CONDITIONAL DAVIS PRICING

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Abstract. We study the set of Davis (marginal utility-based) prices of a financial derivative in the case where the investor has a non-replicable random endowment. We give a new characterization of the set of all such prices, and provide an example showing that even in the simplest of settings - such as Samuelson’s geometric Brownian motion model - the interval of Davis prices is often a non-degenerate subinterval of the set of all no-arbitrage prices. This is in stark contrast to the case with a constant or replicable endowment where non-uniqueness of Davis prices is exceptional. We provide formulas for the endpoints for these prices and illustrate the theory with several examples.

1. Introduction

We consider an investor trading in a frictionless but incomplete financial model with stock price dynamics modeled by a locally bounded semimartingale $S$. The investor receives a random endowment $B > 0$ at a future time $T > 0$ and we seek to price a contingent claim with payoff $\varphi$ at time $T$. In many cases of interest, the interval of arbitrage-free prices of $\varphi$ takes on an extreme form: its open endpoints are given by the (essential) infimum and supremum of $\varphi$. To reduce this interval to a useful size one needs additional economic input which is typically based on demand-and-supply considerations. We refer the reader to monographs [3] and [14] for further general information and thorough historical overviews of various ways to price unspanned payoffs in incomplete models. For example, [4] and its extension [2]...
use so-called good deal bounds based on the Hansen-Jagannathan bound for the Sharpe ratio to reduce the width of the interval of arbitrage-free prices.

One of the most well-known pricing procedures is introduced by Mark Davis in [8], and goes by the name Davis pricing or marginal utility-based pricing. Let us describe briefly, and informally, the conditional version we study in this paper. The main ingredients are a utility function $U:(0,\infty)\rightarrow \mathbb{R}$, a random endowment $B$, and the claim $\varphi$ to be priced. Consider an agent who receives the endowment $B$ at a future time $T > 0$, but also has access to any quantity $q$ of the claim $\varphi$ for the unit price $p$, as well as to a financial market with a zero-interest bond and stock price $S$. In the effort to invest optimally in the resulting market, the agent faces the following optimization problem

$$\sup_{q \in \mathbb{R}} \sup_{\pi \in \mathcal{A}} \mathbb{E}[U(q(\varphi - p) + B + \int_0^T \pi_u dS_u)]$$

with $\pi$ in suitable set $\mathcal{A}$ of trading strategies (defined in Section 2 below). A constant $p$ is then called a conditional Davis price of $\varphi$ (conditional on the presence of the random endowment $B$), if the (first) supremum in (1.1) is attained at $q = 0$. In other words, $p$ is a Davis price if the agent is indifferent between being able to buy or sell any quantity of $\varphi$ at unit price $p$, and not having access to $\varphi$ at all.

Conditional Davis prices as described above can also be seen as a variation of the classical case where the random endowment is replaced by a constant initial wealth $x > 0$, but where the utility function is no longer deterministic. We could consider the random endowment $B$ as a part of the preference structure of the agent, i.e., think of $x \mapsto U(x + B(\omega))$ as a stochastic utility function and view $\mathbb{E}[U(q\varphi + B)]$ as the expected utility of the position $q\varphi$.

One of the original reasons for the introduction of Davis prices was the intuition that Davis prices should be unique. Equivalently, the hope was to reduce the arbitrage-free interval to a single price. Indeed, due to the typical strict concavity of utility functions, one expects at least some non-trivial demand for $\varphi$ ($q > 0$) at prices $p' < p$ and at least some non-trivial supply ($q < 0$) when $p' > p$. Therefore, Theorem 3.1(ii) in [17] was surprising because it gives an example of an incomplete model, a constant endowment $B := x > 0$, and a contingent claim $\varphi$ with a non-trivial interval of Davis prices. This is often treated as a rare pathological case as the construction requires sophisticated functional-theoretic machinery and a hope remained that “most” relevant models do not exhibit such behavior and that Davis pricing can still be used to assign a single price to any “reasonable” contingent claim $\varphi$. As we show in the present paper, even such a hope is unfounded: we will show that in Samuelson’s geometric Brownian motion model with constant coefficients there exists a whole spectrum of explicit
random endowments $B > 0$ and payoffs $\varphi$ with a non-trivial and explicitly computable interval of marginal utility-based prices (both $B$ and $\varphi$ are bounded random variables).

Even though the concept of a Davis price has been around for several decades, previous studies do not cover the conditional case where the endowment $B > 0$ is unspanned beyond some special cases and under strong regularity conditions. For example, [16] define Davis prices in their Remark 1, but only investigate the underlying utility-maximization problem, [17] study Davis prices where the random endowment $B := x > 0$ is constant, and [24] study asymptotic expansions under a stringent decay assumption which forces uniqueness (the decay condition is from Theorem 3.1(i) in [17]). In [32], non-uniqueness of Davis prices is established for “extreme” random endowments $B$ (a class we rule out by assuming that $B \in L^\infty_{++}$).

There exist natural conditions on the market model (see [26]) such that every bounded contingent claim admits a unique Davis price. However, these conditions apply only in the case with spanned endowment and, as we will see, no longer guarantee uniqueness in the presence of a general unspanned random endowment.

The goal of the present paper is to develop the theory of Davis pricing in the conditional case with a bounded random endowment $B > 0$ described above under minimal assumptions, and to look into uniqueness issues such as the one raised in [17].

Assuming throughout that the utility function $U$ is defined on the positive real line our main results fall into several categories:

1. For an arbitrary bounded endowment $B$ we give a new, simple and natural, characterization (Theorem 4.2) of Davis prices of the payoff $\varphi \in L^\infty(P)$ as the set of all numbers of the form $\langle \varphi, \hat{Q} \rangle \in \mathbb{R}$ when $\hat{Q} \in ba(P)$ ranges through the set of finitely-additive minimizers of the associated dual utility problem (as introduced in [6]). Here $\langle \cdot, \cdot \rangle$ denotes the dual pairing between $L^\infty(P)$ and its dual $ba(P) := (L^\infty(P))'$.\footnote{The Davis pricing theory is simpler and with fewer surprises — including non-uniqueness — for utilities defined on all of $\mathbb{R}$. This is primarily due to several simplifications that occur in the dual approach to utility maximization in this setting (see [1]).}

2. We give a whole class of non-pathological non-uniqueness examples (Example 7.3) of Davis prices in the conditional case. In fact, we show that the interval of Davis prices is nontrivial in the standard Samuelson-Black-Scholes model and with any utility function in the power (CRRA) family, as soon as both $B$ and $\varphi$ are non-constant, bounded, and uniformly Lipschitz ($W^{1,\infty}$) functions of an independent Brownian motion.

3. As is well known, questions of uniqueness, existence, and computability of Davis prices are intimately linked to differentiability properties
of the value function $\varphi \mapsto v(\varphi; B)$, where

$$v(\varphi; B) := \sup_{\pi \in A} \mathbb{E}\left[U(B + \varphi + \int_0^T \pi_u dS_u)\right].$$

In this context, we give an example (Example 5.1) showing that this function need not be differentiable, even in “constant” $\varphi$-directions, and even when $U(x) := \log(x)$. More precisely, the mapping $\mathbb{R} \ni x \mapsto v(x; B)$ can fail to be differentiable in the interior of its domain. This is in contrast to Theorem 2.2 in [23] which ensures differentiability of the primal value function $x \mapsto v(x; 0)$ when there is no random endowment. It also provides a counterexample to a statement in [6] (see also the discussion in the Erratum [7]).

(4) On the constructive side, we show that under a mild growth condition on the utility function $U$ around 0, one-sided directional derivatives of $v(\varphi; B)$ always exist and can be characterized as values of a new linear stochastic control problem (Proposition 5.5).

We solve this problem explicitly under the additional assumption of minimal (unique) super-replicability from [29] placed on $B$ and $\varphi$. This gives us explicit formulas for the two endpoints of the interval of Davis prices. As an offshoot, we show additionally that the mapping $\varphi \mapsto v(\varphi; B)$ is differentiable whenever $B$ is minimally super-replicable.

