varepsilon}$ and $\nabla u^{\varepsilon}$, respectively, where $u^{\varepsilon} \in C^{1, \alpha}$ for all $\alpha \in(0,1)$. Passing to the limit in (13.1), we see that $\Delta u^{\varepsilon}$ exists in the distributional sense and is equal to $u^{\varepsilon}+\beta_{\varepsilon}\left(\left|\nabla u^{\varepsilon}\right|^{2}\right)-h$, which is a $C^{0, \alpha}$ function. Elliptic regularity implies that $D^{2} u^{\varepsilon}$ in fact exists in the classical sense and $u^{\varepsilon}$ is $C^{2, \alpha}$. (By bootstrapping, we could conclude that $u^{\varepsilon}$ is $C^{4, \alpha}$ because $h$ is $C^{2,1}$.)

The convexity of $u^{\varepsilon}$ will be established by representing $u^{\varepsilon}$ as the value function of a stochastic control problem with convex cost functions. With $\beta_{\varepsilon}$ defined by (4.2), we define a convex function $g_{\varepsilon}: \mathbb{R}^{2} \rightarrow \mathbb{R}$ and its (convex) Legendre transform $l_{\varepsilon}: \mathbb{R}^{2} \rightarrow \mathbb{R}$ by

$$
\begin{equation*}
g_{\varepsilon}(x) \stackrel{\Delta}{=} \beta_{\varepsilon}\left(|x|^{2}\right), \quad l_{\varepsilon}(y) \stackrel{\Delta}{\Delta} \sup _{x \in \mathbb{R}^{2}}\left\{x \cdot y-g_{\varepsilon}(x)\right\} . \tag{13.11}
\end{equation*}
$$

For every $y \in \mathbb{R}^{2}$,

$$
\begin{equation*}
l_{\varepsilon}(y) \geqq \frac{\varepsilon}{2}|y|^{2}-g_{\varepsilon}\left(\frac{\varepsilon}{2} y\right) \geqq \frac{\varepsilon}{4}|y|^{2} . \tag{13.12}
\end{equation*}
$$

Furthermore, the supremum in the definition of $l_{\varepsilon}$ is attained if $x$ is related to $y$ by $y=2 \beta_{\varepsilon}^{\prime}\left(|x|^{2}\right) x$, i.e.,

$$
\begin{equation*}
l_{\varepsilon}\left(2 \beta_{\varepsilon}^{\prime}\left(|x|^{2}\right) x\right)=2 \beta_{\varepsilon}^{\prime}\left(|x|^{2}\right)|x|^{2}-\beta_{\varepsilon}\left(|x|^{2}\right) \quad \forall x \in \mathbb{R}^{2} . \tag{13.13}
\end{equation*}
$$

A control process is any two-dimensional, absolutely continuous process $\eta$ adapted to the Brownian motion $\left\{W_{t}, \mathscr{F}_{t} ; 0 \leqq t<\infty\right\}$ and satisfying $\eta_{0}=0$ almost sarely. Given an initial state $x \in \mathbb{R}^{2}$, the corresponding state process is

$$
\begin{equation*}
Y_{t} \triangleq x+\sqrt{2} W_{t}-\eta_{t} . \tag{13.14}
\end{equation*}
$$

For each $R>0$, we define the cost corresponding to $\eta$ up to the exit from $B_{R}(0)$ as

$$
v_{\eta}^{\varepsilon, R}(x) \triangleq E^{x} \int_{0}^{T_{R}} e^{-t}\left[h\left(Y_{t}\right)+l_{\varepsilon}\left(\dot{\eta}_{t}\right)\right] d t
$$

where $\tau_{R} \triangleq \inf \left\{t \geqq 0 ;\left|Y_{t}\right| \geqq R\right\}$, and $\dot{\eta}_{t}=(d / d t) \eta_{t}$. The value function up to the exit from $B_{R}(0)$ is

$$
v^{\varepsilon, R}(x) \xlongequal[=]{\underline{\Delta} \inf _{\eta} v_{\eta}^{\varepsilon, R}(x) .}
$$

It is clear that $v^{e, R}(x)$ is nondecreasing in $R$, and

$$
\begin{equation*}
\lim _{R \rightarrow \infty} v^{\varepsilon, R}(x) \leqq v^{\varepsilon}(x) \triangleq \inf _{\eta} E^{x} \int_{0}^{\infty} e^{-t}\left[h\left(Y_{t}\right)+l_{\varepsilon}\left(\dot{\eta}_{t}\right)\right] d t \tag{13.15}
\end{equation*}
$$

where $v^{\varepsilon}$ is the value function for a control problem on $\mathbb{R}^{2}$.
Lemma 13.6. For each $\varepsilon \in(0,1), R>0$, the solution $u^{\varepsilon, R}$ of (13.1), (13.2) agrees with $v^{\varepsilon, R}$ on $B_{R}(0)$.

