Viscosity Solutions for McKean-Vlasov Control on a torus*

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Abstract

An optimal control problem in the space of probability measures, and the viscosity solutions of the corresponding dynamic programming equations defined using the intrinsic linear derivative are studied. The value function is shown to be Lipschitz continuous with respect to a novel smooth Fourier-Wasserstein metric. A comparison result between the Lipschitz viscosity sub and super solutions of the dynamic programming equation is proved using this metric, characterizing the value function as the unique Lipschitz viscosity solution.

Key words: Mean Field Games, Wasserstein metric, Viscosity Solutions, McKean-Vlasov.

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1 Introduction

McKean–Vlasov optimal control is a part of the overarching program of Lasry & Lions [23, 24, 25] as articulated by Lions through his College de France lectures [26], and independently initiated by Huang, Malhamé, & Caines [22]. We refer the reader to the classical book of Carmona & Delarue [8] and to the lecture notes of Cardaliaguet [6] for detailed information and more references.

Main feature of the McKean-Vlasov type optimization is the dependence of its evolution and cost not only on the position of the state but also on its probability distribution, making the set of probability measures as its state space. Thus, the dynamic programming approach results in nonlinear partial differential equations set in the space of probability measures. Without common noise, they are first order Hamilton-Jacobi-Bellman equations, and its Hamiltonian is defined only when derivative of the value function is twice differentiable. In fact, this type of unboundedness is almost always the case for optimal control problems set in infinite dimensional spaces [19] and is the main new technical difficulty.

These dynamic programming equations are analogous to the coupled Hamilton-Jacobi and Fokker-Planck-Kolmogorov systems that characterize the solutions of the mean-field games for which deep regularity results are proved in [7] under some structural conditions. However, in general the dynamic programming equations for the McKean-Vlasov optimal control problems are not expected to admit classical solutions as shown in subsection 4.1 below, and a weak formulation is needed.

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As the maximum principle is still the salient feature in these settings as well, the viscosity solutions of Crandall & Lions [15, 16, 17, 20] is clearly the appropriate choice. However, due to the unboundedness of the Hamiltonian, original definition must be modified. In fact, such modifications of viscosity solutions in infinite dimensional spaces have already been studied extensively, and the book [19] provides an exhaustive account of these results. Still, it is believed that more can be achieved in the context of McKean–Vlasov due to the special structure of the set of probability measures. Indeed, an approach developed by Lions lifts the problems from the Wasserstein space to a regular $\mathbb{L}^2$ space, and then exploits the Hilbert structure to obtain new comparison results. This procedure also delivers the novel *Lions derivative* which has many useful properties, and we refer to [8] for its definition and more information. This method is further developed in several papers including [1, 3, 12, 28, 29]. The choice of the appropriate notion of a derivative is also explored in the recent paper [21], which then utilizes the deep connections to geometry to prove uniqueness results for Hamiltonians that are bounded in the sense discussed above.

Our main goals are to develop a viscosity theory directly on the space of probability measures using the linear derivative, provide a comparison result, and obtain a characterization of the value function as the unique viscosity solution in a certain class of functions. A natural approach towards this goal is to project the problem onto finite-dimensional spaces to leverage the already developed theory on these structures. A second-order problem studied in [14] provides a clear example of this approach as its projections exactly solve the projected finite dimensional equations. However, in general these projections are only approximate solutions, and [13] uses the Ekeland variational principle together with Gaussian smoothed Wasserstein metrics as gauge functions to control the approximation errors. A different technical tool is developed in [4], and [21] studies the pure projection problem. Other approaches include the path-dependent equations used in [33], gradient flows in [11], convergence analysis in [2] and an optimal stopping problem in [31, 32]. Recent paper [10] exploits the semi-convexity, and also provides an extensive survey.

We on the other hand employ the classical viscosity technique of doubling the variables as done in [5] in lieu of projection. The central difficulty of this approach is to appropriately replace the distance-square term $|x-y|^2$ used in the finite dimensional comparison proofs with the square of a metric on the space of measures. Thus, the crucial ingredient of our method is a novel Fourier-based smooth metric whose intriguing properties are studied in Section 5. Our other main results are a comparison between Lipschitz continuous sub and super viscosity solutions, Theorem 4.1 and the Lipschitz continuity of the value function with respect to a weaker metric, Theorem 4.2. Although the Lipschitz property of the value function is rather elementary for the Wasserstein metrics, it requires detailed analysis for the Fourier based ones. Indeed, a technical estimate, Proposition 7.1, on the dependence of the solutions of the McKean–Vlasov stochastic differential equation on the initial distribution is needed for this property.

As our approach contains several new steps, we study the simplest problem that allows us to showcase its details and power concisely. In particular, to ease the notation we omit the dependence of all functions on the time variable which can be added directly. Additionally, dynamics with jumps can be included as done in [5]. The compact structure of the torus is clearly a simplifying feature as well. In our accompanying paper [30] we remove most of these restrictions and study the extension of our method in higher dimensions.

The paper is organized as follow. General structure and notations are given in the next section, in Section 3 we define the problem and state the assumptions. The main results are stated in Section 4. We construct a family of Fourier-Wasserstein metrics in Section 5. The comparison result is proved in Section 6, and the Lipschitz property in Section 7. Standard results of dynamic programming and viscosity property are proved in Section 8 and respectively in Section 9.

2 Notations

In this section, we summarize the notations and known results used in the sequel. We denote the dimension of the ambient space by d, and the finite horizon by $T > 0$. $\mathbb{Z}^d$ is the set of all d-tuples of integers. $\mathbb{T}^d = \mathbb{R}^d/(2\pi\mathbb{Z})^d$ is the d-dimensional torus with the metric given by $|x-y|_{\mathbb{T}^d} := \inf_{k\in\mathbb{Z}^d} |x-y-2k\pi|$. We use a filtered probability space $(\Omega, \mathbb{F} = (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P})$ that supports Brownian motions. We assume that initial filtration $\mathcal{F}_0$ is rich enough so that for any probability measure on $\mathbb{T}^d$, there exists a random variable on Ω whose distribution is equal to this measure.

For a metric space (E, d), $\mathcal{M}(E)$ is the set of all Radon measures on E, and $\mathcal{P}(E)$ denotes the set of all probability measures on E. Let $\mathbb{L}^0(E)$ be the set of all E-valued random variables. For $X \in \mathbb{L}^0(E)$, $\mathcal{L}(X) \in \mathcal{P}(E)$ is the *law of* X.

We denote the set of all continuous real-valued functions on E by $\mathcal{C}(E)$, and the bounded ones by $\mathcal{C}_b(E) \subset \mathcal{C}(E)$. We write $\mathcal{C}(E, d)$ when the dependence on the metric is relevant, and $\mathcal{C}(E \mapsto Y)$ if the range Y is not the real numbers. For a positive integer n, $\mathcal{C}^n(E)$ is the set of n-times continuously differentiable, real-valued functions with the usual norm $\|\cdot\|_{\mathcal{C}^n}$ given by the sum of supremum norms of each derivative of order at most n.

We endow $\mathcal{M}(E)$ with the weak* topology $\sigma(\mathcal{P}(E), \mathcal{C}_b(E))$ and write $\mu_n \rightharpoonup \mu$, when $\lim_{n\to\infty} \mu_n(f) = \mu(f)$ for every $f \in \mathcal{C}_b(E)$. Using the standard (linear) derivative on the convex set $\mathcal{P}(E)$, we say that $\phi \in \mathcal{C}(\mathcal{P}(E))$ is *continuously differentiable* if there exists $\partial_\mu\phi \in \mathcal{C}(\mathcal{P}(E) \mapsto \mathcal{C}(E)))$ satisfying,

$$\phi(\nu) = \phi(\mu) + \int_0^1 \int_E \partial_\mu\phi(\mu + \tau(\nu-\mu))(x)\,(\nu-\mu)(\mathrm{d}x)\,\mathrm{d}\tau, \qquad \forall\,\mu,\nu \in \mathcal{P}(E).$$

We set $\mathcal{O} := (0,T) \times \mathcal{P}(\mathbb{T}^d)$. For $\psi \in \mathcal{C}(\overline{\mathcal{O}})$ and $(t,\mu) \in \mathcal{O}$, $\partial_t\psi(t,\mu)$ denotes the time derivative evaluated at (t,μ), and $\partial_\mu\psi(t,\mu) \in \mathcal{C}(\mathbb{T})$ denotes the derivative in the μ-variable again evaluated at (t,μ). $\mathbb{L}^2(\mathbb{T}^d)$ is the set of measurable functions on $\mathbb{T}^d$ that are square integrable with respect to the Lebesgue measure, with following orthonormal Fourier basis,

$$e_k(x) := (2\pi)^{-\frac{d}{2}}\, e^{ik\cdot x}, \qquad x \in \mathbb{T}^d,\ k \in \mathbb{Z}^d, \tag{2.1}$$

where $i = \sqrt{-1}$ and z^* be the complex conjugate of z. In particular, for any $\gamma \in \mathbb{L}^2(\mathbb{T}^d)$,

$$\gamma = \sum_{k\in\mathbb{Z}^d} F_k(\gamma)\, e_k, \quad \text{where} \quad F_k(\gamma) := \int_{\mathbb{T}^d} \gamma(x) e_k^*(x)\,\mathrm{d}x, \quad k \in \mathbb{Z}^d.$$

Following metrics on $\mathcal{P}(\mathbb{T}^d)$ are given by their dual representations,

$$\rho_\lambda(\mu,\nu) := \sup\{(\mu-\nu)(\psi) \;:\; \psi \in \mathbb{H}_\lambda(\mathbb{T}^d),\ \|\psi\|_\lambda \le 1\}, \qquad \lambda \ge 1,$$
$$\widehat{\rho}_n(\mu,\nu) := \sup\{(\mu-\nu)(\psi) \;:\; \psi \in \mathcal{C}^n(\mathbb{T}^d),\ \|\psi\|_{\mathcal{C}^n} \le 1\}, \qquad n = 1,2,\ldots,$$

where in view of Kantorovich duality, $\widehat{\rho}_1$ is the Wasserstein-one distance, and for $\lambda \ge 1$,

$$\mathbb{H}_\lambda(\mathbb{T}^d) := \{f \in \mathbb{L}^2(\mathbb{T}^d) : \|f\|_\lambda < \infty\}, \qquad \|f\|_\lambda := \Big(\sum_{k\in\mathbb{Z}^d} (1+|k|^2)^\lambda\, |F_k(f)|^2\Big)^{\frac{1}{2}}.$$

A Fourier representation of ρ_λ is derived in Corollary 5.2.

It is well-known that $\mathbb{H}_\lambda$ is the classical Sobolev space with fractional derivatives. Indeed, for any integer $n \ge 1$, $\mathcal{C}^n(\mathbb{T}^d) \subset \mathbb{H}_n(\mathbb{T}^d) = W^{n,2}(\mathbb{T}^d)$, and $\widehat{\rho}_n \le c_n\rho_n$ for some constant c_n. Moreover, by the embedding results, $\mathbb{H}_\lambda(\mathbb{T}^d) \subset C^n(\mathbb{T}^d)$ if $\lambda > n + \frac{d}{2}$. In particular, we set

$$n_*(d) = n_* := 3 + \lfloor \frac{d}{2} \rfloor, \quad \mathcal{C}_* := \mathcal{C}^{n_*}(\mathbb{T}^d), \quad \rho_* := \rho_{n_*}, \quad \widehat{\rho}_* := \widehat{\rho}_{n_*}, \tag{2.2}$$

where $\lfloor a \rfloor$ is the integer part of a real number a. Then, $\mathbb{H}_{n_*}(\mathbb{T}^d) \subset \mathcal{C}^2(\mathbb{T}^d)$.

3 McKean-Vlasov control

In this section, we define the *McKean-Vlasov optimal control* problem and for a general introduction, we refer the reader to Chapter 6 in [8]. Formally, starting from $t \in [0,T]$, the goal is to choose feedback controls $(\alpha_u(\cdot))_{u\in[t,T]}$ so as to minimize

$$\int_t^T \mathbb{E}[\ell(X_u, \mathcal{L}(X_u), \alpha_u(X_u))]\, \mathrm{d}u + \varphi(\mathcal{L}(X_T)),$$

where ℓ is the *running cost*, φ is the *terminal cost*, b, σ are given functions, and with a Brownian motion B, $\mathrm{d}X_u = b(X_u, \mathcal{L}(X_u), \alpha_u(X_u))\mathrm{d}u + \sigma(X_u, \mathcal{L}(X_u), \alpha_u(X_u))\mathrm{d}B_u$.

We continue by defining this problem properly.