The paper is organized as follows. The model is described, the terminology and notation set, standing assumptions imposed, and preliminary analysis of our central utility-maximization problem are in Section 2. In Section 3 we define conditional Davis prices. Section 4 characterizes the conditional Davis prices from the dual point of view and lays out some of the first consequences of this characterization. Directional derivatives of the primal utility-maximization problem are studied in Section 5 which also contains the explicit example of non-smoothness mentioned in (3) above. Section 5 also gives a characterization of the directional derivative in terms of a linear stochastic control problem. Section 6 recalls the definition of unique super-replicability from [29] (now called minimal super-replicability) and provides a family of examples of minimally super-replicable claims. The main result of Section 6 gives an explicit expression for the directional derivative of the utility-maximization value function under the minimal super-replicability condition. This result is subsequently used in Section 7 to give explicit formulas for the interval of conditional Davis prices in a general setting. Finally, we use these formulas in two examples, one of which supports our claim that non-uniqueness of conditional Davis prices occurs even in the simplest of settings.

We conclude the introduction with some notation and terminology used throughout. Given a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0,T]}, \mathbb{P})$, with
time horizon $T \in (0, \infty)$, $L^2$ denotes the family of all progressively measurable processes $\pi$ with $\int_0^T |\pi_u|^2 \, du < \infty$, a.s. For a semimartingale $S$, $L(S)$ denotes the set of all predictable $S$-integrable processes, and the stochastic integral of $\pi \in L(S)$ with respect to $S$ is denoted by either $\pi \cdot S$ or $\int_0^T \pi_u \, dS_u$.

The subspaces of various $L^p$ spaces consisting of nonnegative random variables (or sets of set-functions taking nonnegative values) get a subscript $+$, while $L_+^\infty$ denotes the family of all nonnegative essentially bounded random variables, essentially bounded away from 0, i.e., $L_+^\infty := \cup_{x>0} (x + L_+^\infty)$. The space of finite finitely-additive set-functions on $(\Omega, \mathcal{F})$, absolutely continuous with respect to $\mathbb{P}$, is denoted by $\text{ba}(\mathbb{P})$ (or simply $\text{ba}$ if no confusion can arise). The space of finite $\sigma$-additive measures absolutely continuous with respect to $\mathbb{P}$ is often identified with $L_1^+$ while the space $\text{ba}$ is identified with the topological dual $(L^\infty)^*$ of $L^\infty$. For $\mu \in \text{ba}_+$, $\mu^r$ denotes the regular part of $\mu$, i.e., the (set-wise) largest $\sigma$-additive measure dominated by $\mu$. The singular part $\mu - \mu^r$ is denoted by $\mu^s$, making $\mu = \mu^r + \mu^s$ the Yosida-Hewitt decomposition of $\mu$.

2. The setup and assumptions

2.1. The market model. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0,T]}, \mathbb{P})$ be a filtered probability space which satisfies the usual conditions, and let $\{S_t\}_{t \in [0,T]}$ be a locally bounded semimartingale. $\mathcal{M}$ denotes the set of all $\mathbb{P}$-equivalent countably-additive probability measures $Q$ on $\mathcal{F}$ for which $S$ is a $Q$-local martingale.

**Standing Assumption 2.1** (NFLVR). $\mathcal{M} \neq \emptyset$.

**Remark 2.2.** We assume that the asset-price process $S$ is locally bounded and postulate the existence of a local martingale measure. While it is possible to relax our setting to the non-locally-bounded case (as used in, e.g., [6]), it is not be possible to relax Assumption 2.1 so as to imply the existence of a supermartingale deflator only. Indeed, the presence of a non-replicable endowment $B$ makes the admissibility class which produces only nonnegative wealth processes too small to host an optimizer. This delicate issue is discussed and illustrated on pages 240-241 in [28]. To keep the focus of the current paper on the issues directly related to conditional Davis pricing, we have opted for a set of assumptions which is slightly stronger than absolutely necessary.

In the same spirit, we assume that all random endowments and contingent claims considered in this paper are uniformly bounded, i.e., $B \in L_+^\infty$ and $\varphi \in L^\infty$. Such an assumption may be replaced by a weaker one where we only require them to be bounded in absolute value by a constant plus the outcome of a maximal wealth process. We do not implement that generalization, but simply point the reader to Lemma 1 on p. 848 in [16] for details and terminology.
2.2. Gains and admissibility. The investor’s gains process has the following dynamics

\[(\pi \cdot S)_t := \int_0^t \pi_u dS_u, \quad t \in [0, T],\]

for some \( \pi \in L(S) \). We call \( \pi \in L(S) \) admissible if the gains process is uniformly lower bounded by a constant in which case we write \( \pi \in A \). The set of terminal outcomes (gains) of admissibly strategies is denoted by \( K \), i.e., we define

\[ K := \{(\pi \cdot S)_T : \pi \in A\} .\]

2.3. The primal problem. Let \( U \) be a utility function on \((0, \infty)\) - a strictly concave, strictly increasing, and continuously differentiable function with \( U'(0+) = +\infty \) and \( U'(+\infty) = 0 \). When necessary, we extend the domain of \( U \) artificially to \( \mathbb{R} \) by setting \( U(x) = -\infty \) for \( x < 0 \) and \( U(0) = \inf_{x > 0} U(x) \). Finally, \( U \) is said to be reasonably elastic (as defined in [23]) if

\[ \limsup_{x \to \infty} \frac{xU'(x)}{U(x)} < 1. \]

Even though we will need it for some of our results, we do not impose the condition of reasonable elasticity from the start.

Let \( v \) be defined as in (1.2) with the above definition of the admissible set \( A \). Then, for any \( B \in L^\infty_++ \) we set

\[ \Omega(B) := v(0; B) = \sup_{X \in K} E\left[U\left(B + X\right)\right] \]

where \( v \) is defined in (1.2). In (2.2) we use the convention that \( E[U(B+X)] = -\infty \) if \( E[U(B + X)^+] = +\infty \). Because \( \Omega(B) \geq U(\text{essinf} B) > -\infty \), \( \Omega \) is \((-\infty, \infty]\)-valued on \( L^\infty_+ \). In (2.3) below, we impose a dual properness assumption which among other things ensures that \( \Omega \) is finitely valued on \( L^\infty_+ \).

2.4. The dual utility maximization problem. The set of equivalent local martingale measures \( M \) can be identified - via Radon-Nikodym derivatives with respect to \( \mathbb{P} \) - with a subset of \( L^1(\mathbb{P})^* \) and embedded, naturally, into \( \text{ba}(\mathbb{P}) := L^\infty(\mathbb{P})^* \supset L^1(\mathbb{P}) \). We define \( \overline{M'} \) as the weak-*-closure of \( M \) and we define \( \mathcal{D} \subset \text{ba}(\mathbb{P}) \) as the family of all \( yQ \) where \( y \in (0, \infty) \) are constants and \( Q \in \overline{M'} \). The dual utility functional can now be defined by

\[ \nabla_B(\mu) := \sup_{X \in L^\infty} \left( E[U(B + X)] - \langle \mu, X \rangle \right), \quad \mu \in \text{ba}(\mathbb{P}). \]

In particular, \( \nabla_B \) is convex, lower weak-*-semicontinuous on \( \text{ba}(\mathbb{P}) \) and bounded from below by \( E[U(B)] \in \mathbb{R} \). For the reminder of the paper we impose a properness assumption. While not the weakest possible in our setting, this standard assumption allows us to deal swiftly, and yet with a minimal loss of generality, with several technical points that are not central to the message of the paper:
Standing Assumption 2.3 (Properness). There exist \( y_0 \in (0, \infty) \) and \( Q_0 \in \mathcal{M} \) such that \( \mu_0 := y_0 Q_0 \) satisfies
\[
\mathbb{V}_B(\mu_0) < \infty.
\]

Thanks to a minimal modification of Lemma 2.1 on p. 138 in [33] and the discussion before it, \( \mathbb{V}_B \) admits the following representation
\[
\mathbb{V}_B(\mu) = \mathbb{E} \left[ V \left( \frac{d\mu}{dP} \right) \right] + \langle \mu, B \rangle, \quad \mu \in \mathcal{D},
\]
where \( V \) is the dual utility function (strictly convex) defined by
\[
V(y) := \sup_{x > 0} \left( U(x) - xy \right), \quad y > 0.
\]
Consequently, Fenchel’s inequality and (2.3) guarantee that the primal value function \( U \) satisfies \( U(B) < \infty \) for all \( B \in \mathbb{L}_{++}^{\infty} \). Furthermore, (2.3) also ensures that the corresponding dual value function defined by
\[
\mathbb{W}(B) := \inf_{\mu \in \mathcal{D}} \mathbb{V}_B(\mu), \quad B \in \mathbb{L}_{++}^{\infty},
\]
is finitely valued. For \( B \in \mathbb{L}_{++}^{\infty} \) we let \( \hat{D}(B) \) denote the set of all minimizers, i.e.
\[
\hat{D}(B) = \{ \mu \in \mathcal{D} : \mathbb{W}(B) = \mathbb{V}_B(\mu) \}.
\]
The next two results collect some basic facts we will need in the sequel:

**Lemma 2.4.** Assume that \( B \in \mathbb{L}_{++}^{\infty} \) and that \( \mathbb{A} : \mathcal{D} \to [0, \infty) \) is a nonnegative weak-* lower semicontinuous functional. Then we have:

1. any minimizing sequence for \( \mathbb{V}_B + \mathbb{A} \) is bounded in total mass, and
2. the set of all minimizers of \( \mathbb{V}_B + \mathbb{A} \) is nonempty and weak-* compact.