Proof. Itô's lemma implies that for a given control process $\eta, x \in B_{R}(0)$ and $t \geqq 0$ :

$$
\begin{align*}
E^{x} e^{-t \wedge \tau_{R}} u^{\varepsilon, R}\left(Y_{t \wedge \tau_{R}}\right)= & u^{\varepsilon, R}(x)+E^{x} \int_{0}^{t \wedge \tau_{R}} e^{-s}\left[\beta_{\varepsilon}\left(\left|\nabla u^{\varepsilon, R}\left(Y_{s}\right)\right|^{2}\right)\right. \\
& \left.\left.-h\left(Y_{s}\right)-\nabla u^{\varepsilon, R}\left(Y_{s}\right) \cdot \eta_{s}\right)\right] d s  \tag{13.16}\\
\geqq & u^{\varepsilon, R}(x)-E^{x} \int_{0}^{t \wedge \tau_{R}} e^{-s}\left[h\left(Y_{s}\right)+l_{\varepsilon}\left(\dot{\eta}_{s}\right)\right] d s .
\end{align*}
$$

Letting $t \rightarrow \infty$, we see that $v_{\eta}^{\varepsilon, R}(x) \geqq u^{\varepsilon, r}(x)$ for all all $\eta$, so $v^{\varepsilon, R}(x) \geqq u^{\varepsilon, R}(x)$. However, if $Y^{R}$ is the solution to

$$
Y_{t}^{R}=x-\int_{0}^{t} 2 \beta_{\varepsilon}^{\prime}\left(\left|\nabla u^{\varepsilon, R}\left(Y_{s}^{R}\right)\right|^{2}\right) \nabla u^{\varepsilon, R}\left(Y_{s}^{R}\right) d s+\sqrt{2} W_{t}, \quad 0 \leqq t \leqq \tau_{R}
$$

then the corresponding control process satisfies

$$
\dot{\eta}_{t}^{R}=2 \beta_{\varepsilon}^{\prime}\left(\left|\nabla u^{\varepsilon, R}\left(Y_{t}^{R}\right)\right|^{2}\right) \nabla u^{\varepsilon, R}\left(Y_{t}^{R}\right), \quad 0 \leqq t \leqq \tau_{R},
$$

and equality holds in (13.16) because of (13.13), i.e.,

$$
v_{\eta}^{\varepsilon, R}(x)=u^{\varepsilon, R}(x) \leqq v^{\varepsilon, R}(x),
$$

and thus $u^{\varepsilon, R}(x)=v^{\varepsilon, R}(x)$.
Lemma 13.7. For each $\varepsilon \in(0,1)$, the function $u^{\varepsilon}$ constructed in Lemma 13.5 agrees with the value function $v^{\varepsilon}$ defined in (13.15).

Proof. We have immediately from (13.15) and Lemma 13.6 that $u^{\varepsilon} \leqq v^{\varepsilon}$. For the reverse inequality, let $x \in \mathbb{R}^{2}$ be given and define $Y^{\infty}$ (up to the time of a possible explosion) by

$$
Y_{t}^{\infty}=x-\int_{0}^{t} 2 \beta_{\varepsilon}^{\prime}\left(\left|\nabla u^{\varepsilon}\left(Y_{t}^{\infty}\right)\right|^{2}\right) \nabla u^{\varepsilon}\left(Y_{t}^{\infty}\right) d t+\sqrt{2} W_{t}
$$

Imitating (13.16), we have from Itô's lemma and (13.13) that for every $R>0$,

$$
\begin{equation*}
u^{\varepsilon}(x)=E^{x} \int_{0}^{t \wedge \tau_{R}} e^{-s}\left[h\left(Y_{s}^{\infty}\right)+l_{\varepsilon}\left(\dot{\eta}_{s}^{\infty}\right)\right] d s+E^{x} e^{-t \wedge \tau_{R}} u^{\varepsilon}\left(Y_{t \wedge \tau_{R}}^{\infty}\right) \tag{13.17}
\end{equation*}
$$

where

$$
\dot{\eta}_{t}^{\infty} \triangleq 2 \beta_{\varepsilon}^{\prime}\left(\left|\nabla u^{\varepsilon}\left(Y_{t}^{\infty}\right)\right|^{2}\right) \nabla u^{\varepsilon}\left(Y_{t}^{\infty}\right), \quad \tau_{R} \triangleq \inf \left\{t \geqq 0 ;\left|Y_{t}^{\infty}\right| \geqq R\right\}
$$