3.1 Controlled processes

Suppose that A is a closed Euclidean space and let the *control set* $\mathcal{C}_a$ be a subset of $\mathcal{C}(\mathbb{T}^d \to A)$ containing all constant functions, and the *admissible controls* $\mathcal{A}$ be the set of (deterministic) measurable functions $\boldsymbol{\alpha} : [0,T] \mapsto \mathcal{C}_a$. We denote the value of any $\boldsymbol{\alpha} \in \mathcal{A}$ at time $u \in [0,T]$ by $\alpha_u \in \mathcal{C}_a$. Given functions are the *drift* vector $b = (b_1, \dots, b_d) \in \mathbb{R}^d$, the $d \times d'$ *volatility* matrix $\sigma = (\sigma_{ij})$ with $i = 1, \dots, d$, $j = 1, \dots, d'$, and the costs ℓ, φ. We continue by stating our standing regularity assumptions on these functions,

$$b_i, \sigma_{ij}, \ell : \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d) \times A \mapsto \mathbb{R}, \qquad \varphi : \mathcal{P}(\mathbb{T}^d) \mapsto \mathbb{R}.$$

Recall $\mathcal{C}_*, \rho_*, \widehat{\rho}_*$ of (2.2), and for $\alpha \in \mathcal{C}_a$, $x \in \mathbb{T}$, and $\mu \in \mathcal{P}(\mathbb{T}^d)$, set

$$b^\alpha(x,\mu) := b(x,\mu,\alpha(x)), \quad \sigma^\alpha(x,\mu) := \sigma(x,\mu,\alpha(x)), \quad \ell^\alpha(x,\mu) := \ell(x,\mu,\alpha(x)).$$

Assumption 3.1 (Regularity)**.** There exists $c_a < \infty$ such that for all $\alpha \in \mathcal{C}_a$ and $\mu \in \mathcal{P}(\mathbb{T}^d)$,

$$\|b^\alpha(\cdot,\mu)\|_{\mathcal{C}_*} + \|\sigma^\alpha(\cdot,\mu)\|_{\mathcal{C}_*} + \|\ell^\alpha(\cdot,\mu)\|_{\mathcal{C}_*} \le c_a,$$

and for $h = b, \sigma, \ell, \varphi$,

$$|h(x,\mu,a) - h(x,\nu,a)| \le c_a\ \widehat{\rho}_*(\mu,\nu), \qquad \forall\, x \in \mathbb{T}^d,\ \mu,\nu \in \mathcal{P}(\mathbb{T}^d),\ a \in A.$$

Under this regularity condition, for any $\boldsymbol{\alpha} \in \mathcal{A}$, $t \in [0,T]$, and $\mathcal{F}_t$ measurable, $\mathbb{T}^d$ valued random variable ξ with $\mu = \mathcal{L}(\xi)$, there is a unique $\mathbb{F}$-adapted solution $X_s^{t,\mu,\boldsymbol{\alpha}}$ of the following *McKean-Vlasov stochastic differential equation*,

$$X_s^{t,\mu,\boldsymbol{\alpha}} = \xi + \int_t^s b^{\alpha_u}(X_u^{t,\mu,\boldsymbol{\alpha}}, \mathcal{L}_u^{t,\mu,\boldsymbol{\alpha}})\, \mathrm{d}u + \int_t^s \sigma^{\alpha_u}(X_u^{t,\mu,\boldsymbol{\alpha}}, \mathcal{L}_u^{t,\mu,\boldsymbol{\alpha}})\, \mathrm{d}B_u, \qquad s \in [t,T], \quad (3.1)$$

where $\mathcal{L}_u^{t,\mu,\boldsymbol{\alpha}} = \mathcal{L}(X_u^{t,\mu,\boldsymbol{\alpha}})$, and B is a d' dimensional Brownian motion.

Although the solution $X_u^{t,\mu,\boldsymbol{\alpha}}$ depends on the choice of the initial condition ξ and the Brownian increments $(B_u - B_t)_{u\in[t,T]}$, as the Brownian increments are independent of $\mathcal{F}_t$ and we consider feedback controls, the flow $(\mathcal{L}_u^{t,\mu,\boldsymbol{\alpha}})_{u\in[t,T]}$ depends only on the law $\mu = \mathcal{L}(\xi)$ of the initial condition and not on ξ itself.

Clearly, the existence and uniqueness of solutions of (3.1) can be obtained under weaker assumptions. However, the stronger condition with n_* derivatives is needed for the comparison and the Lipschitz continuity results. We also emphasize that the regularity Assumption 3.1 puts implicit regularity restrictions of the control set $\mathcal{C}_a$ as discussed in Remark 3.2 below.

3.2 Problem

Starting from $(t,\mu) \in \overline{\mathcal{O}}$, the *pay-off* of a control process $\boldsymbol{\alpha} \in \mathcal{A}$ is given by,

$$J(t,\mu,\boldsymbol{\alpha}) := \int_t^T \mathbb{E}[\ell^{\alpha_u}(X_u^{t,\mu,\boldsymbol{\alpha}}, \mathcal{L}_u^{t,\mu,\boldsymbol{\alpha}})]\, \mathrm{d}u + \varphi(\mathcal{L}_T^{t,\mu,\boldsymbol{\alpha}}), \qquad \boldsymbol{\alpha} \in \mathcal{A},\ (t,\mu) \in \overline{\mathcal{O}}. \tag{3.2}$$

Since $\mathbb{E}[\ell^{\alpha_u}(X_u^{t,\mu,\boldsymbol{\alpha}}, \mathcal{L}_u^{t,\mu,\boldsymbol{\alpha}})] = \mathcal{L}_u^{t,\mu,\boldsymbol{\alpha}}(\ell(\cdot, \mathcal{L}_u^{t,\mu,\boldsymbol{\alpha}}, \alpha_u(\cdot)))$, $J(t,\mu,\boldsymbol{\alpha})$ is a function of $\mu = \mathcal{L}(\xi)$ independent of the choice of the initial random variable ξ. Although, this property, called *law-invariance*, holds directly in our setting, in general structures it is quite subtle. We refer to Proposition 2.4 of [18], and Theorem 3.5 in [12] for its general proof, and to Section 6.5 and Definition 6.27 of [8] for a discussion.

Then, the McKean-Vlasov optimal control problem is to minimize the pay-off functional J over $\boldsymbol{\alpha} \in \mathcal{A}$, and the *value function* is given by,

$$v(t,\mu) := \inf_{\boldsymbol{\alpha} \in \mathcal{A}} J(t,\mu,\boldsymbol{\alpha}), \qquad (t,\mu) \in \overline{\mathcal{O}}.$$

Remark 3.2. Suppose that $\mathcal{C}_a = \{\alpha \in \mathcal{C}_*(\mathbb{T}^d \to A) : \ \|\alpha\|_{\mathcal{C}_*} \le c_0\}$ for some constant $c_0 \ge 0$. Consider the class of functions of the form $h(x, \mu(f), a)$ for some $f \in \mathcal{C}_*$, and $h : \mathbb{T}^d \times \mathbb{R} \times A \to \mathbb{R}$ satisfying $\|h(\cdot, y, \cdot)\|_{\mathcal{C}_*} + \|h(x, \cdot, a)\|_{1,\infty} \le c_1$ for every $x \in \mathbb{T}^d$, $y \in \mathbb{R}$, and $a \in A$, for some $c_1 \ge 0$. Then, $h^\alpha(x,\mu) = h(x, \mu(f), \alpha(x))$, and $\|h^\alpha(\cdot,\mu)\|_{\mathcal{C}_*}$ is less than a constant c_a depending on c_0, c_1 and n_*. Also, for every $x \in \mathbb{T}^d$,

$$|h(x,\mu(f),\alpha(x)) - h(x,\nu(f),\alpha(x))| \le c_1 |(\mu-\nu)(f)| \le c_1 \|f\|_{\mathcal{C}_*} \widehat{\rho}_*(\mu,\nu) \le c_1 c_0 \widehat{\rho}_*(\mu,\nu).$$

Hence, this class of functions satisfy the regularity assumption. More generally, under appropriate assumptions functions $h(x, \mu(f_1), \dots, \mu(f_m), a)$ with $f_1, \dots, f_m \in \mathcal{C}_*(\mathbb{T})$, and $h : \mathbb{T}^d \times \mathbb{R}^m \times A \to \mathbb{R}$ also satisfy the regularity assumption with the above control set $\mathcal{C}_a$. We emphasize that even when the coefficients depend on μ only through $\mu(f_1), \dots, \mu(f_m)$ of the measure μ, the value function in general is still infinite dimensional.

Assumptions made above hold in a large class of examples studied in the mean-field games. In particular, for the Kuramoto problem studied in [9], for some constants $\kappa, \sigma > 0$,

$$\ell(\mu, a) = \frac{1}{2} a^2 + \kappa[1 - (\mu(\cos))^2 - (\nu(\sin))^2], \quad b(x,\mu,a) = a, \quad \sigma(a) = \sigma.$$

3.3 Dynamic programming principle

We next state the dynamic programming principle which is central to the viscosity approach to optimal control. A general proof in a different setting is given in [18]. However, the continuity of the value function proved in Section 7, and the standard techniques outlined in [20] allows for a simpler proof that we provide in Section 8.

Theorem 3.3 (Dynamic programming)**.** *For every $\mu \in \mathcal{P}(\mathbb{T}^d)$ and $0 \le t \le \tau \le T$,*

$$v(t,\mu) = \inf_{\boldsymbol{\alpha} \in \mathcal{A}} \int_t^\tau \mathbb{E}[\ell^{\alpha_u}(X_u^{t,\mu,\boldsymbol{\alpha}}, \mathcal{L}_u^{t,\mu,\boldsymbol{\alpha}})]\ \mathrm{d}u + v(\tau, \mathcal{L}_\tau^{t,\mu,\boldsymbol{\alpha}}). \tag{3.3}$$

It is well known that the dynamic programming can be used directly to show that the value function is a viscosity solution of the dynamic programming equation

$$-\partial_t v(t,\mu) = H(\mu, \partial_\mu v(t,\mu)), \qquad t \in [0,T),\ \mu \in \mathcal{P}(\mathbb{T}^d), \tag{3.4}$$

where for $\gamma \in \mathcal{C}^2(\mathbb{T}^d)$, $\mu \in \mathcal{P}(\mathbb{T}^d)$, $x \in \mathbb{T}^d$ and $\alpha \in \mathcal{C}_a$,

$$H(\mu,\gamma) := \inf_{\alpha \in \mathcal{C}_a} \left\{ \mu\left(\ell^\alpha(\cdot,\mu) + \mathcal{M}^{\alpha,\mu}[\gamma](\cdot)\right) \right\},$$

$$\mathcal{M}^{\alpha,\mu}[\gamma](x) := b(x,\mu,\alpha(x))\cdot\partial_x\gamma(x) + \sum_{i,j=1}^{d}\sum_{l=1}^{d'} \sigma_{il}(x,\mu,\alpha(x))\sigma_{jl}(x,\mu,\alpha(x))\partial_{x_ix_j}\gamma(x).$$

The value function also trivially satisfies the following terminal condition,

$$v(T,\mu) = \varphi(\mu), \qquad \forall\ \mu \in \mathcal{P}(\mathbb{T}^d). \tag{3.5}$$

As the value function is not necessarily differentiable, a weak formulation is needed, and we use the notion of viscosity solutions. The definition that we use is exactly the classical one in which the auxiliary test functions are continuously differentiable functions on $\overline{\mathcal{O}} = [0,T]\times\mathcal{P}(\mathbb{T}^d)$, with the linear derivative in $\mathcal{P}(\mathbb{T}^d)$ recalled in Section 2. We continue by specifying the auxiliary functions used in the definition of viscosity solutions.

Definition 3.4. We say that $\psi \in \mathcal{C}(\overline{\mathcal{O}})$ is a *test function,* if ψ is continuously differentiable with $\partial_\mu\psi(t,\mu) \in \mathcal{C}^2(\mathbb{T}^d)$ for every $(t,\mu)\in\overline{\mathcal{O}}$, and the map $(t,\mu)\in\overline{\mathcal{O}} \mapsto H(\mu,\partial_\mu\psi(t,\mu))$ is continuous. We denote the set of all test functions by $\mathcal{C}_s(\overline{\mathcal{O}})$.

Definition 3.5. A continuous function $u \in \mathcal{C}(\overline{\mathcal{O}})$ is a *viscosity subsolution of* (3.4)*,* if every $\psi\in\mathcal{C}_s(\overline{\mathcal{O}})$, $(t_0,\mu_0)\in[0,T)\times\mathcal{P}(\mathbb{T}^d)$, satisfying $(u-\psi)(t_0,\mu_0) = \max_{\overline{\mathcal{O}}}(u-\psi)$, also satisfies

$$-\partial_t\psi(t_0,\mu_0) \le H(\mu_0,\partial_\mu\psi(t_0,\mu_0)).$$

A continuous function $w\in\mathcal{C}(\overline{\mathcal{O}})$ is a *viscosity supersolution of* (3.4)*,* if every $\psi\in\mathcal{C}_s(\overline{\mathcal{O}})$, $(t_0,\mu_0)\in[0,T)\times\mathcal{P}(\mathbb{T}^d)$, satisfying $(w-\psi)(t_0,\mu_0) = \min_{\overline{\mathcal{O}}}(w-\psi)$, also satisfies

$$-\partial_t\psi(t_0,\mu_0) \ge H(\mu_0,\partial_\mu\psi(t_0,\mu_0)).$$

Finally, $v\in\mathcal{C}(\overline{\mathcal{O}})$ is a *viscosity solution of* (3.4)*,* if it is both a sub and a super solution.

