Arbitrage under Knightian uncertainty

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- We consider a financial market without any probabilistic or topological structure but rather with a partial order representing the common beliefs of all agents.
- In this structure, we investigate the proper extensions of the classical notions of arbitrage and viability or the economic equilibrium. Then, study their implications.
- We provide a definition of arbitrage.
- Our contribution is to extend the classical works of Harrison & Kreps'79 and Kreps'81 to incorporate Knightian uncertainty.
- We also complete their work proving an equivalence between viability and no-arbitrage.

Outline

Context

Frank Knight in his 1921 book, *Risk, Uncertainty, and Profit*, formalized the distinction between risk and uncertainty.

▷ According to Knight, risk applies to situations where we do not know the outcome of a given situation, but can accurately measure the odds.
▷ Uncertainty applies to situations where we cannot know all the information we need in order to set accurate odds.

There is a fundamental distinction between the reward for taking a known risk and that for assuming a risk whose value itself is not known.

The Turner Review of May 2009 on

A regulatory response to the global financial crisis

"More fundamentally, however, it is important to realize that the assumption that past distribution patters carry robust inferences for the probability of future patterns is methodologically insecure. It involves applying to the world of social and economics relationships to a technique drawn from the world of physics, . . . it is unclear whether this analogy is valid when applied to economic and social relationships, or whether instead, we need to recognise that we are not dealing with mathematically modelable risk, but with inherent "Knightian uncertainty'. "

Ellsberg

Consider the experiment with two urns containing red and black balls :

- Urn 1 has exactly 50 red and 50 black balls.
- Urn 2 has only red and black balls of unknown number ;

Four random events are described :

- R_1 : Pays \$1 if a Red ball is from from Urn 1 ;
- B_1 : Pays \$1 if a Black ball is from from Urn 1 ;
- R_2 : Pays \$1 if a Red ball is from from Urn 2 ;
- B_2 : Pays \$1 if a Black ball is from from Urn 2.

Most people choose R_1 over R_2 and B_2 ; also B_1 over B_2 and R_2. And they are indifferent between R_1 and B_1 and R_2 and B_2.

No Probability explains this

Suppose we have a subjective probability $\mathbb{P}$.

We have four disjoint events R_1, R_2, B_1, B_2 with

- $\mathbb{P}(R_1) > \mathbb{P}(R_2)$;
- $\mathbb{P}(B_1) > \mathbb{P}(B_2)$;
- $R_1 \cup B_1 = R_2 \cup B_2$ is the whole space, i.e.,

$$\begin{aligned} 1 = \mathbb{P}(R_1 \cup B_1) &= \mathbb{P}(R_1) + \mathbb{P}(B_1) \\ &> \mathbb{P}(R_2) + \mathbb{P}(B_2) = \mathbb{P}(R_2 \cup B_2) = 1. \end{aligned}$$

For the same reason there is no linear pricing.

Consider an agent whose preference relation is given by a Gilboa-Schmeidler utility :

$$U_{GS}(X) := \min\{\mathbb{E}_a[X], \mathbb{E}_b[X]\},$$

where $a < \frac{1}{2} < b$ and $\mathbb{E}_p$ is expectation with a probability $\mathbb{P}_p$ such that $\mathbb{P}_p(R_1) = 0.5$, $\mathbb{P}_b(R_2) = p$.

The induced preference is $X \preceq Y$ if $U_{GS}(X) \leq U_{GS}(Y)$.

This agent is consistent with the observations :

$$U_{GS}(R_1) = 0.5 > U_{GS}(R_2) = a \qquad \Rightarrow \qquad R_1 \succ R_2,$$
$$U_{GS}(R_1) = 0.5 > U_{GS}(B_2) = 1 - b \qquad \Rightarrow \qquad R_1 \succ B_2,$$
$$U_{GS}(B_1) = 0.5 > U_{GS}(R_2) = a \qquad \Rightarrow \qquad B_1 \succ R_2,$$
$$U_{GS}(B_1) = 0.5 > U_{GS}(B_2) = 1 - b \qquad \Rightarrow \qquad B_1 \succ B_2.$$

Now consider a market in which $dS_t = S_t[rdt + \sigma_t dW_t]$, where there is no common agreement on the value of volatility but it is agreed that $\sigma_t \in [a, b]$. Let

$$\Sigma := \{\sigma : [0, T] \to [a, b] \mid \sigma \text{ is adapted } \}.$$

For each $\sigma \in \Sigma$ there is a probability measure $\mathbb{P}^\sigma$ which is the distribution of the corresponding stock price. This structure is similar to the Ellsberg examples, but here all individual measures are mutually orthogonal.