**Proof.** For (1), we let \( \{ \mu_n \}_{n \in \mathbb{N}} \) be a minimizing sequence for \( \mathbb{V}_B + \mathbb{A} \). By the definition of \( \mathcal{D} \), it can be written in the following form
\[
\mu_n = y_n Q_n \text{ where } Q_n \in \overline{\mathcal{M}} \text{ and } \{y_n\}_{n \in \mathbb{N}} \subseteq [0, \infty).
\]
Using the representation (2.4) and the standing assumption (2.3) we get the following estimate:
\[
\mathbb{V}_B(\mu_0) + \mathbb{A}(\mu_0) \geq \limsup_n \left( \mathbb{E}[V(y_n \frac{dQ_n}{dP})] + y_n \langle Q_n, B \rangle \right)
\]
\[
\geq \limsup_n \left( V(y_n) + y_n \text{ ess inf } B \right),
\]
where the first inequality follows from the positivity of \( \mathbb{A} \) and the minimizing property of the sequence \( \{\mu_n\}_{n \in \mathbb{N}} \), and the second one is produced by Jensen’s inequality and the decreasing property of \( V \). Since \( \lim_{y \to \infty} V'(y) = 0 \) and the ess inf \( B > 0 \), we conclude that the sequence \( \{y_n\}_{n \in \mathbb{N}} \) is bounded from above, which, in turn, establishes (1).
For (2), we pick a minimizing sequence \( \{\mu_n\}_{n \in \mathbb{N}} \) and use (1) to establish the existence of a constant \( \bar{y} > 0 \) such that \( \mu_n(\Omega) \leq \bar{y} \) for all \( n \). Therefore, the family \( \{\mu_n\}_{n \in \mathbb{N}} \) lives in bounded subset \( \bar{y}M' \) of \( ba \), and we can use Banach-Alaoglu’s theorem to conclude that there is a subnet \( (\mu_\alpha) \) of \( \{\mu_n\}_{n \in \mathbb{N}} \) with \( \mu_\alpha \to \hat{\mu} \in ba \) in the weak-* sense. It remains to use the weak-* closedness of \( D \) and the lower semicontinuity of the functional \( \mathbb{V}_B + A \) to conclude that \( \hat{\mu} \) is a minimizer over \( D \).

We denote by \( \hat{D} \subseteq D \) the non-empty and closed set of all minimizers for \( \mathbb{V}_B + A \). We use (1) to see that the set \( \{\mu(\Omega) : \mu \in \hat{D}\} \) is bounded. That, in turn, allows us to use Banach-Alaoglu’s theorem to conclude that \( \hat{D} \) is weak-*compact.

**Lemma 2.5.** For \( B \in L^\infty_{++} \), the set \( \hat{D}(B) \) is a nonempty weak-*compact subset of \( ba(\mathbb{P}) \) and there exists a nonnegative random variable \( \hat{Y} = \hat{Y}(B) \) such that \( \mathbb{P}[\hat{Y} > 0] > 0 \) and

\[
\hat{Y} = \frac{d\mu}{d\nu} \quad \text{for all } \mu \in \hat{D}(B).
\]

Furthermore, the strong duality \( \Upsilon(B) = \mathfrak{V}(B) \) holds for all \( B \in L^\infty_{++} \).

**Proof.** The nonemptyness and compactness of \( \hat{D}(B) \) follow directly from Lemma 2.4 with \( A := 0 \).

To see that \( \hat{Y} \neq 0 \) we argue by contradiction and suppose that \( \mathbb{P}[\hat{Y} = 0] = 1 \). In that case \( \mathbb{V}(0) < \infty \) and, so, thanks to Jensen’s inequality, we have \( \mathbb{V}_B(\mu) < \infty \) for all \( \mu \in \mathcal{D} \). In particular, we have for some \( Q \in \mathcal{M} \) and \( \hat{\mu} \in \hat{D}(B) \)

\[
\mathbb{V}_B(\mu^\varepsilon) < \infty, \quad \text{where} \quad \mu^\varepsilon := \varepsilon Q + (1 - \varepsilon)\hat{\mu} , \quad \varepsilon \in [0,1].
\]

Because the regular-part functional is additive we have

\[
\mu^\varepsilon_r = \varepsilon Q + (1 - \varepsilon)\hat{\mu}_r = \varepsilon Q.
\]

Therefore, the minimality of \( \hat{\mu} \) implies that

\[
\mathbb{E}[\mathbb{V}(\varepsilon \frac{d\mu}{d\nu})] + \langle \mu^\varepsilon, B \rangle = \mathbb{V}_B(\mu^\varepsilon) \geq \mathbb{V}_B(\hat{\mu}) \geq \mathbb{V}(0) + \langle \hat{\mu}, B \rangle.
\]

Fatou’s lemma then yields

\[
\langle Q - \hat{\mu}, B \rangle \geq \liminf_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \left( \mathbb{V}(0) - \mathbb{E}\left[ \mathbb{V}\left( \varepsilon \frac{d\mu}{d\nu} \right) \right] \right) = -\mathbb{V}'(0) = +\infty.
\]

This is a contradiction because \( B \in L^\infty_{++} \) ensures that the left-hand-side is finite.

Finally, to establish the strong duality property, we define a nested sequence of weak-*compact dual sets

\[
\mathcal{D}_n := \{ \mu \in \mathcal{D} : \mu(\Omega) \leq n \}, \quad n \in \mathbb{N},
\]

as well as the primal set

\[
\mathcal{C} := (\mathcal{K} - L^0_+) \cap L^\infty = \{ X \in L^\infty : \langle Q, X \rangle \leq 0 \text{ for all } Q \in \mathcal{M} \}.
\]
For a proof of the last identity see, e.g., Corollary 3.4(1) in [30]. As a consequence, we have the following identity for $X \in L_\infty(P)$

\[
\lim_{n \to \infty} \sup_{\mu \in D_n} \langle \mu, X \rangle = \sup_{\mu \in D} \langle \mu, X \rangle = \begin{cases}
0, & X \in \mathcal{C}, \\
+\infty, & X \not\in \mathcal{C}.
\end{cases}
\]

The minimax theorem (see, e.g., Theorem 2.10.2, p. 144 in [34]) can then be used to conclude that

\[
\mathfrak{U}(B) = \lim_{n \to \infty} \inf_{\mu \in D_n} \sup_{X \in L_\infty(P)} \left( \mathbb{E}[U(X + B)] - \langle \mu, X \rangle \right)
= \sup_{X \in \mathcal{C}} \mathbb{E}[U(X + B)],
\]

with the last equality justified by the monotone convergence theorem. □

3. Conditional Davis prices

Definition 3.1. For $B \in L_\infty^+$, a random variable $R \in L_\infty$ is said to be $B$-irrelevant, denoted by $R \in I(B)$, if

\[
U(B + qR) \leq U(B), \quad \forall q \in \mathbb{R}.
\]

(3.1)

**Remark 3.2.** The function $U$ is finite-valued at $B$, as well as in an $L_\infty$-open ball around $B$. Therefore, both sides of (3.1) are real-valued for small enough $q$. Thanks to the concavity of $U$, it is enough to check (3.1) only for $q$ in a neighborhood of 0 to determine whether $R \in I(B)$.

Lemma 3.3. $I(B)$ is a nonempty, weak-* closed linear subspace in $L_\infty$.