4 Main results

Our main result is the characterization of the value function as the unique continuous viscosity solution of the dynamic programming equation (3.4) and the terminal condition (3.5).

Recall the metrics $\rho_*, \widehat{\rho}_*$ of (2.2).

Theorem 4.1 (Comparison)**.** *Suppose that the regularity Assumption 3.1 holds, $u\in\mathcal{C}(\overline{\mathcal{O}})$ is a viscosity subsolutionof* (3.4) *and* (3.5)*, and $w\in\mathcal{C}(\overline{\mathcal{O}})$ is a viscosity supersolution of* (3.4) *and* (3.5)*. If further u or w is Lipschitz continuous in the μ-variable with respect to the metric ρ_*, then $u\le v$ on $\overline{\mathcal{O}}$.*

Above comparison result is proved in Section 6 below.

Theorem 4.2 (Continuity)**.** *Under the regularity Assumption 3.1, there exists a constant $L_v>0$ depending only on the horizon T and the constant c_a of Assumption 3.1, so that*

$$|v(t,\mu)-v(s,\nu)| \le L_v\left[\widehat{\rho}_*(\mu,\nu) + |t-s|^{\frac{1}{2}}\right], \qquad \forall\,\mu,\nu\in\mathcal{P}(\mathbb{T}^d),\ t,s\in[0,T]. \tag{4.1}$$

This continuity result proved in Section 7 below, also implies Lipschitz continuity with respect to ρ_*, since $\widehat{\rho}_* \le c_*\rho_*$ for some constant c_*. The following result follows directly from the standard viscosity theory [20], and its proof is given in Section 9 below.

Theorem 4.3 (Viscosity property)**.** *Under the regularity Assumption 3.1, the value function is a viscosity solution of* (3.4) *in $\mathcal{O}$, satisfying the terminal condition* (3.5)*.*

In particular, any continuous viscosity subsolution is less than or equal to the value function v, and any continuous viscosity supersolution is greater than or equal to v.

Remark 4.4. In the comparison result, we could use any metric ρ_λ with $\lambda > 2+\frac{d}{2}$. However, our proof for Lipschitz continuity requires us to employ the smaller metric $\widehat{\rho}_m$ and only for integer values of m. This combination of the results dictates the global choice $\lambda = n_*$.

4.1 An example

In this subsection, we provide a simple example to illustrate the notation and also the need for viscosity solutions. We take $T = 1$, $d = 1$, $A = \mathbb{R}$, $b(x, \mu, a) = a$, $\sigma \equiv 1$, $\varphi \equiv 0$, and

$$\ell(\mu, a) := \frac{1}{2}a^2 + L(m(\mu)), \quad \text{where} \quad m(\mu) := \int_{\mathbb{T}} x \ \mu(\mathrm{d}x),$$

and $L : [-\pi, \pi] \to \mathbb{R}$ is a given Lipschitz function. It can be shown that the value function of the above problem is independent of the control set $\mathcal{C}_a$, and is given by,

$$v(t, \mu) = w(t, m(\mu)), \qquad (t, \mu) \in \overline{\mathcal{O}},$$

$$w(t, y) := \inf_{\hat{\boldsymbol{\alpha}} \in \widehat{\mathcal{A}}} \hat{J}(t, y, \hat{\boldsymbol{\alpha}}) := \inf_{\hat{\boldsymbol{\alpha}} \in \widehat{\mathcal{A}}} \int_t^1 [\frac{1}{2}(\hat{\alpha}_u)^2 + L(Y_u^{t,y,\hat{\boldsymbol{\alpha}}})] \ \mathrm{d}u, \qquad (t, y) \in [0, 1] \times \mathbb{T},$$

where $\widehat{\mathcal{A}}$ is the set of all measurable maps $\hat{\boldsymbol{\alpha}} : [0, 1] \mapsto \mathbb{T}$, and $Y_u^{t,y,\hat{\boldsymbol{\alpha}}} = y + \int_t^u \hat{\alpha}_s \mathrm{d}s$. It is well known that w is the unique viscosity solution of the Eikonal equation,

$$-\partial_t w(t, y) = -\frac{1}{2}(\partial_y w(t, y))^2 + L(y), \qquad y \in \mathbb{T}, \tag{4.2}$$

and $w(1, \cdot) \equiv 0$. Since w is not always differentiable, we conclude that v is not either, and therefore a weak theory is needed. On the other hand, when w is differentiable, we have

$$\partial_\mu v(t, \mu)(x) = \partial_y w(t, m(\mu))\, x \quad \Rightarrow \quad \partial_x(\partial_\mu v(t, \mu)(x)) = \partial_y w(t, m(\mu)).$$

Hence, by Jensen's inequality,

$$\begin{aligned} H(\mu, \partial_\mu v(t, \mu)) &= \inf_{\alpha \in \mathcal{C}_a} \int_{\mathbb{T}} \left(\frac{1}{2}\alpha(x)^2 + \alpha(x)\partial_y w(t, m(\mu))\right) \mu(\mathrm{d}x) + L(m(\mu)) \\ &\geq \inf_{\alpha \in \mathcal{C}_a} \left\{ \frac{1}{2} \left(\int_{\mathbb{T}} \alpha(x)\mu(\mathrm{d}x) \right)^2 + \left(\int_{\mathbb{T}} \alpha(x)\mu(\mathrm{d}x) \right) \partial_y w(t, m(\mu)) \right\} + L(m(\mu)) \\ &= \inf_{a \in \mathbb{R}} \left\{ \frac{1}{2}a^2 + a\partial_y w(t, m(\mu)) \right\} + L(m(\mu)) \\ &= -\frac{1}{2}(\partial_y w(t, m(\mu)))^2 + L(m(\mu)). \end{aligned}$$

As constant functions $\alpha \equiv a$ are always in $\mathcal{C}_a$, we also have the opposite inequality. Therefore,

$$H(\mu, \partial_\mu v(t, \mu)) = -\frac{1}{2}(\partial_y w(t, m(\mu))^2 + L(m(\mu)),$$

for every $\mathcal{C}_a$. Since $\partial_t v(t, \mu) = \partial_t w(t, m(\mu))$, the Eikonal equation (4.2) implies that when w is differentiable, v is a classical solution of the dynamic programming equation (3.4).

5 Fourier-Wasserstein metrics

In this section, we study the properties of the norms and the metric ρ_λ defined in Section 2. Similar metrics are also defined in [27] using a dual representation with Sobolev functions.

Recall that z^* is the complex conjugate of z, and the orthonormal basis $\{e_k\}_{k \in \mathbb{Z}^d}$, Fourier coefficients $F_k(f)$ are defined in Section 2. For $\mu \in \mathcal{M}(\mathbb{T}^d)$, $k \in \mathbb{Z}^d$, we also set $F_k(\mu) := \mu(e_k^*)$. As $\mathbb{T}^d$ is compact, $F_k(\mu)$ is finite for every k, and $F_0(\mu) = 1$ for all $\mu \in \mathcal{P}(\mathbb{T}^d)$.

For $\lambda \geq 1$, we define a norm on $\mathcal{M}(\mathbb{T}^d)$, dual to $\|\cdot\|_\lambda$ by,

$$|\eta|_\lambda := \sup\{\eta(\psi) \ : \ \psi \in \mathbb{H}_\lambda(\mathbb{T}^d), \ \|\psi\|_\lambda \leq 1\}, \qquad \eta \in \mathcal{M}(\mathbb{T}^d),$$

so that $\rho_\lambda(\mu, \nu) = |\mu - \nu|_\lambda$.

Lemma 5.1. *For $\lambda > \frac{d}{2}$, $\eta \in \mathcal{M}(\mathbb{T}^d)$, $|\eta|_\lambda < \infty$ and has the following dual representation,*

$$|\eta|_\lambda = (\sum_{k\in\mathbb{Z}^d} (1+|k|^2)^{-\lambda} |F_k(\eta)|^2)^{\frac{1}{2}}. \tag{5.1}$$

Proof. We first note as $2\lambda > d$, $c_\lambda := \sum_{k\in\mathbb{Z}^d}(1+|k|^2)^{-\lambda} < \infty$. Let $d(\eta)$ be the expression in the right hand side of (5.1) and $TV(\eta)$ be the total variation of the measure η. Then, $|F_k(\eta)| \le TV(\eta)$ and therefore, $d(\eta) \le \sum_{k\in\mathbb{Z}^d}(1+|k|^2)^{-\lambda}\, TV(\eta) = c_\lambda TV(\eta)$.

For $\psi \in \mathcal{C}(\mathbb{T}^d)$, the Fourier representation $\psi = \sum_{k\in\mathbb{Z}^d} F_k(\psi) e_k$ implies that,

$$\begin{aligned}\eta(\psi) &= \sum_{k\in\mathbb{Z}^d} F_k(\psi)\, \eta(e_k) = \sum_{k\in\mathbb{Z}^d} F_k(\psi)\, F_k^*(\eta) \\ &= \sum_{k\in\mathbb{Z}^d} [(1+|k|^2)^{\frac{\lambda}{2}}\, F_k(\psi)]\, [(1+|k|^2)^{-\frac{\lambda}{2}}\, F_k^*(\eta)] \\ &\le (\sum_{k\in\mathbb{Z}^d} (1+|k|^2)^{\lambda} |F_k(\psi)|^2)^{\frac{1}{2}}\, (\sum_{k\in\mathbb{Z}^d} (1+|k|^2)^{-\lambda}\, |F_k^*(\eta)|^2)^{\frac{1}{2}} = \|\psi\|_\lambda\, d(\eta).\end{aligned} \tag{5.2}$$

In view of the definition of $|\cdot|_\lambda$, $|\eta|_\lambda \le d(\eta)$, for any $\eta \in \mathcal{M}(\mathbb{T}^d)$.

To prove the opposite inequality, fix $\eta \in \mathcal{M}(\mathbb{T}^d)$ and define a function $\tilde\psi$ by,

$$\tilde\psi(x) := \sum_{k\in\mathbb{Z}^d} (1+|k|^2)^{-\lambda}\, F_k(\eta)\, e_k(x), \quad \Rightarrow \quad F_k(\tilde\psi) = (1+|k|^2)^{-\lambda}\, F_k(\eta), \quad k \in \mathbb{Z}^d.$$

Since $c_\lambda < \infty$, $\tilde\psi$ is well-defined. Moreover,

$$\|\tilde\psi\|_\lambda^2 = \sum_{k\in\mathbb{Z}^d} (1+|k|^2)^{\lambda} |F_k(\tilde\psi)|^2 = \sum_{k\in\mathbb{Z}^d} (1+|k|^2)^{-\lambda}\, |F_k(\eta)|^2 = d^2(\eta) < \infty.$$

Hence, $\tilde\psi \in \mathbb{H}_\lambda(\mathbb{T}^d)$, and by (5.2),

$$\eta(\tilde\psi) = \sum_{k\in\mathbb{Z}^d} F_k(\tilde\psi)\, F_k^*(\eta) = \sum_{k\in\mathbb{Z}^d} (1+|k|^2)^{-\lambda}\, |F_k(\eta)|^2 = d^2(\eta) = \|\tilde\psi\|_\lambda\, d(\eta).$$

As $\eta(\tilde\psi) \le |\eta|_\lambda \|\tilde\psi\|_\lambda$ by the definition of $|\cdot|_\lambda$, we have $d(\eta)\|\tilde\psi\|_\lambda = \eta(\tilde\psi) \le |\eta|_\lambda \|\tilde\psi\|_\lambda$. ☐

An immediate corollary is the following.

Corollary 5.2. *For any $\lambda > \frac{d}{2}$, ρ_λ is a metric on $\mathcal{P}(\mathbb{T}^d)$ with a dual representation,*

$$\rho_\lambda(\mu,\nu) = \max\{\ (\mu-\nu)(\psi)\ :\ \|\psi\|_\lambda \le 1\ \} = (\sum_{k\in\mathbb{Z}^d} (1+|k|^2)^{-\lambda}\, |F_k(\mu-\nu)|^2\,)^{\frac{1}{2}}.$$

Proof. The dual representation follows directly from the previous lemma. Suppose that $\rho_\lambda(\mu,\nu) = 0$, then $F_k(\mu) = F_k(\nu)$ for every $k \in \mathbb{Z}^d$. As μ, ν have the same Fourier series, we conclude that $\mu = \nu$. The fact that ρ_λ is a metric now follows from the dual representation. ☐

The following provides a connection between the two metrics we consider. Also with $m = 1$, it implies that the classical Wasserstein one metric $\widehat\rho_1$ is dominated by ρ_1.