Deleting the (nonnegative) second term on the right-hand side of (13.17) and letting $R \rightarrow \infty, t \rightarrow \infty$, we obtain

$$
\begin{equation*}
u^{\varepsilon}(x) \geqq E^{x} \int_{0}^{\tau_{\infty}} e^{-s}\left[h\left(Y_{s}^{\infty}\right)+l_{\varepsilon}\left(\dot{\eta}_{s}^{\infty}\right)\right] d s, \tag{13.18}
\end{equation*}
$$

where $\tau_{\infty} \triangleq \lim _{R \rightarrow \infty} \tau_{R}$ is finite if and only if $Y^{\infty}$ explodes in finite time.
To see that $\tau_{\infty}=\infty$ almost surely, observe that for all $t \geqq 0, R>0$,

$$
\left|\eta_{t \wedge \tau_{R}}^{\infty}\right|^{2}=2 \int_{0}^{t \wedge \tau_{R}} \eta_{s}^{\infty} \cdot \dot{\eta}_{s}^{\infty} d s \leqq \int_{0}^{t \wedge \tau_{R}}\left|\dot{\eta}_{s}^{\infty}\right|^{2} d s+\int_{0}^{t \wedge \tau_{R}}\left|\eta_{s \wedge \tau_{R}}^{\infty}\right|^{2} d s .
$$

Gronwall's inequality implies

$$
\max _{0 \leqq s \leqq t \wedge \tau_{R}}\left|\eta_{s}^{\infty}\right|^{2} \leqq e^{t} \int_{0}^{t \wedge \tau_{R}}\left|\dot{\eta}_{s}^{\infty}\right|^{2} d s \leqq \frac{4 e^{t}}{\varepsilon} \int_{0}^{t \wedge \tau_{R}} l_{\varepsilon}\left(\dot{\eta}_{s}^{\infty}\right) d s
$$

where we have used (13.12). Letting $R \rightarrow \infty$ and taking expectations, we conclude that

$$
E^{x} \sup _{0 \leqq s<t \wedge \tau_{\infty}}\left|\eta_{s}^{\infty}\right|^{2} \leqq E^{x} \frac{4 e^{t}}{\varepsilon} \int_{0}^{t \wedge \tau_{\infty}} l_{\varepsilon}\left(\dot{\eta}_{s}^{\infty}\right) d s \leqq \frac{4 e^{2 t}}{\varepsilon} u^{\varepsilon}(x)<\infty, \quad \forall t \geqq 0 .
$$

But

$$
\sup _{0 \leqq s<t \wedge \tau_{\infty}}\left|Y_{s}^{\infty}\right| \leqq x+\sup _{0 \leqq s<t \wedge \tau_{\infty}}\left|\eta_{s}^{\infty}\right|+\sqrt{2} \max _{0 \leqq s \leqq t}\left|W_{s}\right|
$$

and $\sup _{0 \leqq s<t \wedge \tau_{\infty}}\left|Y_{s}^{\infty}\right|<\infty$ on $\left\{\tau_{\infty} \leqq t\right\}$. It follows that $P^{*}\left\{\tau_{\infty} \leqq t\right\}=0$ for all $t \geqq 0$. Inequality (13.18) can now be restated as

$$
u^{\varepsilon}(x) \geqq E^{x} \int_{0}^{\infty} e^{-s}\left[h\left(Y_{s}^{\infty}\right)+l_{\varepsilon}\left(\dot{\eta}_{s}^{\infty}\right)\right] d s \geqq v^{\varepsilon}(x)
$$

Corollary 13.8. For each $\varepsilon \in(0,1)$, the function $u^{\varepsilon}$ constructed in Lemma 13.5 is convex.

Corollary 13.9. For each $\varepsilon \in(0,1], \lim _{|x| \rightarrow \infty} u^{\varepsilon}(x)=\infty$.
Proof. In light of (2.8), (2.9), (13.12), and (13.15), we have

$$
u^{\varepsilon}(x) \geqq \inf _{\eta} E^{x} \int_{0}^{\infty} e^{-t}\left[\frac{c_{0}}{2}\left|Y_{t}\right|^{2}+\frac{\varepsilon}{4}\left|\dot{\eta}_{t}\right|^{2}\right] d t .
$$

But the right-hand side is the value associated with a linear-quadratic-Gaussian problem, which is easily computed to be $\frac{1}{2} \alpha|x|^{2}+2 \alpha$, where $\alpha$ is the positive root of the quadratic equation $(2 / \varepsilon) \alpha^{2}+\alpha-c_{0}=0$.