**Proof.** The function $U$ is concave at $B$, so $I(B)$ is the set of those directions $R$ with the property that the directional derivative of $U$ in directions $R$ and $-R$ are nonpositive. In other words, we have

\[
\sup_{\mu \in \partial U(B)} \langle R, \mu \rangle \leq 0 \quad \text{and} \quad \sup_{\mu \in \partial U(B)} -\langle R, \mu \rangle \leq 0,
\]

where $\partial U(B) \subseteq ba(P)$ is the super-differential of $U$. Therefore, $I(B)$ is the annihilator of $\partial U(B)$, i.e.,

\[
I(B) = \{ R \in L_\infty : \langle \mu, R \rangle = 0 \quad \text{for all} \quad \mu \in \partial U(B) \},
\]

which implies the statement. □

The following definition is due to Mark Davis and originates in [8] and is equivalent to the definition given in the Introduction above:

**Definition 3.4.** A number $p \in \mathbb{R}$ is said to be a $B$-conditional Davis price and simply a conditional Davis price if $B$ is clear from the context, for a contingent claim $\varphi \in L_\infty$ if

\[
\varphi - p \text{ is $B$-irrelevant}.
\]

The set of all $B$-conditional Davis prices of $\varphi$ is denoted by $P(\varphi|B)$. 
Equivalently, \( p \in P(\varphi | B) \) if and only if
\[
U(B + q(\varphi - p)) \leq U(B), \quad \forall q \in \mathbb{R}.
\]

Before we embark on an in-depth study of conditional Davis prices in the subsequent sections, we establish some of their most fundamental properties here. We start with a definition which matches that of [32, pp. 605-606]:

**Definition 3.5.** A constant \( p \in \mathbb{R} \) is called an arbitrage-free price of \( \varphi \) if the following implication holds for all \( \pi \in \mathcal{A} \) and \( q \in \mathbb{R} \):
\[
q(\varphi - p) + \int_0^T \pi_u dS_u \geq 0 \implies q(\varphi - p) + \int_0^T \pi_u dS_u = 0, \mathbb{P}\text{-a.s.}
\]

**Proposition 3.6.** Each conditional Davis price \( p \) for a contingent claim \( \varphi \in \mathbb{L}^\infty \) is also an arbitrage-free price of \( \varphi \).

**Proof.** Let \( p \) be a conditional Davis price and suppose, to the contrary in (3.3), that we can find \( q \in \mathbb{R} \) and \( \pi \in \mathcal{A} \) such that the nonnegative random variable
\[
A := q(\varphi - p) + \int_0^T \pi_u dS_u
\]
is strictly positive with strictly positive probability, i.e., \( \mathbb{P}[A \geq 0] = 1 \) and \( \mathbb{P}[A > 0] > 0 \). Then, for \( n \in \mathbb{N} \), the inequality (3.2) implies that
\[
U(B) \geq U(B + nq(\varphi - p)) \geq \mathbb{E}\left[U(B + nq(\varphi - p) + n \int_0^T \pi_u dS_u)\right] = \mathbb{E}[U(B + nA)].
\]
It remains to let \( n \to \infty \) and use the monotone convergence theorem to reach a contradiction with the fact that \( U(B) < \infty \).

We also extend here the standard characterization of Davis prices in terms of perturbed value functions to the conditional case. Given \( B \in \mathbb{L}^\infty_{++} \) and \( \varphi \in \mathbb{L}^\infty \) we let the function \( u : \mathbb{R}^2 \to [-\infty, \infty) \) be defined by
\[
u(\varepsilon, x) := U(B + x + \varepsilon \varphi),
\]
and let its supergradient at \((0, 0)\) be denoted by \( \partial u(0, 0) \).

**Proposition 3.7.** For each \((\delta, y) \in \partial(0, 0)\) we have \( y > 0 \) and
\[
P(\varphi | B) = \{ \delta/y : (\delta, y) \in \partial u(0, 0) \}.
\]

**Proof.** Thanks to the assumption that \( B \in \mathbb{L}^\infty_{++} \), \( u \) is concave and finite-valued in some neighborhood of \((0, 0)\), and the first statement follows from the fact that \( x \mapsto u(0, x) \) is strictly increasing on its effective domain which contains \( 0 \) in its interior.

Definition 3.4 translates into the following statement:
\[
p \in P(\varphi | B) \text{ if and only if } u(0, 0) \geq u(\varepsilon, -\varepsilon p) \text{ for all } \varepsilon.
\]
By concavity, this is equivalent to the nonpositivity of the directional derivative of $u$, at $(0, 0)$ in the directions $(-1, p)$ and $(1, -p)$, i.e.

$$\inf_{(\delta, y) \in \partial u(0, 0)} -\delta + py \leq 0 \quad \text{and} \quad \inf_{(\delta, y) \in \partial u(0, 0)} \delta - py \leq 0.$$ 

By the convexity of the supergradient, this is equivalent to existence of a pair $(\delta, y) \in \partial u(0, 0)$ such that $py = \delta$. □

**Remark 3.8.** Eq. (3.6) is often used to explain the relationship between differentiability of the function $u$ and the uniqueness of the Davis price in the unconditional setting. Indeed, when $B$ is replicable, the function $u(0, x)$ is differentiable in the variable $x$ (by [23, Theorem 2.1, p. 909]) so that all elements $(\delta, y)$ of $\partial u(0, 0)$ have the same $y$, given by $y = \frac{\partial}{\partial x} u(0, x)|_{x=0}$. Therefore, we have multiple Davis prices if and only if we have multiple values of the $\delta$-components in elements of the supergradient $\partial u(0, 0)$. That occurs if and only if the left and the right derivative of $\varepsilon \mapsto u(0, \varepsilon)$ at 0 do not match, which is, in turn, equivalent to the lack of differentiability at 0.

The case with $B$ unspanned is a more subtle. As we will see in Example [5.1] below, when a nonreplicable random endowment is present, $u$ is no longer necessarily differentiable in $x$. That means that both $y$ and $\delta$ are potentially allowed to vary across different elements $(\delta, y)$ of the subgradient. It may happen, however, that each such pair has the same quotient, making the Davis price unique. A simple example is when (as in Example [5.1]) $u$ is not differentiable in $x$ at $x = 0$, and $\varphi := 1$. Then $(x, \varepsilon) \mapsto u(x, \varepsilon) = u(x + \varepsilon, 0)$ is clearly not differentiable at $(0, 0)$, but $\varphi := 1$ has a unique conditional Davis price (the latter claim follows from Corollary [6.9] below).

**4. Characterization of Conditional Davis Prices**

**4.1. A dual characterization.** The dual characterization of the set of conditional Davis prices in Theorem [4.2] below rests on the following, simple, lemma:

**Lemma 4.1.** A random variable $R \in L^\infty$ is $B$-irrelevant if and only if

$$\inf_{\mu \in D} \left( \nabla_B(\mu) + |\langle \mu, R \rangle| \right) = \inf_{\mu \in D} \nabla_B(\mu). \quad (4.1)$$

**Proof.** Because $I(B)$ is a vector space, we can scale $R$ so that, without loss generality, we can assume that $B \pm R \in L^\infty_{++}$. Then, by the minimax theorem (see Theorem 2.10.2, p. 144 in [34]), we have

$$\inf_{\mu \in D} \left( \nabla_B(\mu) + |\langle \mu, R \rangle| \right) = \inf_{\mu \in D} \sup_{|q| \leq 1} \left( \nabla_B(\mu) + q|\mu, R| \right) = \sup_{|q| \leq 1} \inf_{\mu \in D} \left( \nabla_B(\mu) + q|\mu, R| \right) = \sup_{|q| \leq 1} \nabla_B(\mu + qR).$$

The same equality with $R = 0$, implies that (4.1) is equivalent to

$$\mu(B) = \sup_{|q| \leq 1} \mu(B + qR).$$
which is, in turn, equivalent to \( R \in \mathcal{I}(B) \) by Remark 3.2.

By Lemma 2.5, we have \( \mu(\Omega) > 0 \) for each \( \mu \in \hat{D}(B) \). Therefore, the family

\[
\hat{D}_0(B) := \{ \frac{1}{\mu(\Omega)} \mu : \mu \in \hat{D}(B) \}
\]

is a well-defined nonempty family of finitely-additive probabilities. We now have everything set up for our main characterization of conditional Davis prices:

**Theorem 4.2.** For \( \varphi \in \mathbb{L}^\infty(\mathbb{P}) \) the following two statements are equivalent

1. \( p \in P(\varphi|B) \), i.e., \( p \) is a \( B \)-conditional Davis price of \( \varphi \).
2. \( p = \langle Q, \varphi \rangle \), for some \( Q \in \hat{D}_0(B) \).