For a construction of these measures and other properties, see Soner, Touzi & Zhang (2012,2013) and Nutz (2013), Nutz & Soner (2012).

- As in the previous example, most markets with Knightian uncertainty demand a cloud of measures $\mathcal{Q}$.
- The quasi-sure order in this setting is natural :

$$X \leq_{\mathcal{Q}} Y \quad \Leftrightarrow \quad \mathbb{Q}(X \leq Y) = 1, \ \forall \mathbb{Q} \in \mathcal{Q}.$$

- The positive cone $\mathcal{K}$ is given by,

$$\begin{aligned} R \in \mathcal{K} \quad &\Leftrightarrow \quad R >_{\mathcal{Q}} 0, \\ &\Leftrightarrow \quad R \geq_{\mathcal{Q}} 0 \text{ and } \exists \mathbb{Q}^* \in \mathcal{Q} \text{ s.t. } \mathbb{Q}^*(R > 0) > 0. \end{aligned}$$

- In most Knigthian uncertain markets, $\mathcal{Q}$ is non-dominated. Then, the space $\mathcal{K}$ is " too large".

Foundations of Arbitrage

- Two papers : Harrison & Kreps (1979) and Kreps (1981) laid the economic foundations of arbitrage by connecting it to economic viability.
- Harrison & Kreps (1979) works on $\mathcal{L}^2(\Omega, \mathbb{P})$ with a given $\mathbb{P}$ and proves that viability is equivalent to the existence of a risk-neutral measure.
- Kreps (1981) considers a more abstract set-up.

- The commodity space (the set of all claims) $\mathcal{H}$ is a topological space.
- $\mathcal{K}$ is its positive cone.
- $\mathcal{M}$ is a subspace of $\mathcal{H}$ and prices on $\mathcal{M}$ are given by a linear map π.

Definition (Kreps '81)

$(\mathcal{H}, \mathcal{M}, \pi)$ is viable if there is a representative agent or equivalently a preference relation $\preceq$ that is complete, convex, continuous such that :

1. *$m \preceq 0$ for every $m \in \mathcal{M}$ with $\pi(m) \leq 0$.*
   This is an equilibrium condition ; it demands that the representative agents is able to choose optimal claim among all claims in $\mathcal{M}$ with budget constraint of zero. Note that the set of all agents is large. In particular, they have endowments.
2. *$\preceq$ is strictly increasing on $\mathcal{K}$.*
   This condition eliminates arbitrage.

Theorem (Harrison & Kreps'79, Kreps'81)

A market is viable if and only if its extendable, i.e., if there exists a linear, continuous, strictly monotone $\varphi : \mathcal{H} \to \mathbb{R}$*, which extends* π *:*

$$\varphi(m) = \pi(m) \;\; \forall m \in \mathcal{M}, \qquad \varphi(k) > 0, \quad \forall \; k \in \mathcal{K}.$$

The extension φ is the equivalent risk neutral measure. In this context strict monotonicity replaces equivalence.

There is no simple equivalence to no-arbitrage in these papers. Delbaen & Schachermayer proves this equivalence for markets with risk.

There is a filtered probability space $(\Omega, \mathbb{F}, \mathbb{P})$ and semi-martingale S representing the stock price process.

- Arbitrage is an admissible, predicable process so that $(H \cdot S)$ is bounded from below and $(H \cdot S)_T \geq 0$, $\mathbb{P}$-a.s., and $\mathbb{P}((H \cdot S)_T > 0) > 0$.
- A Free Lunch with Vanishing Risk is a sequence of admissible, predicable processes H^n so that $f_n := (H \cdot S)_T$ satisfies

$$f_n^- \to 0 \text{ uniformly}, \quad \|f_n - f\|_{\mathcal{L}^\infty(\Omega,\mathbb{P})} \to 0 \quad \text{and} \quad \mathbb{P}(f > 0) > 0.$$

Under risk (i.e., when the common order is $\mathbb{P}$ almost-sure withn a given probability measure), the following two are same :

- There is no Free Lunch with Vanishing Risk (NFLVR) if there is no $f_n := (H \cdot S)_T$ satisfying

$$f_n^- \to 0 \ \text{ uniformly}, \quad \|f_n - f\|_{\mathcal{L}^\infty(\Omega,\mathbb{P})} \to 0 \quad \text{and} \quad \mathbb{P}(f > 0) > 0.$$

- For every ξ with $\mathbb{P}(\xi \geq 0) = 1$ and $\mathbb{P}(\xi > 0) > 0$,

$$\mathcal{D}(\xi) := \inf\left\{x \in \mathbb{R} \ : \ \exists H \ \text{ so that } \ x + (H \cdot S)_T \geq \xi \ a.s. \ \right\} > 0.$$

Kardaras & Karatzas and Herdegen define viability exactly as above but for general markets.