Lemma 5.3. *For any integer $m \ge 1$, there exists $c_{m,d} > 0$, such that $\widehat\rho_m(\mu,\nu) \le c_{m,d}\, \rho_m(\mu,\nu)$ for every $\mu, \nu \in \mathcal{P}(\mathbb{T}^d)$.*

Proof. Fix the $m \ge 1$ and let $D^m\psi$ be the m-th order derivatives of $\psi \in \mathcal{C}^m(\mathbb{T}^d)$. Then, since $|k|^{2m}|F_k(\psi)|^2 = |F_k(D^m\psi)|^2$,

$$\sum_{k\in\mathbb{Z}^d} |k|^{2m}|F_k(\psi)|^2 = \sum_{k\in\mathbb{Z}^d} |F_k(D^m\psi)|^2 = \|D^m\psi\|^2_{\mathbb{L}^2(\mathbb{T}^d)} \le d^m\ (2\pi)^d \|\psi\|^2_{\mathcal{C}^m(\mathbb{T}^d)}.$$

As $(1+|k|^2)^m \le 2^m(1+|k|^{2m})$, for any $k \in \mathbb{Z}^d$, $\psi \in \mathcal{C}^m(\mathbb{T}^d)$,

$$\|\psi\|^2_m \le 2^m \sum_{k\in\mathbb{Z}^d} |F_k(\psi)|^2 + 2^m \sum_{k\in\mathbb{Z}^d} |k|^{2m}|F_k(\psi)|^2 \le c^2_{m,d}\|\psi\|^2_{\mathcal{C}^m(\mathbb{T}^d)},$$

where $c^2_{m,d} = 2^m[1 + d^m(2\pi)^d]$. Hence,

$$\begin{aligned}
\widehat{\rho}_m(\mu,\nu) &= \sup\{(\mu-\nu)(\psi)\ :\ \|\psi\|_{\mathcal{C}^m(\mathbb{T}^d)} \le 1\} \\
&\le \sup\{(\mu-\nu)(\psi)\ :\ \psi \in \mathcal{C}^m(\mathbb{T}^d),\ \|\psi\|_m \le c_{m,d}\} \\
&\le \sup\{(\mu-\nu)(\psi)\ :\ \psi \in \mathbb{H}_m(\mathbb{T}^d),\ \|\psi\|_m \le c_{m,d}\} = c_{m,d}\rho_m(\mu,\nu).
\end{aligned}$$

□

Our next result is on the differentiability of ρ_λ. Recall the test functions $\mathcal{C}_s(\overline{\mathcal{O}})$ of Definition 3.4, $n_*(d)$ of (2.2), and the basis e_k of Section 2.

Lemma 5.4. *Fix $\lambda > \frac{d}{2}$, $\nu \in \mathcal{P}(\mathbb{T}^d)$ and set $h(\mu) := \frac{1}{2}\rho^2_\lambda(\mu,\nu)$. Then,*

$$\partial_\mu h(\mu)(x) = \sum_{k\in\mathbb{Z}^d} (1+|k|^2)^{-\lambda}\, F_k(\mu-\nu)\, e^*_k(x), \qquad x \in \mathbb{T}^d,$$

and $\|\partial_\mu h(\mu)\|_\lambda = \rho_\lambda(\mu,\nu)$. Moreover, if $\lambda = n_*(d)$, then $\partial_\mu h(\mu) \in \mathcal{C}^2(\mathbb{T}^d)$.

Proof. Fix $\nu \in \mathcal{P}(\mathbb{T}^d)$. For each $k \in \mathbb{Z}^d$, set $a_k(\mu) := \frac{1}{2}|F_k(\mu-\nu)|^2$. Then, we directly calculate that $\partial_\mu a_k(\mu)(\cdot) = F_k(\mu-\nu)\, e^*_k(\cdot)$. Then, for any $x \in \mathbb{T}^d$,

$$\partial_\mu h(\mu)(x) = \sum_{k\in\mathbb{Z}^d} (1+|k|^2)^{-\lambda}\ \partial_\mu a_k(\mu)(x) = \sum_{k\in\mathbb{Z}^d} (1+|k|^2)^{-\lambda}\, F_k(\mu-\nu)\, e^*_k(x).$$

The above formula implies that $F_k(\partial_\mu h(\mu)) = (1+|k|^2)^{-\lambda}F^*_k(\mu-\nu)$ for every $k \in \mathbb{Z}^d$. Hence,

$$\|\partial_\mu h(\mu)\|^2_\lambda = \sum_{k\in\mathbb{Z}^d} (1+|k|^2)^{\lambda}\, |F_k(\partial_\mu h(\mu))|^2 = \sum_{k\in\mathbb{Z}^d} (1+|k|^2)^{-\lambda}\, |F_k(\mu-\nu)|^2 = \rho^2_\lambda(\mu,\nu).$$

In view of the Sobolev embedding of $\mathbb{H}_{n_*}(\mathbb{T}^d)$ into $\mathcal{C}^2(\mathbb{T}^d)$, $\partial_\mu h(\mu) \in \mathcal{C}^2(\mathbb{T}^d)$. □

6 Comparison

In this section we prove Theorem 4.1 in several steps. Recall the test functions $\mathcal{C}_s(\overline{\mathcal{O}})$ of Definition 3.4, and n_*, ρ_* of (2.2). Then, $2(n_*-2) \ge d+1$, and consequently,

$$c(d) := \sum_{k\in\mathbb{Z}^d} (1+|k|^2)^{2-n_*} < \infty. \tag{6.1}$$

Step 1 (Set-up). Let u, w be as in the statement of the theorem. Towards a contraposition suppose that $\sup_{\overline{\mathcal{O}}}(u-w) > 0$. We fix a sufficiently small $\delta > 0$ satisfying

$$l := \max_{(t,\mu)\in\overline{\mathcal{O}}} \{(u-w)(t,\mu) - \delta(T-t)\} > 0.$$

Set $\bar{u}(t,\mu) := u(t,\mu) - \delta(T-t)$. Then, $\bar{u}$ is a continuous viscosity subsolution of

$$- \partial_t \bar{u}(t,\mu) = H(\mu, \partial_\mu \bar{u}(t,\mu)) - \delta. \tag{6.2}$$

Step 2 (Doubling the variables). For $\epsilon > 0$, set

$$\Phi_\epsilon(t,\mu,s,\nu) := \bar{u}(t,\mu) - w(s,\nu) - \frac{1}{2\epsilon}\left(\rho_*^2(\mu,\nu) + (t-s)^2\right).$$

As $\overline{\mathcal{O}}$ is compact and $\bar{u}, w$ are continuous, there exists $(t_\epsilon, s_\epsilon, \mu_\epsilon, \nu_\epsilon) \in \overline{\mathcal{O}} \times \overline{\mathcal{O}}$ satisfying

$$\Phi_\epsilon(t_\epsilon,\mu_\epsilon,s_\epsilon,\nu_\epsilon) = \max_{\overline{\mathcal{O}}\times\overline{\mathcal{O}}} \Phi_\epsilon \geq l > 0.$$

Set $M := \max \bar{u}$, $m := \min v$, $\zeta_\epsilon := \rho_*^2(\mu_\epsilon,\nu_\epsilon) + (t_\epsilon - s_\epsilon)^2$, so that

$$0 \leq \zeta_\epsilon \leq 2\epsilon\ (M + m - l). \tag{6.3}$$

Step 3 (Letting ϵ to zero). Since $\overline{\mathcal{O}}$ is compact, there is a subsequence $\{(t_\epsilon,\mu_\epsilon,s_\epsilon,\nu_\epsilon)\} \subset \overline{\mathcal{O}} \times \overline{\mathcal{O}}$, denoted by ϵ again, and $(t^*,\mu^*,s^*,\nu^*) \in \overline{\mathcal{O}} \times \overline{\mathcal{O}}$, such that

$$\mu_\epsilon \rightharpoonup \mu^*, \quad \nu_\epsilon \rightharpoonup \nu^*, \quad t_\epsilon \to t^*, \quad s_\epsilon \to s^*, \qquad \text{as } \ \epsilon \downarrow 0.$$

By (6.3) it is clear that $t^* = s^*$, and $\rho_*(\mu^*,\nu^*) = 0$. Then, by Lemma 5.3, $\mu^* = \nu^*$.

If t^* were to be equal to T, by the terminal condition (3.5), we would have

$$0 < l \leq \liminf_{\epsilon\downarrow 0}\ \Phi_\epsilon(t_\epsilon,\mu_\epsilon,s_\epsilon,\nu_\epsilon) \leq \lim_{\epsilon\downarrow 0} [\bar{u}(t_\epsilon,\mu_\epsilon) - w(s_\epsilon,\nu_\epsilon)] = \bar{u}(T,\mu^*) - w(T,\mu^*) \leq 0.$$

Hence, $t^* < T$ and $t_\epsilon, s_\epsilon < T$ for all sufficiently small $\epsilon > 0$.
Step 4 (Distance estimate). Without loss of generality, suppose that w is Lipschitz, i.e,

$$|w(t,\mu) - w(t,\nu)| \leq \frac{1}{2} L_w\, \rho_*(\mu,\nu), \qquad \mu,\nu \in \mathcal{P}(\mathbb{T}^d),\ t \in [0,T].$$

Then, for each $\epsilon > 0$,

$$\begin{aligned}\bar{u}(t_\epsilon,\mu_\epsilon) - w(s_\epsilon,\nu_\epsilon) - \frac{1}{2\epsilon}\,\zeta_\epsilon &= \Phi_\epsilon(t_\epsilon,\mu_\epsilon,s_\epsilon,\nu_\epsilon) \geq \Phi_\epsilon(t_\epsilon,\mu_\epsilon,s_\epsilon,\mu_\epsilon) \\ &= \bar{u}(t_\epsilon,\mu_\epsilon) - w(s_\epsilon,\mu_\epsilon) - \frac{1}{2\epsilon}\,(t_\epsilon - s_\epsilon)^2.\end{aligned}$$

Therefore, $\rho_*^2(\mu_\epsilon,\nu_\epsilon) = \zeta_\epsilon - (t_\epsilon - s_\epsilon)^2 \leq 2\epsilon\,[w(s_\epsilon,\mu_\epsilon) - w(s_\epsilon,\nu_\epsilon)] \leq 2\epsilon\, L_w\, \rho_*(\mu_\epsilon,\nu_\epsilon)$. Hence,

$$\rho_*(\mu_\epsilon,\nu_\epsilon) \leq \epsilon\, L_w \qquad \forall\, \epsilon > 0. \tag{6.4}$$

Step 5 (Viscosity property). Set

$$\psi_\epsilon(t,\mu) := \frac{1}{2\epsilon}[\rho_*^2(\mu,\nu_\epsilon) + (t - s_\epsilon)^2], \qquad \phi_\epsilon(s,\nu) := -\frac{1}{2\epsilon}[\rho_*^2(\mu_\epsilon,\nu) + (t_\epsilon - s)^2].$$

By Lemma 5.4, both $\partial_\mu \psi_\epsilon(t,\mu), \partial_\nu \phi_\epsilon(t,\mu) \in \mathcal{C}^2(\mathbb{T}^d)$. Moreover, by the regularity Assumption 3.1, maps $(t,\mu) \mapsto H(\mu, \partial_\mu \psi_\epsilon(t,\mu))$, and $(t,\nu) \mapsto H(\nu, \partial_\mu \phi_\epsilon(t,\nu))$ are continuous. Hence, ψ_ϵ and ϕ_ϵ are smooth test functions. Set

$$\kappa_\epsilon(x) := \partial_\mu \psi_\epsilon(t_\epsilon,\mu_\epsilon)(x) = \partial_\mu \phi_\epsilon(s_\epsilon,\nu_\epsilon)(x) = \frac{1}{\epsilon}\sum_{k\in\mathbb{Z}^d} \frac{F_k(\mu_\epsilon - \nu_\epsilon)}{(1+|k|^2)^{n_*}}\ e_k^*(x), \qquad x \in \mathbb{T}^d.$$

Also, $\bar{u}(t,\mu) - \psi_\epsilon(t,\mu)$ is maximized at t_ϵ, μ_ϵ. Since $t_\epsilon < T$, $\psi_\epsilon \in \mathcal{C}_s(\overline{\mathcal{O}})$ and $\bar{u}$ is a viscosity subsolution of (6.2), then

$$-\frac{t_\epsilon - s_\epsilon}{\epsilon} \le H(\mu_\epsilon, \kappa_\epsilon) - \delta.$$

By the viscosity property of w, a similar argument implies that

$$-\frac{t_\epsilon - s_\epsilon}{\epsilon} \ge H(\nu_\epsilon, \kappa_\epsilon).$$

We subtract the above inequalities to arrive at

$$0 < \delta \le H(\mu_\epsilon, \kappa_\epsilon) - H(\nu_\epsilon, \kappa_\epsilon). \tag{6.5}$$

Step 6 (Estimation). Since $H(\mu, \kappa_\epsilon) = \inf_{\alpha \in \mathcal{C}_a} \{\mu(\ell^\alpha(\cdot,\mu) + \mathcal{M}^{\alpha,\mu}[\kappa_\epsilon](\cdot))\}$,

$$|H(\mu_\epsilon, \kappa_\epsilon) - H(\nu_\epsilon, \kappa_\epsilon)| \le \sup_{\alpha \in \mathcal{C}_a} \mathcal{T}_\epsilon^\alpha + \sup_{\alpha \in \mathcal{C}_a} \mathcal{I}_\epsilon^\alpha + \sup_{\alpha \in \mathcal{C}_a} \mathcal{J}_\epsilon^\alpha,$$

where

$$\begin{aligned}
\mathcal{T}_\epsilon^\alpha &:= |\mu_\epsilon(\ell^\alpha(\cdot,\mu_\epsilon)) - \nu_\epsilon(\ell^\alpha(\cdot,\nu_\epsilon))| \\
\mathcal{I}_\epsilon^\alpha &:= |(\mu_\epsilon - \nu_\epsilon)(\mathcal{M}^{\alpha,\mu_\epsilon}[\kappa_\epsilon](\cdot))| \\
\mathcal{J}_\epsilon^\alpha &:= |\nu_\epsilon(\mathcal{M}^{\alpha,\mu_\epsilon}[\kappa_\epsilon](\cdot) - \mathcal{M}^{\alpha,\nu_\epsilon}[\kappa_\epsilon](\cdot))| .
\end{aligned}$$

Step 7 (Estimating $\mathcal{T}_\epsilon^\alpha$). By the regularity Assumption 3.1 and the estimate (6.4),

$$\begin{aligned}
|\mu_\epsilon(\ell^\alpha(\cdot,\mu_\epsilon)) - \nu_\epsilon(\ell^\alpha(\cdot,\nu_\epsilon))| &\le |(\mu_\epsilon - \nu_\epsilon)(\ell^\alpha(\cdot,\mu_\epsilon))| + |\nu_\epsilon(\ell^\alpha(\cdot,\mu_\epsilon) - \ell^\alpha(\cdot,\nu_\epsilon))| \\
&\le \rho_*(\mu_\epsilon,\nu_\epsilon)\, \|\ell^\alpha(\cdot,\mu_\epsilon)\|_{\mathcal{C}_*} + \sup_{x \in \mathbb{T}^d} |\ell^\alpha(x,\mu_\epsilon) - \ell^\alpha(x,\nu_\epsilon)| \\
&\le 2c_a\, \rho_*(\mu_\epsilon,\nu_\epsilon) \le 2c_a L_w\, \epsilon.
\end{aligned}$$

Hence, we have $\lim_{\epsilon \downarrow 0} \sup_{\alpha \in \mathcal{C}_a} \mathcal{T}_\epsilon^\alpha = 0$.