In particular, \( P(\varphi|B) \) is a nonempty compact subinterval of \( \mathbb{R} \).

**Proof.** (1) \( \Rightarrow \) (2):

By Lemma 2.4 with \( A(\mu) := |\langle \mu, \varphi - p \rangle| \), the functional \( \mu \mapsto \mathbb{V}_B(\mu) + |\langle \mu, \varphi - p \rangle| \) admits a minimizer \( \hat{\mu} \).

By Lemma 4.1 the same \( \hat{\mu} \) must minimize the functional \( \mu \mapsto \mathbb{V}_B(\mu) \), as well, and, so, \( \hat{\mu} \in \hat{D}(B) \) and \( \langle \hat{\mu}, \varphi - p \rangle = 0 \).

(2) \( \Rightarrow \) (1): Suppose that \( p \) is such that \( \langle \mu^*, \varphi - p \rangle = 0 \), for some \( \mu^* \in \hat{D}(B) \). Then, for any \( \mu \), we have

\[
\mathbb{V}_B(\mu^*) + |\langle \mu^*, \varphi - p \rangle| = \mathbb{V}_B(\mu^*) \leq \mathbb{V}_B(\mu) \leq \mathbb{V}_B(\mu) + |\langle \mu, \varphi - p \rangle|,
\]

and Lemma 4.1 can be used.

Finally, Lemma 2.4 and the continuity of the map \( \mu \mapsto \langle \mu, \varphi \rangle \) ensure the set \( P(\varphi|B) \) is compact.

\( \square \)

**4.2. First consequences.** A reinterpretation in the setting of portfolios with convex constraints leads to the following dual characterization:

**Corollary 4.3.** Suppose that \( U \) is reasonably elastic. Then, for each constant \( c \geq 0 \) and each \( R \in \mathbb{L}^\infty \) we have

\[
\inf_{\mu \in \mathcal{D}} \left( \mathbb{V}_B(\mu) + c|\langle \mu, R \rangle| \right) = \inf_{y \geq 0, Q \in \mathcal{M}} \left( \mathbb{V}_B(yQ) + c|\langle yQ, R \rangle| \right).
\]

**Proof.** Let \( \mathcal{C} := (\mathcal{K} - \mathbb{L}_0) \cap \mathbb{L}^\infty \) and let \( \mathcal{C}' \) be the family of all random variables \( X' \in \mathbb{L}^\infty \) of the form

\[
X' = B + X + qR, \quad X \in \mathcal{C}, \quad q \in [-c, c].
\]
The support function $\alpha_{C'}$ for the set $C'$ is then given by
\[
\alpha_{C'}(\mu) = \sup_{X' \in C'} \left( \langle \mu, X \rangle + q(\mu, R) \right)
\]
where $q \in [-c, c]$. Therefore,
\[
\inf_{\mu \in D} \left( V_B(\mu) + c|\langle \mu, R \rangle| \right) = \inf_{\mu \in \text{ba}(P)} \left( V_0(\mu) + \alpha_{C'}(\mu) \right).
\]
Moreover, the set $C$ is weak-$\ast$-closed by Theorem 4.2 in [9]; hence, so is $C'$. Hence, the assumptions of Proposition 3.14, p. 686 of [30] are satisfied (via Corollary 3.4, p. 679 in [30]) and, so, the infimum on the right-hand side of (4.3) can be replaced by an infimum over $\sigma$-additive measures.

Our next two consequences of Theorem 4.2 provide a partial generalization and an alternative method of proof for Theorem 3.1, p. 206 in [17].

**Proposition 4.4.** Suppose that $B \in L^\infty_+, U$ is reasonably elastic and the dual problem (2.5) admits a non-$\sigma$-additive optimizer. Then there exists $A \in \mathcal{F}$ such that $\varphi = 1_A$ has multiple $B$-conditional Davis prices.

**Proof.** Let $\{\mu_n\}_{n \in \mathbb{N}}$ be a minimizing sequence for $\inf_{\mu \in D} V_B(\mu)$. By Corollary 4.3 we can assume that each $\mu$ is countably additive. Moreover, Lemma 2.4 part (1) guarantees that its total-mass sequence $\{\mu_n(\Omega)\}_{n \in \mathbb{N}}$ is bounded. By extracting a subsequence, we can assume that $\{\mu_n(\Omega)\}_{n \in \mathbb{N}}$ converges towards a constant $y \geq 0$. By Lemma 2.5, this conclusion can be strengthened to $y > 0$.

We suppose first that $\{\mu_n\}_{n \in \mathbb{N}}$ is not weak-$\ast$-convergent. Then, two of its convergent subnets will have different limits, and both of these will be elements of $\hat{D}(B)$ with the same total mass $y > 0$. Hence, the set $\hat{D}_0(B)$ of (4.2) is not a singleton, and, by Theorem 4.2, there exists $\varphi = 1_A$, with $A \in \mathcal{F}$, with two different conditional Davis prices.

If, on the other hand, $\{\mu_n\}_{n \in \mathbb{N}}$ converges to $\hat{\mu}$ in the weak-$\ast$-sense, then $\hat{\mu} \in \hat{D}(B)$. Furthermore, by the Vitali-Hahn-Saks theorem (see [12], Corollary 8 on p.159) the limit $\hat{\mu}$ is countably additive. Hence, the set $\hat{D}(B)$ will have at least two different elements - one countably additive and one not. Then a random variable $\varphi = 1_A$ with two different conditional Davis prices can be constructed as above.

The next consequence of Theorem 4.2 gives a sufficient condition (analogous to that of Theorem 3.1 on p. 206 of [17]) for the uniqueness of conditional Davis prices. Before we state it, we recall that, under the condition
of reasonable elasticity, the authors of [6] show there exists a process \( \hat{\pi} \in A \) such that \( \hat{X} := (\hat{\pi} \cdot S)_T + B \) satisfies

\[
\mathbb{E}[U(\hat{X})] = \mathcal{U}(B) \quad \text{and} \quad U'(\hat{X}) = \frac{d\hat{\mu}}{d\mathbb{P}},
\]

where \( \hat{\mu} \in \hat{D}(B) \). The random variable \( \hat{X} \) with this property is \( \mathbb{P} \)-a.s. unique.

**Corollary 4.5.** Suppose \( U \) is reasonably elastic and that \( |\varphi| \leq c\hat{X} \), for some constant \( c \geq 0 \), where \( \hat{X} \) is as in (4.4). Then the set \( P(\varphi|B) \) of B-conditional Davis prices for \( \varphi \in L^\infty \) is a singleton.

**Proof.** In view of Lemma 2.5 and Theorem 4.2 it will be enough to show that \( \langle \hat{\mu}^r, \hat{X} \rangle = 0 \), for each \( \hat{\mu} \in \hat{D}(B) \). This, in turn, follows directly from the first part of Equation (4.7) in [6]. \( \square \)

5. **Directional derivatives of the primal value function**

Our next task is to study directional differentiability of the primal utility-maximization value function \( \mathcal{U} \) defined by (2.2). Its relevance in the context of Davis pricing has been noted by several authors (including Davis in [8]), and we comment on it explicitly in Remark 3.8 above. We also use the obtained results in the later sections to give a workable characterization of the interval of conditional Davis prices. First we show, by means of an example, that differentiability - even in the most “benign” directions - cannot be expected in general. Then we give a characterization of the one-sided directional derivative in terms of a linear control problem. We hope that both our counterexample and the later characterization hold some independent interest outside of the context of Davis pricing.

5.1. **An example of nondifferentiability.** Our next example shows that the set \( \hat{D}(B) \) of dual minimizers may contain measures with different total masses. In other words, \( \hat{\mu}(\Omega) \) may not be constant over \( \hat{\mu} \in \hat{D}(B) \). Consequently, \( \mathcal{U} \) may fail to be differentiable even in “constant directions” in the sense that \( \varepsilon \to \mathcal{U}(B + \varepsilon) \) may fail to be differentiable at \( \varepsilon = 0 \). Once we introduce the concept of minimal superreplicability in the next section, we will see how it can be used to regain differentiability in certain cases of interest.

For simplicity and concreteness, we base the example on Example 5.1’ in [23], and use the following notation and conventions: All random variables \( X \) will be defined on the sample space \( \Omega := \mathbb{N}_0 = \mathbb{N} \cup \{0\} \), measures are identified with sequences in \( \ell^1_+ \) and for \( Q = (q_n) \in \ell^1_+ \) we write \( \langle Q, X \rangle \) for \( \sum q_n X_n \) whenever \( X = (X_n) \in \ell^\infty \).