- Kreps viability is equivalent to extendability.
- If a market is viable, then there is a strictly positive, linear measure. Hence, Kreps' definition of viability is not always compatible with Knightian uncertainty as such markets often do not have such functionals.
- In general markets, we want to modify Krebs definition appropriately so that its is equivalent to arbitrage.

Our Approach

Financial Market

- $\mathcal{H} = \mathcal{B}_b$ is the set of all bounded, Borel measurable random variables and any $X \in \mathcal{H}$ represents a claim.
- $\leq$ is a partial order on $\mathcal{H}$.

Examples

1. *If a probability measure is given, then* $X \leq Y$ *iff* $X \leq Y, \ \mathbb{P} - a.s.$
2. *If a cloud probability measures* $\mathcal{Q}$ *is given, then* $X \leq Y$ *iff* $X \leq Y, \ \mathcal{Q} - q.s.$

$\mathcal{A}$ is the set of a all preference relations (i.e., complete and transitive) satisfying

- monotone with respect to $\leq$, respects the common order ;
- convex, risk averse ;
- weakly continuous, i.e., for any sequence of real numbers $c_n \downarrow 0$,

$$-c_n + X \preceq Y, \quad \Rightarrow \quad X \preceq Y.$$

Example (Gilboa-Schmeidler)

Given a cloud $\mathcal{Q}$, and a utility function $u : \mathbb{R} \to \mathbb{R}$, set

$$U_{GS}(X) := \inf_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[u(X)].$$

Then define the preference relation by, $X \preceq Y$ if $U_{GS}(X) \leq U_{GS}(Y)$.

To complete the structure, we bring in the traded assets $\mathcal{I}$ (analog of $(\mathcal{M}, \pi)$ in Kreps'81) and also potential arbitrages, $\mathcal{R}$.

Marketed space $(\mathcal{H}, \leq, \mathcal{I}, \mathcal{R})$ is given by,

- $(\mathcal{H}, \leq)$ is an ordered vector space.
- $\mathcal{I}$ is a cone of claims that are liquidly traded with zero initial cost.
  Examples of $\mathcal{I}$ are stochastic integrals and/or liquidly traded options.
- $\mathcal{R}$ is an arbitrary convex subset of $\mathcal{H}^+$. We call it as the set of relevant contracts. We assume all positive constants are in $\mathcal{R}$.
- The set $\mathcal{R}$ determines arbitrage and could be equal to $\mathcal{H}^+$.

Example

This is related to Bouchard & Nutz (2015) and Biagini, Bouchard, Kardaras & Nutz (2017).

Let $\mathcal{P}$ be a cloud, $\leq_{\mathcal{P}}$ be the quasi-sure order and $\mathcal{R} = \mathcal{H}^+$, i.e.,

$$R \in \mathcal{R} \quad \Leftrightarrow \quad \inf_{\mathbb{P} \in \mathcal{P}} \mathbb{P}(R \geq 0) = 1, \quad \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{P}(R > 0) > 0.$$

If $\mathcal{P}$ is non-dominated, there exists no linear map φ on $\mathcal{B}_b(\Omega, \mathcal{P})$ satisfying $\varphi(R) > 0$ for every $R \in \mathcal{R}$. Hence, the market is not viable according to the definition of Kreps. We want to change that.

Viability

Definition (Viability)

A market $(\mathcal{H}, \leq, \mathcal{I}, \mathcal{R})$ is viable if there exists a set of heterogenous agents $\widehat{\mathcal{A}} \subset \mathcal{A}$ so that

1. $\ell \prec 0$, *for every* $\ell \in \mathcal{I}$ *and* $\preceq \in \widehat{\mathcal{A}}$.
2. *For every* $R \in \mathcal{R}$, *there is* $\preceq_R \in \widehat{\mathcal{A}}$ *so that* $0 \prec_R R$.