Step 8 (Estimating $\mathcal{I}_\epsilon$). For $x \in \mathbb{T}^d$, $\mu \in \mathcal{P}(\mathbb{T}^d)$, $\alpha \in \mathcal{C}_a$, and $k \in \mathbb{Z}^d$ set

$$\beta_k^\alpha(x,\mu) := \mathcal{M}^{\alpha,\mu}[e_k^*](x) = -[k \cdot b^\alpha(x,\mu) + a_k^\alpha(x,\mu)] e_k^*(x),$$

where for $x \in \mathbb{T}^d, \mu \in \mathcal{P}(\mathbb{T}^d), \alpha \in \mathcal{C}_a, k \in \mathbb{Z}^d$,

$$a_k^\alpha(x,\mu) := \frac{1}{2} \sum_{i,j=1}^{d} \sum_{l=1}^{d'} \sigma_{il}(x,\mu,\alpha(x)) \sigma_{jl}(x,\mu,\alpha(x)) k_i k_j. \tag{6.6}$$

Then,

$$\mathcal{M}^{\alpha,\mu_\epsilon}[\kappa_\epsilon](x) = \frac{1}{\epsilon} \sum_{k \in \mathbb{Z}^d} \frac{1}{(1+|k|^2)^{n_*}}\; F_k(\mu_\epsilon - \nu_\epsilon)\; \beta_k^\alpha(x,\mu_\epsilon).$$

This in turn implies that

$$\begin{aligned}
\mathcal{I}_\epsilon^\alpha &\le \frac{1}{\epsilon} \sum_{k \in \mathbb{Z}^d} \frac{1}{(1+|k|^2)^{n_*}}\; |F_k(\mu_\epsilon - \nu_\epsilon)|\;\; |(\mu_\epsilon - \nu_\epsilon)(\beta_k^\alpha(\cdot,\mu_\epsilon))| \\
&\le \frac{1}{\epsilon}\, (\sum_{k \in \mathbb{Z}^d} \frac{|F_k(\mu_\epsilon - \nu_\epsilon)|^2}{(1+|k|^2)^{n_*}})^{\frac{1}{2}}\; (\sum_{k \in \mathbb{Z}^d} \frac{((\mu_\epsilon - \nu_\epsilon)(\beta_k^\alpha(\cdot,\mu_\epsilon)))^2}{(1+|k|^2)^{n_*}})^{\frac{1}{2}} \\
&\le \frac{\rho_*(\mu_\epsilon,\nu_\epsilon)}{\epsilon}\, (\sum_{k \in \mathbb{Z}^d} (1+|k|^2)^{2-n_*}\; \beta_{k,\epsilon}^2\;)^{\frac{1}{2}},
\end{aligned}$$

where

$$\beta_{k,\epsilon} := (1+|k|^2)^{-1} \sup_{\alpha\in\mathcal{C}_a} |(\mu_\epsilon - \nu_\epsilon)(\beta_k^\alpha(\cdot,\mu_\epsilon))|\,, \qquad \in \mathbb{Z}.$$

Again by Assumption 3.1, $|\beta_{k,\epsilon}| \le c_a + c_a^2$, and $\beta_{k,\epsilon}^\alpha$ is Lipschitz continuous with a Lipschitz constant c_k uniformly in α. Hence, by Kantorovich duality $\beta_{k,\epsilon} \le c_k \widehat{\rho}_1(\mu_\epsilon,\nu_\epsilon)$. As $\mu_\epsilon - \nu_\epsilon$ converges weakly to zero, we conclude that $\beta_{k,\epsilon}$ also converges to zero for every $k \in \mathbb{Z}$. Also $c(d) = \sum_{k=1}^\infty (1+|k|^2)^{2-n_*}$ is finite by (6.1), and we have argued that $|\beta_{k,\epsilon}|$ is uniformly bounded. Hence, we may use dominated convergence to conclude that the sequence $\sum_{k=1}^\infty (1+|k|^2)^{2-n_*}\beta_{k,\epsilon}^2$ converges to zero as $\epsilon \downarrow 0$. Then, by (6.4),

$$\lim_{\epsilon\downarrow 0} \sup_{\alpha\in\mathcal{C}_a} \mathcal{I}_\epsilon^\alpha \le \lim_{\epsilon\downarrow 0} L_w \Big(\sum_{k=1}^\infty (1+|k|^2)^{2-n_*}\ \beta_{k,\epsilon}^2\ \Big)^{\frac{1}{2}} = 0.$$

Step 9 (Estimating $\mathcal{J}_\epsilon$). The definition of $\mathcal{J}_\epsilon^\alpha$ imply that

$$\mathcal{J}_\epsilon^\alpha \le \sup_{x\in\mathbb{T}^d} \{|\mathcal{M}^{\alpha,\mu_\epsilon}[\kappa_\epsilon](x) - \mathcal{M}^{\alpha,\nu_\epsilon}[\kappa_\epsilon](x)|\}.$$

Let a_k^α be as in (6.6), and for $\alpha \in \mathcal{C}_a$, $x \in \mathbb{T}^d$, $k \in \mathbb{Z}^d$, set

$$\begin{aligned}\gamma_{k,\epsilon}^\alpha(x) &:= \mathcal{M}^{\alpha,\mu_\epsilon}[e_k^*](x) - \mathcal{M}^{\alpha,\nu_\epsilon}[e_k^*](x)\\ &= k\cdot[b^\alpha(x,\nu_\epsilon) - b^\alpha(x,\mu_\epsilon)]e_k^*(x) + [a_k^\alpha(x,\nu_\epsilon) - a_k^\alpha(x,\mu_\epsilon)]e_k^*(x).\end{aligned}$$

By the regularity Assumption 3.1, there exists c_2 such that

$$\sup_{x\in\mathbb{T}^d} |\gamma_{k,\epsilon}^\alpha(x)| \le c_2(1+|k|^2)\widehat{\rho}_*(\mu_\epsilon,\nu_\epsilon), \qquad \forall \alpha\in\mathcal{C}_a,\ k\in\mathbb{Z}^d.$$

Hence, for every $\boldsymbol{\alpha} \in \mathcal{A}$,

$$\begin{aligned}\mathcal{J}_\epsilon^\alpha &\le \frac{1}{\epsilon}\sum_{k\in\mathbb{Z}^d} \frac{|F_k(\mu_\epsilon-\nu_\epsilon)|}{(1+|k|^2)^{n_*}} \sup_{x\in\mathbb{T}^d} |\gamma_{k,\epsilon}^\alpha(x)|\\ &\le \frac{c_2}{\epsilon}\Big(\sum_{k\in\mathbb{Z}^d} \frac{|F_k(\mu_\epsilon-\nu_\epsilon)|^2}{(1+|k|^2)^{n_*}}\Big)^{\frac{1}{2}} \Big(\sum_{k\in\mathbb{Z}^d} (1+|k|^2)^{2-n_*}\Big)^{\frac{1}{2}} \widehat{\rho}_*(\mu_\epsilon,\nu_\epsilon)\\ &\le c_2 L_w\ c(d)\,\widehat{\rho}_*(\mu_\epsilon,\nu_\epsilon) =: \hat{c}\ \widehat{\rho}_*(\mu_\epsilon,\nu_\epsilon),\end{aligned}$$

where $c(d)$ is as in (6.1). Therefore, $\lim_{\epsilon\downarrow 0}\ \sup_{\alpha\in\mathcal{C}_a} \mathcal{J}_\epsilon^\alpha \le \hat{c}\lim_{\epsilon\downarrow 0}\ \widehat{\rho}_*(\mu_\epsilon,\nu_\epsilon) = 0.$

Step 10 (Conclusion). By (6.5) and above steps, $0 < \delta \le \lim_{\epsilon\downarrow 0} [H(\mu_\epsilon,\kappa_\epsilon) - H(\nu_\epsilon,\kappa_\epsilon)] \le 0$. This clear contradiction implies that $\max_{\overline{\mathcal{O}}}(u-w) \le 0$. □

7 Lipschitz continuity

In this section, we prove Theorem 4.2.

7.1 Regularity in space

We first prove the continuous dependence of the solutions of the McKean-Vlasov stochastic differential equation (3.1) on its initial data.

Proposition 7.1. *Suppose that the regularity Assumption 3.1 holds. Then, there exists* $\hat{c} > 0$ *depending on* T *and the constant* c_a *of Assumption 3.1, such that*

$$\widehat{\rho}_*(\mathcal{L}_u^{t,\mu,\boldsymbol{\alpha}}, \mathcal{L}_u^{t,\nu,\boldsymbol{\alpha}}) \le \hat{c}\,\widehat{\rho}_*(\mu,\nu), \quad \forall\ 0\le t\le u\le T,\ \mu,\nu\in\mathcal{P}(\mathbb{T}^d),\ \boldsymbol{\alpha}\in\mathcal{A}.$$

Proof. We complete the proof in several steps.

Step 1 (Setting). We fix $t \in [0,T]$, $\mu, \nu \in \mathcal{P}(\mathbb{T}^d)$, $\boldsymbol{\alpha} \in \mathcal{A}$, and set

$$Y_u := X_u^{t,\mu,\boldsymbol{\alpha}}, \ \ \mu_u := \mathcal{L}_u^{t,\mu,\boldsymbol{\alpha}}, \qquad Z_u := X_u^{t,\nu,\boldsymbol{\alpha}}, \ \ \nu_u := \mathcal{L}_u^{t,\nu,\boldsymbol{\alpha}}, \qquad u \in [t,T].$$

By the definition of $\widehat{\rho}_*$, we need to prove the following estimate for every $u \in [t,T]$,

$$(\mu_u - \nu_u)(\psi) \le \hat{c}\, \widehat{\rho}_*(\mu,\nu)\, \|\psi\|_{\mathcal{C}_*}, \qquad \forall\, \psi \in \mathcal{C}_*.$$

Step 2 (SDEs). For $x \in \mathbb{T}^d$, let Y^x, Z^x be the solutions of the stochastic differential equations,

$$\begin{aligned} Y_u^x &= x + \int_t^u \left[b^{\alpha_s}(Y_s^x, \mu_s)\mathrm{d}s + \sigma^{\alpha_s}(Y_s^x, \mu_s)\mathrm{d}B_s \right], \\ Z_u^x &= x + \int_t^u \left[b^{\alpha_s}(Z_s^x, \nu_s)\mathrm{d}s + \sigma^{\alpha_s}(Z_s^x, \nu_s)\mathrm{d}B_s \right]. \end{aligned}$$

Set $L_u^\mu(x) := \mathbb{E}[\psi(Y_u^x)]$, and $L_u^\nu(x) := \mathbb{E}[\psi(Z_u^x)]$. Then, by conditioning, we have

$$\mu_u(\psi) = \mathbb{E}[\psi(Y_u)] = \mu(L_u^\mu), \qquad \nu_u(\psi) = \mathbb{E}[\psi(Z_u)] = \nu(L_u^\nu).$$

Therefore,

$$(\mu_u - \nu_u)(\psi) = (\mu - \nu)(L_u^\mu) + \nu(L_u^\mu - L_u^\nu) =: \mathcal{I}_u(\psi) + \mathcal{J}_u(\psi).$$