**Example 5.1.** We start by recalling the elements of (a special case of) the one period Example 5.1’ in [23] where \( \Omega := \mathbb{N}_0 = \mathbb{N} \cup \{0\} \) and \( \mathbb{P} = (p_n) \) with

\[
p_0 := \frac{3}{4}, \quad p_n := \frac{2^{-n}}{1} \quad \text{for} \quad n \in \mathbb{N}.
\]
The one-period stock-price increment $\Delta S = (\Delta S_n)$ is defined as follows
\[
\Delta S_0 := 1 \text{ and } \Delta S_n := \frac{1-n}{n} \text{ for } n \in \mathbb{N}.
\]

With $U := \log$, the primal problem is defined by
\[
u(x) := \sup_{\pi \in [-x,x]} \mathbb{E}[U(x + \pi \Delta S)], \quad x > 0.
\]

Let $Q$ denote the set of all finite martingale measures, i.e.,
\[
Q := \{Q \in \ell^1_+ : \langle Q, \Delta S \rangle = 0\},
\]
and let $M := \{Q \in Q : \langle Q, 1 \rangle = 1\}$. Because $V(y) = -1 - \log(y)$, the dual problem is given by
\[
v(y) := \inf_{Q \in M} \mathbb{E}[V(y \frac{dQ}{dP})] = V(y) + v^*, \quad \text{where } v^* := \inf_{Q \in M} \mathbb{E}[-\log(\frac{dQ}{dP})].
\]

Let $(Q^N)_N \subset M$ be a minimizing sequence for $v^*$ (equivalently, for $v(y)$). We claim that $(Q^N, f)$ cannot converge in $\mathbb{R}$ for each $f \in \ell^\infty$. Indeed, if it did, the sequence $(Q^N)_N$ - interpreted as a sequence in $\ell^1_+$ - would be weakly Cauchy in $\ell^1$. By a theorem of Steinhaus (see [35, Corollary 14, p. 140]) which states that $\ell^1$ is weakly sequentially complete, the sequence $(Q^N)_N$ would admit a weak limit $Q^*$ in $\ell^1_+$. Since any weak limit of any minimizing sequence must also belong to $M$, $Q^*$ would necessarily be minimizer for $v^*$. However, as shown in Example 5.1' in [23], this contradicts the strict supermartingale property of the (unique) dual log-optimizer.

As a consequence of the above, for a given minimizing sequence $(Q^N)_N$, there exists a random variable $H \in \ell^\infty$ such that
\[
\langle Q^N, H \rangle \text{ does not converge in } \mathbb{R} \text{ as } N \to \infty.
\]

Because $\langle Q^N, 1 \rangle = 1$, for each $N$, we can assume that $H \geq 1$. Moreover, there exist two subsequences $(Q^{1,N})_N$ and $(Q^{2,N})_N$ of $(Q^N)_N$ such that the limits
\[
y_1 = \lim_N \langle Q^{1,N}, H \rangle \text{ and } y_2 = \lim_N \langle Q^{2,N}, H \rangle \text{ exist with } y_1 \neq y_2.
\]

With $H$ as above, we define $B := 1/H$ and a new stock price process with increments
\[
\Delta \tilde{S} := B \Delta S,
\]
and then consider the log-utility maximization problem with the random endowment $B$ and the stock-price increments $\Delta \tilde{S}$. The associated dual
problem is given by

\[ \tilde{v}(y) := \inf_{\tilde{Q} \in \tilde{\mathcal{M}}} \mathbb{E}[V(y \frac{d\tilde{Q}}{dP})] + y \langle \tilde{Q}, B \rangle \]

\[ = -1 + \inf_{\tilde{Q} \in \tilde{\mathcal{M}}} \left( \mathbb{E}[-\log(y \frac{d\tilde{Q}}{dP})] + y \langle \tilde{Q}, B \rangle \right) \]

\[ = -1 + \mathbb{E}[\log(B)] + \inf_{\tilde{Q} \in \tilde{\mathcal{M}}} \left( \mathbb{E}[V(y \frac{d\tilde{Q}}{dP} B)] + y \langle \tilde{Q} B, 1 \rangle \right), \quad y > 0, \]

where

\[ \tilde{Q} := \left\{ \tilde{Q} \in \ell_1^+ : \langle \tilde{Q}, \Delta \tilde{S} \rangle = 0 \right\} \quad \text{and} \quad \tilde{\mathcal{M}} := \left\{ \tilde{Q} \in \tilde{Q} : \langle \tilde{Q}, 1 \rangle = 1 \right\}. \]

Because \( \tilde{Q} \in \tilde{Q} \) if and only if \( Q = \tilde{Q} B \in Q \), we have

\[ \inf_{y > 0} \tilde{v}(y) = \mathbb{E}[\log(B)] + \inf_{y > 0} \left( v(y) + y \right) \]

\[ = \mathbb{E}[\log(B)] + \inf_{y > 0} \left( V(y) + y + v^* \right) \]

\[ = \mathbb{E}[\log(B)] + v^*. \]

By using the minimizing sequences \((Q^{1,N})_N, (Q^{2,N})_N\) constructed above we define the sequence of probability measures

\[ \tilde{Q}^{i,N} := \frac{Q^{i,N} H}{(Q^{i,N} H, 1)} \in \tilde{\mathcal{M}} \quad \text{for} \quad i = 1, 2. \]

We can use (5.2) and the fact that \((Q^{i,N})_N, i = 1, 2\), are minimizing sequences for \( v^* \) to see

\[ \mathbb{E}[V(y_1 \frac{d\tilde{Q}^{1,N}}{d\tilde{P}})] + y_1 \langle \tilde{Q}^{1,N}, B \rangle = \mathbb{E}[V(y \frac{dQ^{1,N}}{dP})] + y \langle Q^{1,N} H, 1 \rangle \]

\[ = \mathbb{E}[V(y \frac{dQ^{1,N}}{dP})] + y \left( \frac{y}{(Q^{1,N} H, 1)} - \mathbb{E}[\log(\frac{dQ^{1,N}}{dP})] \right) \]

\[ \rightarrow \mathbb{E}[\log(B)] + v^* \]

\[ = \inf_{y > 0} \tilde{v}(y). \]

Clearly, \( \tilde{v}(y_i) \geq \inf_{y > 0} \tilde{v}(y) \), for \( i = 1, 2 \), which implies that \( \tilde{Q}^{i,N} \) is a minimizing sequence for \( \tilde{v}(y_i) \). Therefore, \( \tilde{v}(y_1) = \tilde{v}(y_2) = \inf_{y > 0} \tilde{v}(y) \) which

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\( ^2 \)It has been shown in [30, Lemma 3.12] that under the reasonable asymptotic elasticity condition, infimization over the set of countably-additive martingale measures - as opposed to its finitely-additive enlargement as in [3] - leads to the same value function.
implies that \( \tilde{v} \) is constant on \([y_1, y_2]\). This, in turn, implies, that the conjugate function to \( \tilde{v} \) fails to be differentiable at 0 (indeed, the entire segment \([y_1, y_2]\) belongs to its superdifferential at zero).

**Remark 5.2.**

1. The construction of the random endowment \( B \) in Example 5.1 above rests on the weak sequential completeness property of \( \ell^1 \) which, in fact, holds for any \( L^1 \)-space. Example 5.1 above is therefore generic in the sense that it can be applied to any model which produces nontrivial singular components in the dual optimizer for the log-investor (with constant endowment). This implies that there also exist random endowments in the Brownian setting of Example 5.1 in [23] which produce a non-differentiable primal utility function.

2. Example 5.1 seems to contradict the claimed continuous differentiability of the primal value function stated in Theorem 3.1(i) in [6]. With the notation from Example 5.1 we can define the primal utility function

\[
\tilde{u}(x) := \sup_{\pi \in \mathbb{R}} \mathbb{E}[U(x + \pi \Delta \tilde{S} + B)], \quad x \in \mathbb{R},
\]

where we use the convention \( \mathbb{E}[U(x + \pi \Delta \tilde{S} + B)] = -\infty \) if \( \mathbb{E}[U(x + \pi \Delta \tilde{S} + B)^-] = +\infty \). Then \( \tilde{u} \) is not differentiable at \( x = 0 \) which is an interior point in \( \tilde{u} \)'s domain.