Lemma 13.10. There is a constant $C_{2}$, independent of $\varepsilon$, such that for every $\varepsilon \in(0,1)$, the function $u^{\varepsilon}$ constructed in Lemma 13.5 satisfies (4.6).

Proof. Let $\nu$ be a unit vector and define $u_{\nu \nu}^{\varepsilon} \triangleq\left(D^{2} u\right) \nu \cdot \nu$. It suffices to produce a constant $C_{2}$, independent of $\varepsilon$ and $\nu$, such that

$$
u_{\nu \nu}^{\varepsilon} \leqq C_{2}\left(1+u^{\varepsilon}\right) .
$$

We begin by differentiating (4.3) to obtain

$$
\begin{gather*}
h_{\nu \nu}=u_{\nu \nu}^{\varepsilon}-\Delta u_{\nu \nu}^{\varepsilon}+2 \beta_{\varepsilon}^{\prime}\left(\left|\nabla u^{\varepsilon}\right|^{2}\right)\left(\nabla u_{\nu \nu}^{\varepsilon} \cdot \nabla u^{\varepsilon}+\left|\left(D^{2} u^{\varepsilon}\right) \nu\right|^{2}\right) \\
+4 \beta_{\varepsilon}^{\prime \prime}\left(\left|\nabla u^{\varepsilon}\right|^{2}\right)\left(D^{2} u^{\varepsilon} \nabla u^{\varepsilon} \cdot \nu\right)^{2}  \tag{13.19}\\
\geqq u_{\nu \nu}^{\varepsilon}-\Delta u_{\nu \nu}^{\varepsilon}+2 \beta_{\varepsilon}^{\prime}\left(\left|\nabla u^{\varepsilon}\right|^{2}\right) \nabla u_{\nu \nu}^{\varepsilon} \cdot \nabla u^{\varepsilon} .
\end{gather*}
$$

Let $x^{\varepsilon}$ be a minimizing point for $u^{\varepsilon}$, choose $p>0$ satisfying (4.4), (4.5), choose $C_{0}>0$ to satisfy (2.8), let $\delta>0$ be given, and define the auxiliary function

$$
\varphi_{\delta}(x)=u_{\nu \nu}^{\varepsilon}(x)-C_{0} u^{\varepsilon}(x)-\delta\left|x-x^{\varepsilon}\right|^{p+2} .
$$

This function attains its maximum at some point $y^{\delta}$, where we have

$$
\begin{gather*}
0=\nabla \varphi_{\delta}\left(y^{\delta}\right)=\nabla u_{\nu \nu}^{\varepsilon}\left(y^{\delta}\right)-C_{0} \nabla u^{\varepsilon}\left(y^{\delta}\right)-\delta(p+2)\left|y^{\delta}-x^{\varepsilon}\right|^{p}\left(y^{\delta}-x^{\varepsilon}\right),  \tag{13.20}\\
0 \geqq \Delta \varphi_{\delta}\left(y^{\delta}\right)=\Delta u_{\nu \nu}^{\varepsilon}\left(y^{\delta}\right)-C_{0} \Delta u^{\varepsilon}\left(y^{\delta}\right)-\delta(p+2)^{2}\left|y^{\delta}-x^{\varepsilon}\right|^{p} . \tag{13.21}
\end{gather*}
$$