Step 3 ($\mathcal{I}_u$ estimate). By the regularity Assumption 3.1, there exists a constant $\hat{c}_1$ satisfying

$$\|b^{\alpha_u}(\cdot, \mu_u)\|_{\mathcal{C}_*} + \|\sigma^{\alpha_u}(\cdot, \mu_u)\|_{\mathcal{C}_*} \le \hat{c}_1, \qquad \forall\ u \in [t,T].$$

Hence, the map $x \in \mathbb{T}^d \to Y_u^x$ is n_* times differentiable. Therefore, $L_u^\mu \in \mathcal{C}_*$ and there exists a constant $\hat{c}_2 > 0$ depending only on c_a of Assumption 3.1, satisfying,

$$\|L_u^\mu\|_{\mathcal{C}_*} \le \hat{c}_2\, \|\psi\|_{\mathcal{C}_*}, \qquad \forall\ u \in [t,T], \mu \in \mathcal{P}(\mathbb{T}^d).$$

This implies that

$$\mathcal{I}_u(\psi) = (\mu - \nu)(L_u^\mu) \le \hat{c}_2\, \widehat{\rho}_*(\mu,\nu)\, \|\psi\|_{\mathcal{C}_*}.$$

Step 4 ($\mathcal{J}_u$ estimate). By definitions, $\mathcal{J}_\le \sup_x |L_u^\mu(x) - L_u^\nu(x)|$, and

$$|L_u^\mu - L_u^\nu| \le \mathbb{E}[|\psi(Y_u^x) - \psi(Z_u^x)|] \le \mathbb{E}[|Y_u^x - Z_u^x|]\, \|\psi\|_1 \le (\mathbb{E}[(Y_s^x - Z_s^x)^2])^{\frac{1}{2}}\, \|\psi\|_*.$$

For $x \in \mathbb{T}^d$, and set $m_s^2(x) := \mathbb{E}[(Y_s^x - Z_s^x)^2]$. We directly estimate that

$$m_u^2(x) \le 2T \int_t^u \mathbb{E}[(b^{\alpha_s}(Y_s^x, \mu_s) - b^{\alpha_s}(Z_s^x, \nu_s))^2]\mathrm{d}s + 2\int_t^u \mathbb{E}[|\sigma^{\alpha_s}(Y_s^x, \mu_s) - \sigma^{\alpha_s}(Z_s^x, \nu_s)|^2]\mathrm{d}s.$$

By the regularity Assumption 3.1,

$$|b^{\alpha_s}(Y_s^x, \mu_s) - b^{\alpha_s}(Z_s^x, \nu_s)| \le c_a \left[|Y_s^x - Z_s^x| + \widehat{\rho}_*(\mu_s, \nu_s) \right].$$

Same estimate also holds for $|\sigma^{\alpha_s}(Y_s^x, \mu_s) - \sigma^{\alpha_s}(Z_s^x, \nu_s)|$. Hence, there exists a constant $\hat{c}_3 > 0$, independent of x, satisfying, $m_u^2 \le \hat{c}_3 \int_t^u [m_s^2 + \widehat{\rho}_*(\mu_s, \nu_s)^2]\mathrm{d}s$ for every $u \in [t,T]$. By Grönwall's inequality, there exists $\hat{c}_4 > 0$ satisfying $m_u^2 \le \hat{c}_4^2 \int_t^u \widehat{\rho}_*(\mu_s, \nu_s)^2 \mathrm{d}s$. Hence,

$$\mathcal{J}_u \le (\mathbb{E}[(Y_s^x - Z_s^x)^2])^{\frac{1}{2}}\, \|\psi\|_* \le \hat{c}_4 \left(\int_t^u \widehat{\rho}_*(\mu_s, \nu_s)^2\ \mathrm{d}s \right)^{\frac{1}{2}} \|\psi\|_{\mathcal{C}_*}, \qquad \forall u \in [t,T].$$

Step 5 (Conclusion). By the previous steps,

$$(\mu_u - \nu_u)(\psi) \le \left(\hat{c}_2\, \widehat{\rho}_*(\mu,\nu) \,+ \hat{c}_4 \, \left(\int_t^u \widehat{\rho}_*(\mu_s,\nu_s)^2 \, \mathrm{d}s \right)^{\frac{1}{2}} \right) \|\psi\|_{\mathcal{C}_*}, \qquad \forall\, \psi \in \mathcal{C}_*.$$

Since above holds for every $\psi \in \mathcal{C}_*$, the definition of $\widehat{\rho}_*$ implies that

$$\widehat{\rho}_*(\mu_u,\nu_u) \le \hat{c}_2\, \widehat{\rho}_*(\mu,\nu) \,+ \hat{c}_4 \, \left(\int_t^u \widehat{\rho}_*(\mu_s,\nu_s)^2 \, \mathrm{d}s \right)^{\frac{1}{2}}, \qquad \forall\, u \in [t,T].$$

Hence,

$$\widehat{\rho}_*(\mu_u,\nu_u)^2 \le 2\hat{c}_2^2\, \widehat{\rho}_*(\mu,\nu)^2 \,+ 2\hat{c}_4^2 \int_t^u \widehat{\rho}_*(\mu_s,\nu_s)^2 \mathrm{d}s, \qquad \forall\, u \in [t,T].$$

Again by Grönwall, $\widehat{\rho}_*(\mu_u,\nu_u)^2 \le \hat{c}^2\, \widehat{\rho}_*(\mu,\nu)^2$ for some $\hat{c} > 0$, for all $u \in [t,T]$. □

The following is an immediate consequence of the above estimate.

Lemma 7.2. *Under the regularity Assumption 3.1, there exists* $L_1 > 0$ *such that*

$$|J(t,\mu,\boldsymbol{\alpha}) - J(t,\nu,\boldsymbol{\alpha})| \le L_1\, \widehat{\rho}_*(\mu,\nu) \qquad \forall\, \boldsymbol{\alpha} \in \mathcal{A},\ \mu,\nu \in \mathcal{P}(\mathbb{T}^d),\ t \in [0,T].$$

Consequently,

$$|v(t,\mu) - v(t,\nu)| \le L_1\, \widehat{\rho}_*(\mu,\nu) \qquad \forall\, \mu,\nu \in \mathcal{P}(\mathbb{T}^d),\ t \in [0,T].$$

Proof. We fix $\boldsymbol{\alpha} \in \mathcal{A}$, $\mu,\nu \in \mathcal{P}(\mathbb{T}^d)$, $t \in [0,T]$, and use the same notation as in Proposition 7.1. For $u \in [t,T]$, the regularity Assumption 3.1 implies that

$$\begin{aligned}
&|\mathbb{E}[\ell^{\alpha_u}(Y_u,\mu_u) - \ell^{\alpha_u}(Z_u,\nu_u)]| \\
&\qquad \le |\mathbb{E}[\ell^{\alpha_u}(Y_u,\mu_u) - \ell^{\alpha_u}(Z_u,\mu_u)]| + |\mathbb{E}[\ell^{\alpha_u}(Z_u,\mu_u) - \ell^{\alpha_u}(Z_u,\nu_u)]| \\
&\qquad \le |(\mu_u - \nu_u)(\ell^{\alpha_u}(\cdot,\mu_u))| + c_a\, \widehat{\rho}_*(\mu_u,\nu_u) \\
&\qquad \le \widehat{\rho}_*(\mu_u,\nu_u) \|\ell^{\alpha_u}(\cdot,\mu_u)\|_{\mathcal{C}_*} + c_a\, \widehat{\rho}_*(\mu_u,\nu_u) \\
&\qquad \le 2c_a\, \widehat{\rho}_*(\mu_u,\nu_u) \le 2c_a\, \hat{c}\, \widehat{\rho}_*(\mu,\nu).
\end{aligned}$$

We now directly estimate using the above to obtain the following inequalities,

$$\begin{aligned}
|J(t,\mu,\boldsymbol{\alpha}) - J(t,\nu,\boldsymbol{\alpha})| &\le \int_t^T |\mathbb{E}[\ell^{\alpha_u}(Y_u,\mu_u) - \ell^{\alpha_u}(Z_u,\nu_u)]| \, \mathrm{d}u \;+\; |\mathbb{E}[\varphi(\mu_T) - \varphi(\nu_T)]| \\
&\le 2c_a\, \hat{c}\,(T-t)\, \widehat{\rho}_*(\mu,\nu) + c_a\, \widehat{\rho}_*(\mu_T,\nu_T) \\
&\le c_a\, \hat{c}\,(2(T-t)+1) \widehat{\rho}_*(\mu,\nu).
\end{aligned}$$

As $|v(t,\mu) - v(t,\nu)| \le \sup_{\boldsymbol{\alpha} \in \mathcal{A}} |J(t,\mu,\boldsymbol{\alpha}) - J(t,\nu,\boldsymbol{\alpha})|$, the proof of the lemma is complete. □

7.2 Time Regularity

Proposition 7.3. *Suppose that the regularity Assumption 3.1 holds. Then, there exists* $L_2 > 0$ *depending on* T *and the constant* c_a *in Assumption 3.1, such that*

$$|v(t,\mu) - v(\tau,\mu)| \le L_2\, |t-\tau|^{\frac{1}{2}}, \quad \forall\ t,\tau \in [0,T],\ \mu \in \mathcal{P}(\mathbb{T}^d).$$

Proof. Fix $0 \le t \le \tau \le T$, $\mu \in \mathcal{P}(\mathbb{T}^d)$, $\boldsymbol{\alpha} \in \mathcal{A}$, and set $h := \tau - t$. With an arbitrary constant $a_* \in A$, we define

$$\tilde{\alpha}_u(\cdot) := \begin{cases} \alpha_{u+h}(\cdot) & \text{if } u \in [t, T-h], \\ a_* & \text{if } u \in [T-h, T]. \end{cases}$$

It is clear that $\tilde{\boldsymbol{\alpha}} \in \mathcal{A}$. Set

$$\tilde{\mu}_u := \mathcal{L}_u^{t,\mu,\tilde{\boldsymbol{\alpha}}}, \quad u \in [t,T], \quad \text{and} \quad \mu_u := \mathcal{L}_u^{\tau,\mu,\boldsymbol{\alpha}}, \quad u \in [\tau, T].$$

Then, $\tilde{\mu}_u = \mu_{u+h}$ for every $u \in [t, T-h]$. In particular,

$$\mathbb{E}[\ell^{\tilde{\alpha}_u}(X_u^{t,\mu,\tilde{\boldsymbol{\alpha}}})] = \mathbb{E}[\ell^{\alpha_u}(X_{u+h}^{\tau,\mu,\boldsymbol{\alpha}})], \qquad \forall\, u \in [t, T-h].$$

Since $\mu_T = \tilde{\mu}_{T-h} = \mathcal{L}(X_{T-h}^{t,\mu,\tilde{\boldsymbol{\alpha}}})$, and $\tilde{\mu}_T = \mathcal{L}(X_T^{t,\mu,\tilde{\boldsymbol{\alpha}}})$,

$$\widehat{\rho}_1(\tilde{\mu}_T, \mu_T) \le \mathbb{E}[|X_T^{t,\mu,\tilde{\boldsymbol{\alpha}}} - X_{T-h}^{t,\mu,\tilde{\boldsymbol{\alpha}}}|] \le \left(\mathbb{E}[(X_T^{t,\mu,\tilde{\boldsymbol{\alpha}}} - X_{T-h}^{t,\mu,\tilde{\boldsymbol{\alpha}}})^2]\right)^{\frac{1}{2}}.$$

As b, σ are bounded by c_a, there is $\tilde{c}_1 > 0$ satisfying, $\widehat{\rho}_1(\tilde{\mu}_T, \mu_T) \le \tilde{c}_1 \sqrt{h}$. Therefore,

$$|\varphi(\tilde{\mu}_T) - \varphi(\mu_T)| \le c_a \widehat{\rho}_*(\tilde{\mu}_T, \mu_T) \le c_a \widehat{\rho}_1(\tilde{\mu}_T, \mu_T) \le \tilde{c}_1\, c_a \sqrt{h}.$$

Above estimate imply that for any $\boldsymbol{\alpha} \in \mathcal{A}$,

$$\begin{aligned} v(t,\mu) - J(\tau,\mu,\boldsymbol{\alpha}) &\le J(t,\mu,\tilde{\boldsymbol{\alpha}}) - J(\tau,\mu,\boldsymbol{\alpha}) \\ &= \int_{T-h}^{T} \mathbb{E}[\ell^{\tilde{\alpha}_u}(X_u^{t,\mu,\tilde{\boldsymbol{\alpha}}})]\,\mathrm{d}u + \varphi(\tilde{\mu}_T) - \varphi(\mu_T) \le c_a h + \tilde{c}_1\, c_a \sqrt{h}. \end{aligned}$$

Hence,

$$v(t,\mu) - v(\tau,\mu) = \sup_{\boldsymbol{\alpha} \in \mathcal{A}} \left(v(t,\mu) - J(t,\mu,\boldsymbol{\alpha})\right) \le c_a h + \tilde{c}_1\, c_a \sqrt{h}.$$

We prove the opposite inequality by using the control

$$\hat{\alpha}_u(\cdot) := \begin{cases} \alpha_{u-h}(\cdot) & \text{if } u \in [h, T], \\ a_* & \text{if } u \in [0, h]. \end{cases}$$

Again $\hat{\boldsymbol{\alpha}} \in \mathcal{A}$, and we set

$$\hat{\mu}_u := \mathcal{L}_u^{\tau,\mu,\hat{\boldsymbol{\alpha}}}, \quad u \in [\tau, T], \quad \text{and} \quad \mu_u := \mathcal{L}_u^{t,\mu,\boldsymbol{\alpha}}, \quad u \in [t, T].$$

Then $\hat{\mu}_u = \mu_{u-h}$ for every $u \in [\tau, T]$ and $\hat{\mu}_T = \mu_{T-h}$. Following the above steps *mutatis mutandis*, we obtain the following inequality for any $\boldsymbol{\alpha} \in \mathcal{A}$,

$$\begin{aligned} v(\tau,\mu) - J(t,\mu,\alpha) &\le J(\tau,\mu,\hat{\boldsymbol{\alpha}}) - J(t,\mu,\alpha) \\ &= -\int_{t}^{\tau} \mathbb{E}[\ell^{\tilde{\alpha}_u}(X_u^{t,\mu,\boldsymbol{\alpha}})]\,\mathrm{d}u + \varphi(\hat{\mu}_t) - \varphi(\mu_T) \le c_a h + \tilde{c}_1\, c_a \sqrt{h}. \end{aligned}$$

Hence,

$$v(\tau,\mu) - v(t,\mu) = \sup_{\boldsymbol{\alpha} \in \mathcal{A}} \left(v(\tau,\mu) - J(t,\mu,\boldsymbol{\alpha})\right) \le c_a h + \tilde{c}_1\, c_a \sqrt{h}.$$

□

8 Dynamic Programming

In this section we prove Theorem 3.3. For a general result but in a different setting, we refer the reader to [18].