5.2. **A characterization via a linear stochastic control problem.**

Even though the superdifferential of \( \mathcal{U} \) at \( B \) consists of finitely-additive measures related to the solution of the dual problem, it is possible to give a characterization of directional derivatives without any recourse to finite additivity. This is the most attractive feature of our linear characterization in Proposition 5.5 below; moreover, as we shall see later, it also leads to explicit computations in many cases. The price we pay is the increased complexity of the linearized problem’s domain.

Throughout the remainder of the paper we impose the following assumption, where \( \hat{X} \) is the primal optimizer characterized by (4.4), and whose existence is guaranteed by the assumption of reasonable elasticity:

**Assumption 5.3.** \( U \) is reasonably elastic and there exists a constant \( b > 0 \) such that

\[
\hat{X} U' \left( (1 - b) \hat{X} \right) \in L^1(\mathbb{P}).
\]

**Remark 5.4.** Assumption 5.3 holds automatically if \( U \) belongs to the class of CRRA (power) utilities

\[
U(x) := \frac{p^p}{p}, \quad \text{for} \quad p \in (-\infty, 1) \setminus \{0\} \quad \text{or} \quad U(x) := \log(x), \quad x > 0.
\]

\[3\] The authors first learned from Pietro Siorpaes about the potential lack of correctness of Remark 4.2 in [6]. We also refer the reader to Erratum [7] for further discussions.
More generally, suppose that $U$ admits an upper bound on the relative risk-aversion, i.e., if $U \in C^2(0, \infty)$ and there exists a constant $c \in (0, \infty)$ such that

$$-\frac{x U''(x)}{U'(x)} \leq c \text{ for all } x > 0.$$  

It follows directly that $x^c U''(x) + c x^{c-1} U'(x) \geq 0 \text{ for all } x > 0$ which implies that the function $x^c U'(x)$ is nondecreasing on $(0, \infty)$. Therefore, $(\frac{1}{2} x)^c U''(\frac{1}{2} x) \leq x^c U''(x)$, and so, $U''(\frac{1}{2} x) \leq 2^c U''(x)$ for all $x > 0$. By combined this inequality with the first-order condition (4.4) we see

$$\hat{X} U'(\frac{1}{2} \hat{X}) \leq 2^c \hat{X} U'(\hat{X}) = 2^c \hat{X} d \mu \frac{d \mu}{d P}.$$  

Consequently, we have

$$\mathbb{E}[\hat{X} U'(\frac{1}{2} \hat{X})] \leq 2^c \mathbb{E}[\hat{X} d \mu \frac{d \mu}{d P}] \leq 2^c \langle \hat{X}, \hat{\mu} \rangle < \infty.$$  

Hence, (5.4) holds with $b := \frac{1}{2}$.

Given the optimizer $\hat{\pi} \in A$ and the random variable $\hat{X}$, we let $\Delta(\varphi) := \cup_{\varepsilon > 0} \Delta^\varepsilon(\varphi)$ where $\Delta^\varepsilon(\varphi)$ denotes the class of all $\delta \in L(S)$, such that

$$\hat{\pi} + \varepsilon \delta \in A \text{ and } \hat{X} + \varepsilon (\varphi + (\delta \cdot S)T) \geq 0.$$  

Because $A$ is a convex cone and $\hat{X} \geq 0$, the family $\Delta^\varepsilon(\varphi)$ is nonincreasing in $\varepsilon \geq 0$ in the sense

$$\varepsilon_1 \leq \varepsilon_2 \Rightarrow \Delta^{\varepsilon_2}(\varphi) \subseteq \Delta^{\varepsilon_1}(\varphi).$$  

Similarly, the family $\Delta^\varepsilon(\varphi)$ is nondecreasing in $\varphi \in L^\infty$ in the sense

$$\varphi_1 \leq \varphi_2 \Rightarrow \Delta^\varepsilon(\varphi_1) \subseteq \Delta^\varepsilon(\varphi_2).$$

Proposition 5.5. Under Assumption 5.3 we have for $\varphi \in L^\infty(\mathbb{P})$

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} (\mathcal{U}(B + \varepsilon \varphi) - \mathcal{U}(B)) = \sup_{\delta \in \Delta(\varphi)} \mathbb{E}[\hat{Y}((\delta \cdot S)T + \varphi)],$$  

where $\hat{Y} := \frac{d \hat{\mu}}{d \mathbb{P}}$.

Proof. For small enough $\varepsilon > 0$ we can find $\pi^\varepsilon \in A$ such that $X^\varepsilon = (\pi^\varepsilon \cdot S)T + B + \varepsilon \varphi$ has the property that

$$\mathbb{E}[U(X^\varepsilon_T)] \geq \mathcal{U}(B + \varepsilon \varphi) - \varepsilon^2.$$  

For such an $\varepsilon > 0$ we define

$$\delta^\varepsilon = \frac{1}{\varepsilon} \left( \pi^\varepsilon - \hat{\pi} \right).$$  

Since $\hat{\pi} + \varepsilon \delta^\varepsilon = \pi^\varepsilon \in A$, the first part of (5.5) above holds. To see that the second part of (5.5) holds, we note that $X + \varepsilon((\delta^\varepsilon \cdot S)T + \varphi) = X^\varepsilon$ and
\[ \mathbb{E}[U(X^\varepsilon)] > -\infty \] which implies \( \hat{X} + \varepsilon((\delta^\varepsilon \cdot S)_T + \varphi) \geq 0 \). Therefore, we have \( \delta^\varepsilon \in \Delta^\varepsilon(\varphi) \). The concavity of \( \Psi \) then implies that

\[
\Psi(B + \varepsilon \varphi) \leq \mathbb{E}[U(X^\varepsilon)] + \varepsilon^2
\]

\[
\leq \mathbb{E}[U(\hat{X})] + \varepsilon \mathbb{E}[U'(\hat{X})((\delta^\varepsilon \cdot S)_T + \varphi)] + \varepsilon^2
\]

\[
\leq \Psi(B) + \varepsilon \sup_{\delta \in \Delta^\varepsilon(\varphi)} \mathbb{E}[\hat{Y}((\delta \cdot S)_T + \varphi)] + \varepsilon^2
\]

This produces the upper bound inequality

\[
\limsup_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \left( \Psi(B + \varepsilon \varphi) - \Psi(B) \right) \leq \sup_{\delta \in \Delta(\varphi)} \mathbb{E}[\hat{Y}((\delta \cdot S)_T + \varphi)].
\]

To prove the opposite inequality, we pick \( \varepsilon_0 > 0 \) and \( \delta \in \Delta^\varepsilon(\varphi) \), so that

\[
\hat{\pi} + \varepsilon_0 \delta \in \mathcal{A} \quad \text{and} \quad \hat{X} + \varepsilon_0 D \geq 0, \quad \text{where}
\]

\[
D = (\delta \cdot S)_T + \varphi.
\]

Because \( b > 0 \), we also have

\[
\hat{X} + b \varepsilon_0 D \geq (1 - b)\hat{X}.
\]

Therefore, for \( \varepsilon \in (0, \varepsilon_1) \) with \( \varepsilon_1 := b \varepsilon_0 \) we have

\[
(5.9) \quad \hat{X} + \varepsilon D \geq (1 - b)\hat{X} > 0.
\]

The concavity of \( U \) implies that for \( \varepsilon \in (0, \varepsilon_1) \) we have

\[
U(\hat{X} + \varepsilon D) \geq U(\hat{X}) + \varepsilon Y^\varepsilon D \text{ where } Y^\varepsilon = U'(\hat{X} + \varepsilon D).
\]

Therefore, for \( \varepsilon \in (0, \varepsilon_1) \) we obtain

\[
\Psi(B + \varepsilon \varphi) \geq \mathbb{E}[U(\hat{X} + \varepsilon D)] \geq \Psi(B) + \varepsilon \mathbb{E}[Y^\varepsilon D].
\]

In order to pass \( \varepsilon \) to zero we note that \( (5.9) \) gives

\[
(5.10) \quad \left( Y^\varepsilon D \right)^{\frac{1}{\varepsilon}} \leq U'(((1 - b)\hat{X}) D^- \leq U'(((1 - b)\hat{X}) \frac{1}{\varepsilon_0} \hat{X},
\]

which is integrable by assumption. The uniform bound in \( (5.10) \) allows us to use Fatou’s lemma together with \( Y^\varepsilon \to U'(\hat{X}) =: \hat{Y} \), \( \mathbb{P} \)-a.s., to conclude that

\[
\liminf_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \left( \Psi(B + \varepsilon \varphi) - \Psi(B) \right) \geq \liminf_{\varepsilon \searrow 0} \mathbb{E}[Y^\varepsilon D] \geq \mathbb{E}[\hat{Y} D]. \quad \square
\]