- First condition is equilibrium.
- The second condition replaces the strict monotonicity of Kreps. Also it is related to the set of risk-neutral measures $\mathcal{Q}$ being equivalent to $\mathcal{P}$ as in Bouchard & Nutz and the notion of arbitrage used there.

Arbitrage and Equivalence

Definition (Arbitrage)

A traded claim $\ell \in \mathcal{I}$ *is* *an arbitrage* *if there exists* $R \in \mathcal{R}$,

$$\ell \geq R.$$

Definition (Free Lunch with Vanishing Risk)

A sequence of traded claims $\{\ell^n\}_n \in \mathcal{I}$ *is called* *a free lunch with vanishing risk* *if there exists* $R \in \mathcal{R}$ *and a sequence of real numbers* $c_n \to 0$ *so that*

$$\ell^n + c_n \geq R, \quad \forall\ n = 1, 2, \ldots.$$

We define the super-replication functional by,

$$\mathcal{D}(X) := \inf \{ c \in \mathbb{R} \ : \ \exists \, \ell \in \mathcal{I} \ \text{ so that } \ c + \ell \geq X \ \}.$$

This is a convex functional and is Lipschitz in the supremum norm.

Lemma

There are no-free-lunches-with vanishing-risk (NFLVR), if and only if $\mathcal{D}(R) > 0$ for all $R \in \mathcal{R}$.

In our setting, Kardaras & Karatzas definition of viability is no arbitrage. We next show that it is equivalent to our notion of viability.

The super-replication functional

$$\mathcal{D}(X) := \inf \{c \in \mathbb{R} \ : \ \exists \, \ell \in \mathcal{I} \ \text{ so that } \ c + \ell \geq X \ \}$$

is convex, proper and Lipschtiz continuous. Also it is homogenous, i.e., $\mathcal{D}(\lambda X) = \lambda \mathcal{D}(X)$ for every $\lambda > 0$. By Fenchel-Moreau

$$\mathcal{D}(X) := \sup_{\varphi \in \mathcal{Q}} \ \varphi(X),$$

where

$$\mathcal{Q} = \{\varphi \in ba(\Omega) \ : \ \varphi(X) \leq \mathcal{D}(X), \ \ \forall \ X \in \mathcal{H}\}.$$

Theorem (Burzoni, Riedel, Soner, 2017)

A financial market is viable if and only if there are no free lunches with vanishing risk.

Proof : Suppose NFLVR holds. Then, $\mathcal{D}(R) := \sup_{\varphi \in \mathcal{Q}} \varphi(R) > 0$ for each $R \in \mathcal{R}$. In particular, $\mathcal{Q} \neq \emptyset$.

Consider heterogenous agents $\{\preceq_\varphi\}_{\varphi \in \mathcal{Q}}$ given by
$X \preceq_\varphi Y$ if $\varphi(X) \leq \varphi(Y)$. Then,

- $\forall \ell \in \mathcal{I}$, $\mathcal{D}(\ell) \leq 0$. Hence, $\varphi(\ell) \leq 0 \Rightarrow \ell \preceq_\varphi 0 \; \forall \varphi \in \mathcal{Q}$.
- $\forall R \in \mathcal{R}$, $\mathcal{D}(R) > 0$. Hence, $\exists \varphi_R \in \mathcal{Q}$ so that $\varphi_R(R) > 0 \Rightarrow R \succ_{\varphi_R} 0$.

Proof : Other direction

- Suppose the market is viable and towards a contraposition assume that $\ell^n + c_n \geq R^*$ for some $R^* \in \mathcal{R}$ and $c_n \to 0$. Then, $-c_n + R^* \leq \ell^n$.
- Choose $\preceq^* \in \widehat{\mathcal{A}}$ so that $0 \prec^* R^*$.
- Since $\preceq^*$ is monotone, $-c_n + R^* \preceq^* \ell^n$.
- Moreover, by viability $-c_n + R^* \preceq^* \ell^n \preceq^* 0$.
- By weak continuity, $-c_n + R^* \preceq^* 0 \quad \Rightarrow \quad R^* \preceq^* 0$.
- This contradicts $0 \prec^* R^*$.