Proof of Theorem 3.3. We fix $(t,\mu) \in \overline{\mathcal{O}}$, $\tau \in [t,T]$, and set

$$Q(\boldsymbol{\alpha}) := \int_t^\tau \mathbb{E}[\ell^{\alpha_s}(X_s^{t,\mu,\boldsymbol{\alpha}}, \mathcal{L}_s^{t,\mu,\boldsymbol{\alpha}})]\mathrm{d}s + v(\tau, \mathcal{L}_\tau^{t,\mu,\boldsymbol{\alpha}}), \qquad \boldsymbol{\alpha} \in \mathcal{A}.$$

Then, the dynamic programming principle can be stated as $v(t,\mu) = \inf_{\boldsymbol{\alpha}\in\mathcal{A}} Q(\boldsymbol{\alpha})$. Recall that $v(t,\mu) = \inf_{\boldsymbol{\alpha}\in\mathcal{A}} J(t,\mu,\boldsymbol{\alpha})$. For any $\boldsymbol{\alpha} \in \mathcal{A}$, and $s \in [\tau, T]$, Markov property implies that $X_s^{t,\mu,\boldsymbol{\alpha}} = X_s^{\tau,\mathcal{L}_\tau^{t,\mu,\boldsymbol{\alpha}},\boldsymbol{\alpha}}$, and consequently $\mathcal{L}_s^{t,\mu,\boldsymbol{\alpha}} = \mathcal{L}_s^{\tau,\mathcal{L}_\tau^{t,\mu,\boldsymbol{\alpha}},\boldsymbol{\alpha}}$. Hence,

$$\begin{aligned}
&\int_\tau^T \mathbb{E}[\ell^{\alpha_s}(X_s^{t,\mu,\boldsymbol{\alpha}}, \mathcal{L}_s^{t,\mu,\boldsymbol{\alpha}})]\ \mathrm{d}s + \varphi(\mathcal{L}_T^{t,\mu,\boldsymbol{\alpha}})\\
&\qquad = \int_\tau^T \mathbb{E}[\ell^{\alpha_s}(X_s^{\tau,\mathcal{L}_\tau^{t,\mu,\boldsymbol{\alpha}},\boldsymbol{\alpha}}, \mathcal{L}_s^{\tau,\mathcal{L}_\tau^{t,\mu,\boldsymbol{\alpha}},\boldsymbol{\alpha}})]\ \mathrm{d}s + \varphi(\mathcal{L}_T^{\tau,\mathcal{L}_\tau^{t,\mu,\boldsymbol{\alpha}},\boldsymbol{\alpha}})\\
&\qquad = J(\tau, \mathcal{L}_\tau^{\tau,\mathcal{L}_\tau^{t,\mu,\boldsymbol{\alpha}},\boldsymbol{\alpha}}, \boldsymbol{\alpha}) \geq v(\tau, \mathcal{L}_\tau^{\tau,\mathcal{L}_\tau^{t,\mu,\boldsymbol{\alpha}},\boldsymbol{\alpha}}).
\end{aligned}$$

This implies that

$$\begin{aligned}
J(t,\mu,\boldsymbol{\alpha}) &= \int_t^\tau \mathbb{E}[\ell^{\alpha_s}(X_s^{t,\mu,\boldsymbol{\alpha}}, \mathcal{L}_s^{t,\mu,\boldsymbol{\alpha}})]\ \mathrm{d}s + \left(\int_\tau^T \mathbb{E}[\ell^{\alpha_s}(X_s^{t,\mu,\boldsymbol{\alpha}}, \mathcal{L}_s^{t,\mu,\boldsymbol{\alpha}}))]\ \mathrm{d}s + \varphi(\mathcal{L}_T^{t,\mu,\boldsymbol{\alpha}})\right)\\
&\geq \int_t^\tau \mathbb{E}[\ell^{\alpha_s}(X_s^{t,\mu,\boldsymbol{\alpha}}, \mathcal{L}_s^{t,\mu,\boldsymbol{\alpha}}))]\ \mathrm{d}s + v(\tau, \mathcal{L}_\tau^{\tau,\mathcal{L}_\tau^{t,\mu,\boldsymbol{\alpha}},\boldsymbol{\alpha}}) = Q(\boldsymbol{\alpha}).
\end{aligned}$$

Therefore, $v(t,\mu) = \inf_{\boldsymbol{\alpha}\in\mathcal{A}} J(t,\mu,\boldsymbol{\alpha}) \geq \inf_{\boldsymbol{\alpha}\in\mathcal{A}} Q(\boldsymbol{\alpha})$.

To prove the opposite inequality, we fix $\epsilon > 0$, and set $\delta := \epsilon/(4L_1)$. By Lemma 7.2, whenever $\widehat{\rho}_*(\nu,\eta) \leq \delta$, we have $|J(\tau,\nu,\boldsymbol{\alpha}) - J(\tau,\eta,\boldsymbol{\alpha})| \leq \epsilon/4$, for every $\boldsymbol{\alpha} \in \mathcal{A}$, and also $|v(\tau,\nu) - v(\tau,\eta)| \leq \epsilon/4$. Consider a covering of $\mathcal{P}(\mathbb{T}^d)$ given by

$$\mathcal{B}(\nu) := \{\eta \in \mathcal{P}(\mathbb{T}^d)\ :\ \widehat{\rho}_*(\nu,\eta) < \delta\ \}, \qquad \nu \in \mathcal{P}(\mathbb{T}^d).$$

It is clear that each $\mathcal{B}(\nu)$ is an open set as $\widehat{\rho}_*$ is continuous with respect to the weak* topology. Then, since $\mathcal{P}(\mathbb{T}^d)$ is weak* compact, there exits $\{\nu_j\}_{j=1,\dots,n} \subset \mathcal{P}(\mathbb{T}^d)$ such that $\mathcal{P}(\mathbb{T}^d) = \cup_{j=1}^n \mathcal{B}(\nu_j)$. Set $\mathcal{B}_1 := \mathcal{B}(\nu_1)$, and and recursively define

$$\mathcal{B}_{j+1} := \mathcal{B}(\nu_{j+1}) \setminus \cup_{l=1}^j \mathcal{B}_l, \qquad j = 1, \dots, n-1,$$

so that $\{\mathcal{B}_j\}_{j=1,\dots,n}$ forms a disjoint covering of $\mathcal{P}(\mathbb{T}^d)$. Moreover, for any $\nu \in \mathcal{B}_j \subset \mathcal{B}(\nu_j)$, $\widehat{\rho}_*(\nu,\nu_j) \leq \delta$, and therefore,

$$|v(\tau,\nu) - v(\tau,\nu_j)| \leq \frac{\epsilon}{4}, \quad \text{and} \quad |J(\tau,\nu,\boldsymbol{\alpha}) - J(\tau,\nu_j,\boldsymbol{\alpha})| \leq \frac{\epsilon}{4}, \quad \forall\ \boldsymbol{\alpha} \in \mathcal{A}.$$

For each j, choose $\boldsymbol{\alpha}^j \in \mathcal{A}$ so that $J(\tau,\nu_j,\boldsymbol{\alpha}^j) \leq v(\tau,\nu_j) + \frac{\epsilon}{4}$. Then,

$$J(\tau,\nu,\boldsymbol{\alpha}^j) \leq J(\tau,\nu_j,\boldsymbol{\alpha}^j) + \frac{\epsilon}{4} \leq v(\tau,\nu_j) + \frac{\epsilon}{2} \leq v(\tau,\nu) + \frac{3\epsilon}{4}, \qquad \forall\ \nu \in \mathcal{B}_j. \tag{8.1}$$

We choose $\boldsymbol{\alpha}^* \in \mathcal{A}$ satisfying $Q(\boldsymbol{\alpha}^*) \leq \inf_{\boldsymbol{\alpha}\in\mathcal{A}} Q(\boldsymbol{\alpha}) + \frac{\epsilon}{4}$, and define a control process $\boldsymbol{\alpha}^\epsilon$ by,

$$\alpha_u^\epsilon(x) = \begin{cases} \alpha_u^*(x), & \text{if } u \in [t,\tau),\\ \sum_{j=1}^n \alpha_u^j(x)\chi_{\mathcal{B}_j}(\mathcal{L}_\tau^{t,\mu,\boldsymbol{\alpha}^*}), & \text{if } u \in [\tau,T], \end{cases} \qquad x \in \mathbb{T}^d.$$

As $\boldsymbol{\alpha}^*$ and $\boldsymbol{\alpha}^\epsilon$ agree on $[t,\tau]$, we have $\mathcal{L}_u^{t,\mu,\boldsymbol{\alpha}^*} = \mathcal{L}_u^{t,\mu,\boldsymbol{\alpha}^\epsilon}$ for all $u \in [t,\tau]$. Hence,

$$\inf_{\boldsymbol{\alpha}\in\mathcal{A}} Q(\boldsymbol{\alpha}) + \frac{\epsilon}{4} \geq Q(\boldsymbol{\alpha}^*) = \int_t^\tau \mathbb{E}[\ell^{\alpha_s^*}(X_s^{t,\mu,\boldsymbol{\alpha}^*}, \mathcal{L}_s^{t,\mu,\boldsymbol{\alpha}^*})]\mathrm{d}s + v(\tau, \mathcal{L}_\tau^{t,\mu,\boldsymbol{\alpha}^*}) = Q(\boldsymbol{\alpha}^\epsilon).$$

Moreover, by the definition of $\boldsymbol{\alpha}^\epsilon$ and (8.1),

$$\begin{aligned} v(\tau, \mathcal{L}_\tau^{t,\mu,\boldsymbol{\alpha}^*}) &= \sum_{j=1}^n v(\tau, \mathcal{L}_\tau^{t,\mu,\boldsymbol{\alpha}^*})\chi_{\mathcal{B}_j}(\mathcal{L}_\tau^{t,\mu,\boldsymbol{\alpha}^*}) \geq \sum_{j=1}^n J(\tau, \mathcal{L}_\tau^{t,\mu,\boldsymbol{\alpha}^*}, \boldsymbol{\alpha}^j)\chi_{\mathcal{B}_j}(\mathcal{L}_\tau^{t,\mu,\boldsymbol{\alpha}^*}) - \frac{3\epsilon}{4} \\ &= J(\tau, \mathcal{L}_\tau^{t,\mu,\boldsymbol{\alpha}^*}, \boldsymbol{\alpha}^\epsilon) - \frac{3\epsilon}{4}. \end{aligned}$$

Hence,

$$\begin{aligned} \inf_{\boldsymbol{\alpha}\in\mathcal{A}} Q(\boldsymbol{\alpha}) + \epsilon \geq Q(\boldsymbol{\alpha}^\epsilon) + \frac{3\epsilon}{4} &= \int_t^\tau \mathbb{E}[\ell^{\alpha_s^*}(X_s^{t,\mu,\boldsymbol{\alpha}^*}, \mathcal{L}_s^{t,\mu,\boldsymbol{\alpha}^*})]\mathrm{d}s + \Big(v(\tau, \mathcal{L}_\tau^{t,\mu,\boldsymbol{\alpha}^*}) + \frac{3\epsilon}{4}\Big) \\ &\geq \int_t^\tau \mathbb{E}[\ell^{\alpha_s^*}(X_s^{t,\mu,\boldsymbol{\alpha}^*}, \mathcal{L}_s^{t,\mu,\boldsymbol{\alpha}^*})]\mathrm{d}s + J(\tau, \mathcal{L}_\tau^{t,\mu,\boldsymbol{\alpha}^\epsilon}, \boldsymbol{\alpha}^\epsilon) \\ &= J(t,\mu,\boldsymbol{\alpha}^\epsilon) \geq v(t,\mu). \end{aligned}$$

□

9 Viscosity property

In this section, we prove the viscosity property of the value function. Although the below proof follows the standard one very closely, we provide it for completeness.