Substituting (4.3) into (13.21) and using (13.19), we obtain

$$
\begin{align*}
0 \geqq & u_{\nu \nu}^{\varepsilon}\left(y^{\delta}\right)+2 \beta_{\varepsilon}^{\prime}\left(\left|\nabla u^{\varepsilon}\left(y^{\delta}\right)\right|^{2}\right) \nabla u_{\nu \nu}^{\varepsilon}\left(y^{\delta}\right) \cdot \nabla u^{\varepsilon}\left(y^{\delta}\right) \\
& -h_{\nu \nu}\left(y^{\delta}\right)-C_{0} u^{\varepsilon}\left(y^{\delta}\right)-C_{0} \beta_{\varepsilon}\left(\left|\nabla u^{\varepsilon}\left(y^{\delta}\right)\right|^{2}\right) \\
& +C_{0} h\left(y^{\delta}\right)-\delta(p+2)^{2}\left|y^{\delta}-x^{\varepsilon}\right|^{p} \\
= & \varphi_{\delta}\left(y^{\delta}\right)+2 \beta_{\varepsilon}^{\prime}\left(\left|\nabla u^{\varepsilon}\left(y^{\delta}\right)\right|^{2}\right) \nabla u_{\nu \nu}^{\varepsilon}\left(y^{\delta}\right) \cdot \nabla u^{\varepsilon}\left(y^{\delta}\right) \\
& -h_{\nu \nu}\left(y^{\delta}\right)-C_{0} \beta_{\varepsilon}\left(\left|\nabla u^{\varepsilon}\left(y^{\delta}\right)\right|^{2}\right)+C_{0} h\left(y^{\delta}\right)  \tag{13.22}\\
& -\delta(p+2)^{2}\left|y^{\delta}-x^{\varepsilon}\right|^{p}+\delta\left|y^{\delta}-x^{\varepsilon}\right|^{p+2} \\
\geqq & \varphi_{\delta}\left(y^{\delta}\right)+2 \beta_{\varepsilon}^{\prime}\left(\left|\nabla u^{\varepsilon}\left(y^{\delta}\right)\right|^{2}\right) \nabla u_{\nu \nu}^{\varepsilon}\left(y^{\delta}\right) \cdot \nabla u^{\varepsilon}\left(y^{\delta}\right) \\
& -C_{0}\left(1+h\left(y^{\delta}\right)\right)-C_{0} \beta_{\varepsilon}\left(\left|\nabla u^{\varepsilon}\left(y^{\delta}\right)\right|^{2}\right)+C_{0} h\left(y^{\delta}\right) \\
& -2 \delta p^{(p / 2)}(p+2)^{(p+2) / 2}
\end{align*}
$$

because of (2.8) and the fact that

$$
-\delta(p+2)^{2} r^{p}+\delta r^{p+2} \geqq-2 \delta p^{(p / 2)}(p+2)^{(p+2) / 2} \quad \forall r \geqq 0 .
$$

But (13.20) implies that

$$
\begin{align*}
\nabla u_{\nu \nu}^{\varepsilon}\left(y^{\delta}\right) \cdot \nabla u^{\varepsilon}\left(y^{\delta}\right) & =C_{0}\left|\nabla u^{\varepsilon}\left(y^{\delta}\right)\right|^{2}+\delta(p+2)\left|y^{\delta}-x^{\varepsilon}\right|^{p}\left(y^{\delta}-x^{\varepsilon}\right) \cdot \nabla u^{\varepsilon}\left(y^{\delta}\right) \\
& \geqq C_{0}\left|\nabla u^{\varepsilon}\left(y^{\delta}\right)\right|^{2} \tag{13.23}
\end{align*}
$$

because $u^{\varepsilon}$ is convex and attains its minimum at $x^{\varepsilon}$. Substitution of (13.23) into (13.22) yields

$$
\begin{align*}
0 \geqq \varphi_{\delta}\left(y^{\delta}\right) & +2 C_{0} \beta_{\varepsilon}^{\prime}\left(\left|\nabla u^{\varepsilon}\left(y^{\delta}\right)\right|^{2}\right)\left|\nabla u^{\varepsilon}\left(y^{\delta}\right)\right|^{2}-C_{0} \beta_{\varepsilon}\left(\left|\nabla u^{\varepsilon}\left(y^{\delta}\right)\right|^{2}\right)  \tag{13.24}\\
& -C_{0}-2 \delta p^{(p / 2)}(p+2)^{(p+2) / 2} .
\end{align*}
$$

The convexity of $\beta_{\varepsilon}$ implies that

$$
\beta_{\varepsilon}^{\prime}(r) r \geqq \beta_{\varepsilon}(r)-\beta_{\varepsilon}(0)=\beta_{\varepsilon}(r) \quad \forall r \geqq 0,
$$

so (13.24) reduces to

$$
\varphi_{\delta}(x) \leqq \varphi_{\delta}\left(y^{\delta}\right) \leqq C_{0}+2 \delta p^{(p / 2)}(p+2)^{(p+2) / 2} \quad \forall x \in \mathbb{R}^{2} .
$$

Letting $\delta \downarrow 0$, we obtain

$$
u_{\nu \nu}^{\varepsilon}(x) \leqq C_{0}\left(1+u^{\varepsilon}(x)\right) \quad \forall x \in \mathbb{R}^{2} .
$$

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