$\square$

Theorem (Burzoni, Riedel, Soner, 2017)

The following are equivalent :

- *Market is viable ;*
- *There no arbitrages (NFLVR) ;*
- *There exists a sublinear martingale measure with full support.*

In particular, one such sublinear martingale measure the Choquet capacity

$$\mathcal{E}(X) := \sup_{\varphi \in \mathcal{Q}} \varphi(X).$$

Examples, Implications

In these class of problems, one fixes a probability space $(\Omega, \mathbb{F}, \mathbb{P})$ and stock price process S. Then,

- The partial order $\leq$ is given through $\mathbb{P}$ almost sure inequalities.
- $\mathcal{R}$ is the set of $\mathbb{P}$ almost-surely non-negative functions that are not equal to zero.

Here there is a single representative agent whose preference is induced by a risk neutral measure.

he fact that martingale measures are countably additive is a deep result and depends on results from stochastic integration.

In this case we fix a measurable space $(\Omega, \mathbb{F})$ and a family of probability measures $\mathcal{P}$. Then,

- $\leq$ is given through $\mathcal{P}$ quasi-sure inequalities.
- The choice of $\mathcal{R}$ is important. The following is used in the literature (e.g., Bouchard & Nutz)but other choices are possible as well, $R \in \mathcal{R}$ if

$$\inf_{\mathbb{P}\in\mathcal{P}} \mathbb{P}(R \geq 0) = 1, \quad \text{and} \quad \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{P}(R > 0) > 0.$$

The result is the existence of bounded additive measures $\mathcal{Q}$ consistent with $\mathcal{I}$ and with full support property, i..e, for every $R \in \mathcal{R}$ there is $\varphi_R \in \mathcal{Q}$ so that $\varphi_R(R) > 0$.

In the preceding setting of the Knightian uncertainty, our result can be viewed a new version of the weak efficient market hypothesis. Indeed, the consistency with economic equilibrium (i.e., viability) is equivalent to the existence of a set of martingale measures $\mathcal{Q}$ that are equivalent to the original priors $\mathcal{P}$.

However, under the sublinear expectation (or equivalently the Choquet capacity), the discounted asset prices are only symmetric martingales.

In summary, from weakest to strongest we have

- one point arbitrage : strictly positive only at one point Riedel ;
- open arbitrage : strictly positive on an open set Burzoni, Fritelli & Maggis, and Dolinsky, Soner ;
- Vienna arbitrage : strictly positive everywhere ;
- uniform arbitrage : uniformly positive. This is the strongest possible ; Bartl, Cheredito & Kupper, and Dolinsky, Soner ;

To eliminate uniform arbitrage one finitely additive martingale measure suffices. While for one point arbitrage, for every point there needs to be a martingale measure which charges that point.

This is related to the notion of smooth ambiguity by Klibanoff, Marinacci, Mukerji, 2005.

- $\mathfrak{P} = \mathfrak{P}(\Omega)$ is the set of all probability measures on $(\Omega, \mathbb{F})$.
- Let μ be probability measure on $\mathfrak{P}$ (i.e., a measure on measures)
- The partial order is then given by, $X \le Y$ provided that

$$\mu\left(\mathbb{P} \in \mathfrak{P} \; : \; \mathbb{P}(X \le Y) = 1\right) = 1.$$

We say $R \in \mathcal{R}$ if

$$\mu\left(\mathbb{P} \in \mathfrak{P} \;:\; \mathbb{P}(R \geq 0) = 1\right) = 1, \quad \text{and}$$
$$\mu\left(\mathbb{P} \in \mathfrak{P} \;:\; \mathbb{P}(R > 0) > 0\right) > 0.$$

Moreover, a Borel set $N \subset \Omega$ is μ polar if

$$\mu\left(\mathbb{P} \in \mathfrak{P} \;:\; \mathbb{P}(N) = 0\right) = 1.$$

Let $\mathcal{N}_\mu$ be the set of all μ polar sets.

Then NFLVR and viability is equivalent to existence of a set of risk neutral measures $\mathcal{Q}$ so that $\mathcal{Q}$ polar sets is equal to $\mathcal{N}_\mu$.

Conclusions

- We have showed that for partial equilibrium to extend to the whole space, an appropriate no-arbitrage notion is necessary and sufficient.
- We extend the classical work of Harrison & Kreps by relaxing the strict monotonicity condition. This relaxation allows us to incorporate Knightian uncertainty.
- Under choices of market, strong and weak efficient market hypothesis follow from our results. In particular, Gilboa-Schmeidler type utilities result from viability.

Viability and Arbitrage under Uncertainty

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THANK YOU FOR YOUR ATTENTION