The following version of the Itô's formula along flows of measures follows from Proposition 5.102 of [8]. Recall that $X^{t,\mu,\boldsymbol{\alpha}}$ is the solution of (3.1), $\mathcal{L}_u^{t,\mu,\boldsymbol{\alpha}} = \mathcal{L}(X_u^{t,\mu,\boldsymbol{\alpha}})$, and the operator $\mathcal{M}^{a,\mu}$ is defined in subsection 3.3.

Lemma 9.1. *For every $\psi \in \mathcal{C}_s(\mathbb{T}^d)$, $(t,\mu) \in \overline{\mathcal{O}}$, $u \in [t,T]$, and $\boldsymbol{\alpha} \in \mathcal{A}$,*

$$\psi(u, \mathcal{L}_u^{t,\mu,\boldsymbol{\alpha}}) = \psi(t,\mu) + \int_t^u \Big(\partial_t\psi(s, \mathcal{L}_s^{t,\mu,\boldsymbol{\alpha}}) + \mathbb{E}[\mathcal{M}^{\alpha_s, \mathcal{L}_s^{t,\mu,\boldsymbol{\alpha}}}[\partial_\mu\psi(s, \mathcal{L}_s^{t,\mu,\boldsymbol{\alpha}})](X_s^{t,\mu,\boldsymbol{\alpha}})]\Big)\,\mathrm{d}s.$$

9.1 Subsolution

Suppose that for $(t_0,\mu_0) \in [0,T) \times \mathcal{P}(\mathbb{T}^d)$ and test function $\psi \in \mathcal{C}_s(\overline{\mathcal{O}})$,

$$0 = (v-\psi)(t_0,\mu_0) = \max_{\overline{\mathcal{O}}}(v-\psi).$$

For $\alpha \in \mathcal{C}_a$, set

$$k^\alpha(t,x,\mu) := \ell(x,\mu,\alpha(x)) + \mathcal{M}^{\alpha,\mu}[\partial_\mu\psi(t,\mu)])](x), \qquad t \in [0,T],\ x \in \mathbb{T}^d,\ \mu \in \mathcal{P}(\mathbb{T}^d).$$

As $H(\mu_0, \partial_\mu\psi(t_0,\mu_0)) = \inf_{\alpha\in\mathcal{C}_a} \mu_0(k^\alpha(t_0,\cdot,\mu_0))$, for any $\epsilon > 0$ there is $\alpha^* \in \mathcal{C}_a$ satisfying,

$$\mu_0(k^{\alpha^*}(t_0,\cdot,\mu_0)) \leq H(\mu_0, \partial_\mu\psi(t_0,\mu_0)) + \epsilon.$$

Set $\alpha_u^* \equiv \alpha_*$ and let $X_u^* := X_u^{t_0,\mu_0,\boldsymbol{\alpha}^*}$ and $\mu_u^* := \mathcal{L}_u^{t_0,\mu_0,\boldsymbol{\alpha}^*}$ for $u \in [t_0,T]$. Since $v \leq \psi$, dynamic programming principle Theorem 3.3 with $\tau = t_0 + h \leq T$ implies that

$$v(t_0,\mu_0) \leq \int_{t_0}^{t_0+h} \mathbb{E}[\ell(X_s^*, \mu_s^*, \alpha^*(X_s^*)]\mathrm{d}s + \psi(t_0+h, \mu_{t_0+h}^*).$$

By Lemma 9.1,

$$\psi(t_0+h,\mu^*_{t_0+h}) = \psi(t_0,\mu_0) + \int_{t_0}^{t_0+h} \Big(\partial_t\psi(s,\mu^*_s) + \mathbb{E}[\mathcal{M}^{\alpha^*,\mu^*_s}[\partial_\mu\psi(s,\mu^*_s)](X^*_s)]\Big)\,\mathrm{d}s.$$

Since $\psi(t_0,\mu_0) = v(t_0,\mu_0)$, above inequalities imply that

$$0 \le \frac{1}{h}\int_{t_0}^{t_0+h} \Big(\partial_t\psi(s,\mu^*_s) + \mathbb{E}[k^{\alpha^*}(s,X^*_s,\mu^*_s)]\Big)\,\mathrm{d}s. \tag{9.1}$$

We now let h tend to zero to arrive at the following inequality,

$$-\partial_t\psi(t_0,\mu_0) \le \mathbb{E}[k^{\alpha^*}(t_0,X_{t_0},\mu_{t_0})] = \mu_0(k^{\alpha^*}(t_0,\cdot,\mu_0)) \le H(\mu_0,\partial_\mu\psi(t_0,\mu_0)) + \epsilon.$$

9.2 Supersolution

Suppose that for $(t_0,\mu_0) \in [0,T)\times\mathcal{P}(\mathbb{T}^d)$ and a test function $\psi \in \mathcal{C}_s(\overline{\mathcal{O}})$,

$$0 = (v-\psi)(t_0,\mu_0) = \min_{\overline{\mathcal{O}}}(v-\psi).$$

We may assume that the minimum is strict. Towards a counterposition, suppose that

$$-\partial_t\psi(t_0,\mu_0) < H(\mu_0,\partial_\mu\psi(t_0,\mu_0)) = \inf_{\alpha\in\mathcal{C}_a}\{\mu_0(k^\alpha,\cdot,\mu_0))\},$$

where $k^\alpha(t,x,\mu) = \ell^\alpha(x,\mu) + \mathcal{M}^{\alpha,\mu}[\partial_\mu\psi(t,\mu)](x)$ is as in the previous subsection. By Definition 3.4 of test functions $\mathcal{C}_s(\mathcal{O})$, the map $(t,\mu)\in\overline{\mathcal{O}} \mapsto H(\mu,\partial_\mu\psi(t,\mu))$ is continuous. Therefore, there exists $\delta>0$ and a neighborhood $\mathcal{B}\subseteq\overline{\mathcal{O}}$ of (t_0,μ_0) such that

$$-\partial_t\psi(s,\mu) + \delta \le H(\mu,\partial_\mu\psi(t,\mu)) = \inf_{\alpha\in\mathcal{C}_a}\{\mu(k^\alpha(t,\cdot,\mu))\}, \qquad \forall\,(t,\mu)\in\mathcal{B}.$$

For $\boldsymbol{\alpha}\in\mathcal{A}$, set $X^{\boldsymbol{\alpha}}_s := X^{t_0,\mu_0,\boldsymbol{\alpha}}_s$, $\mu^{\boldsymbol{\alpha}}_s := \mathcal{L}^{t_0,\mu_0,\boldsymbol{\alpha}}_s$, and consider the (deterministic) time

$$\tau^{\boldsymbol{\alpha}} := \inf\{s\in[t_0,T] : (s,\mu^{\boldsymbol{\alpha}}_s)\notin\mathcal{B}\},$$

so that for every $s\in[t_0,\tau^{\boldsymbol{\alpha}})$, $(s,\mu^{\boldsymbol{\alpha}}_s)\in\mathcal{B}$, and consequently

$$\mu^{\boldsymbol{\alpha}}_s(k^{\alpha_s}(s,\cdot,\mu^{\boldsymbol{\alpha}}_s)) \ge H(\mu^{\boldsymbol{\alpha}}_s,\partial_\mu\psi(s,\mu^{\boldsymbol{\alpha}}_s)) \ge -\partial_t\psi(s,\mu^{\boldsymbol{\alpha}}_s) + \delta.$$

As $\mathbb{E}[k^{\alpha_s}(s,X^{\boldsymbol{\alpha}}_s,\mu^{\boldsymbol{\alpha}}_s)] = \mu^{\boldsymbol{\alpha}}_s(k^{\alpha_s}(s,\cdot,\mu^{\boldsymbol{\alpha}}_s))$,

$$\int_{t_0}^{\tau^{\boldsymbol{\alpha}}} (\mathbb{E}[k^{\alpha_s}(X^{\boldsymbol{\alpha}}_s,\mu^{\boldsymbol{\alpha}}_s)] + \partial_t\psi(s,\mu^{\boldsymbol{\alpha}}_s)\,)\,\mathrm{d}s \ge \delta(\tau^{\boldsymbol{\alpha}} - t_0).$$

Then, by Lemma 9.1, we obtain the following inequality,

$$\begin{aligned}
\psi(\tau^{\boldsymbol{\alpha}},\mu^{\boldsymbol{\alpha}}_{\tau^{\boldsymbol{\alpha}}}) &= \psi(t_0,\mu_0) + \int_{t_0}^{\tau^{\boldsymbol{\alpha}}} (\partial_t\psi(s,\mu^{\boldsymbol{\alpha}}_s) + \mathbb{E}[\mathcal{M}^{\alpha_s,\mu^{\boldsymbol{\alpha}}_s}[\partial_\mu\psi(s,\mu^{\boldsymbol{\alpha}}_s)](X^{\boldsymbol{\alpha}}_s)\;])\mathrm{d}s \\
&= \psi(t_0,\mu_0) + \int_{t_0}^{\tau^{\boldsymbol{\alpha}}} (\partial_t\psi(s,\mu^{\boldsymbol{\alpha}}_s) + \mathbb{E}[k^{\alpha_s}(s,X^{\boldsymbol{\alpha}}_s,\mu^{\boldsymbol{\alpha}}_s)] - \mathbb{E}[\ell^{\alpha_s}(X^{\boldsymbol{\alpha}}_s,\mu^{\boldsymbol{\alpha}}_s)])\,\mathrm{d}s \\
&\ge \psi(t_0,\mu_0) - \int_{t_0}^{\tau^{\boldsymbol{\alpha}}} \mathbb{E}[\ell^{\alpha_s}(X^{\boldsymbol{\alpha}}_s,\mu^{\boldsymbol{\alpha}}_s)]\,\mathrm{d}s + \delta(\tau^{\boldsymbol{\alpha}} - t_0).
\end{aligned}$$

Since $v \geq \psi$ and $\psi(t_0, \mu_0) = v(t_0, \mu_0)$, above implies that

$$v(t_0, \mu_0) \leq \int_{t_0}^{\tau^{\boldsymbol{\alpha}}} \mathbb{E}[\ell^{\alpha_s}(X_s^{\boldsymbol{\alpha}}, \mu_s^{\boldsymbol{\alpha}})] \, \mathrm{d}s + v(\tau^{\boldsymbol{\alpha}}, \mu_{\tau^{\boldsymbol{\alpha}}}^{\boldsymbol{\alpha}}) - g(\boldsymbol{\alpha}), \qquad \forall \, \boldsymbol{\alpha} \in \mathcal{A}.$$

where $g(\boldsymbol{\alpha}) := \delta(\tau^{\boldsymbol{\alpha}} - t_0) + (v(\tau^{\boldsymbol{\alpha}}, \mu_{\tau^{\boldsymbol{\alpha}}}^{\boldsymbol{\alpha}}) - \psi(\tau^{\boldsymbol{\alpha}}, \mu_{\tau^{\boldsymbol{\alpha}}}^{\boldsymbol{\alpha}}))$. We now claim that

$$\delta_0 := \inf_{\boldsymbol{\alpha} \in \mathcal{A}} g(\boldsymbol{\alpha}) > 0.$$

Indeed, since $v \geq \psi$, if $\tau^{\boldsymbol{\alpha}} = T$, then $g(\boldsymbol{\alpha}) \geq \delta(T - t_0)$. On the other hand if $\tau^{\boldsymbol{\alpha}} < T$, then $(\tau^{\boldsymbol{\alpha}}, \mu_{\tau^{\boldsymbol{\alpha}}}^{\boldsymbol{\alpha}}) \in \partial\mathcal{B}$. As $\mathcal{B}$ is compact and $(t_0, \mu_0) \notin \partial\mathcal{B}$ is the strict minimizer of $v - \psi$, we have

$$(v - \psi)(\tau^{\boldsymbol{\alpha}}, \mu_{\tau^{\boldsymbol{\alpha}}}^{\boldsymbol{\alpha}}) \geq \inf_{(t,\mu) \in \partial\mathcal{B}} (v - \psi)(t, \mu) > 0.$$

Hence, $\delta_0 > 0$ and the above inequalities imply that for *every* $\boldsymbol{\alpha} \in \mathcal{A}$,

$$v(t_0, \mu_0) \leq \int_{t_0}^{\tau^{\boldsymbol{\alpha}}} \mathbb{E}[\ell^{\alpha_s}(X_s^{\boldsymbol{\alpha}}, \mu_s^{\boldsymbol{\alpha}})] \mathrm{d}s + v(\tau^{\boldsymbol{\alpha}}, \mu_{\tau^{\boldsymbol{\alpha}}}^{\boldsymbol{\alpha}}) - \delta_0.$$

This contradiction to dynamic programming implies that $-\psi_t(t_0, \mu_0) \geq H(\mu_0, \partial_\mu \psi(t_0, \mu_0))$. ☐

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