The following example highlights the delicate structure of the linear optimization problem on the right-hand side of \( (5.8) \): it may admit several minimizers, each of which corresponds to a strict local martingale.
Example 5.6. Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space supporting two independent Brownian motions \((Z, W)\) and we let \(\{\mathcal{F}_t\}_{t \in [0,T]}\) be their augmented filtration up to some maturity \(T > 0\). We define the stock price dynamics to be
\[
dS_t := S_t(\lambda_t dt + dZ_t), \quad S_0 > 0,
\]
where the process \(\lambda \in \mathcal{L}^2\) is such that the minimal martingale density (see [15] for the definition and further discussion)
\[
\mathcal{E}(-\lambda \cdot Z)_t := e^{-\int_0^t \lambda_u dZ_u - \frac{1}{2} \int_0^t \lambda_u^2 du}, \quad t \in [0, T],
\]
fails the martingale property even though the set \(\mathcal{M}\) of equivalent local martingale measures is nonempty (an example of such a process \(\lambda\) can be found in Theorem 2.1 in [11], but its exact form is not important for this example). As a consequence,
the log-investor’s dual utility optimizer \(\hat{Y}_t := \hat{Y}_0 \mathcal{E}(-\lambda \cdot Z)_t\), where \(\hat{Y}_0 > 0\) is a Lagrange multiplier, is a strict local martingale (see Example 5.1 and Proposition 5.1 in [23] for details). We consider the constant endowment case where \(B := \hat{X}_0 := 1\) and the claim \(\varphi\) is a positive constant. The fact that we are working with the log-utility function implies that \(\hat{Y}_0 = 1\), which in our notation translates to
\[
\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} (\mathcal{U}(B + \varepsilon \varphi) - \mathcal{U}(B)) = \varphi.
\]
The log-utility function satisfies Assumption 5.3 which allows us to use Proposition 5.5 to conclude that
\[
\sup_{\delta \in \Delta(\varphi)} \mathbb{E}[\hat{Y}_T((\delta \cdot S)_T + \varphi)] = \varphi.
\]
For any \(\delta \in \Delta(\varphi)\) for which the local martingale \(\hat{Y}_T((\delta \cdot S)_T)\) is a martingale, we have \(\mathbb{E}[\hat{Y}_T((\delta \cdot S)_T + \varphi)] = \mathbb{E}[\hat{Y}_T \varphi]\). Since \(\hat{Y}\) is a strict local martingale, this value \(\mathbb{E}[\hat{Y}_T \varphi] = \varphi \mathbb{E}[\hat{Y}_T]\) is strictly smaller than the value \(\varphi\) of the supremum in (5.12). On the other hand, the supremum in (5.12) is attained at any \(\delta \in \Delta(\varphi)\) which satisfies the requirement \(\mathbb{E}[\hat{Y}_T((\delta \cdot S)_T + \varphi)] = \varphi\). This requirement is, in turn, satisfied by any \(\delta \in \Delta(\varphi)\) such that \((\delta \cdot S)_T + \varphi = \varphi \hat{X}_T\).

6. Minimally superreplicable random variables

While the linear control problem of Proposition 5.5 provides a useful characterization of \(\mathcal{U}\)'s directional derivatives, the linear problem seems to be difficult to solve explicitly in full generality. The present section outlines a relevant class of payoffs \(\varphi\) for which such a tractable solution is, indeed, available. It involves the concept of minimal superreplicability, which is similar to the notion of unique superreplicability from Condition (B1) in [29].

Definition 6.1. A random variable \(\psi \in \mathbb{L}^\infty(\mathbb{P})\) is said to be
(1) **replicable** if there exists a constant $\psi_0 \in \mathbb{R}$ and $\pi_\psi \in \mathcal{A} \cap (-\mathcal{A})$ such that

$$\psi = \psi_0 + (\pi_\psi \cdot S)_T.$$ 

(2) **minimally superreplicable (by $\Psi$)** if $\Psi \in L^\infty(\mathbb{P})$ is replicable, $\Psi \geq \psi$, and

$$x + (\pi \cdot S)_T \geq \psi \Rightarrow x + (\pi \cdot S)_T \geq \Psi$$

for all $x \in \mathbb{R}$ and $\pi \in \mathcal{A}$.

**Remark 6.2.**

(1) The need to use uniformly bounded gains processes for replication purposes such as in Definition 6.1(1) has long been recognized; see, e.g., Definition 1.15 in [4] and the first part of Remark 3.2 in [17].

(2) The representation in Definition 6.1(1) of a replicable claim $\psi$ in terms of $(\psi_0, \pi_\psi)$ is unique. Moreover, the process $(\pi_\psi \cdot S)_t$ is a bounded $\mathbb{Q}$-martingale for each $\mathbb{Q} \in \mathcal{M}$. Consequently, because each $\mu \in \mathcal{D}$ is the weak-* limit of a net $y_\alpha \mathbb{Q}_\alpha$ with $y_\alpha \in [0, \infty)$ and $\mathbb{Q}_\alpha \in \mathcal{M}$, we have

$$\langle \mu, (\pi_\psi \cdot S)_t \rangle = \lim_{\alpha} y_\alpha \langle \mathbb{Q}_\alpha, (\pi_\psi \cdot S)_t \rangle = 0.$$ 

(3) Provided it exists, the random variable $\Psi$ in Definition 6.1(2) is unique. If $\Psi = \Psi_0 + (\pi_\psi \cdot S)_T$ minimally superreplicates $\psi$, we have the representation

$$\Psi_0 = \sup_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}^\mathbb{Q}[\psi].$$

(4) Minimal superreplicability is scale invariant: If $\psi$ is minimally superreplicable by $\Psi$, then $\alpha \psi$ is minimally superreplicable by $\alpha \Psi$ for $\alpha \geq 0$. It is also invariant under translation by replicable random variables. In particular, replicable random variables are minimally superreplicable.

**Example 6.3.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space supporting two independent Brownian motions $(\beta, W)$ and we let $\{\mathcal{F}_t\}_{t \in [0,T]}$ be their augmented filtration up to some maturity $T > 0$. We let $S$ be the Itô process

$$dS_t := S_t \sigma_t (\lambda_t dt + d\beta_t), \quad S_0 > 0,$$

where $\sigma, \lambda \in \mathcal{L}^2$ are such that NFLVR holds. We focus on payoffs of the form $\varphi = \varphi(W_T)$, where $\varphi : \mathbb{R} \to \mathbb{R}$ is a bounded uniformly Lipschitz function. To show that such contingent claims $\varphi(W_T)$ are minimally superreplicable by the constant $\sup_a \varphi(a)$, we start by assuming that

$$x + (\pi \cdot S)_T \geq \varphi(W_T) \text{ a.s.},$$

for some $x \in \mathbb{R}$ and some $\pi \in \mathcal{A}$. Then, for each $t \in [0, T)$ we have

$$x + (\pi \cdot S)_t \geq \text{esssup}_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}^\mathbb{Q}_t [x + (\pi \cdot S)_T | \mathcal{F}_t] \geq \text{esssup}_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}^\mathbb{Q}_t [\varphi(W_T) | \mathcal{F}_t].$$
Lemma 6.4 below gives conditions under which the limit as $t \uparrow T$ of the right-hand side of (6.2) equals $\sup_a \varphi(a)$. When these conditions are met, the continuity of the paths of the stochastic integral with respect to $S$ implies that $x + (\pi \cdot S)t \geq \sup_a \varphi(a)$. This, in turn, confirms that $\varphi(W_T)$ is minimally superreplicable by the constant $\sup_a \varphi(a)$.

**Lemma 6.4.** In the setting of Example 6.3 above with $\varphi : \mathbb{R} \to \mathbb{R}$ bounded and uniformly Lipschitz, assume that there exists a nonnegative (deterministic) function $f \in L^1([0,T])$ and a predictable process $\nu(0) \in L^2$ such that

1. $|\nu(0)| \leq f(u)$, for Lebesgue-almost all $u \in [0,T]$, $\mathbb{P}$-a.s., and
2. the stochastic exponential $Z^{(0)} := \mathcal{E}(\lambda \cdot \beta - \nu(0) \cdot W)_T$ is the Radon-Nikodym density of some $Q^{(0)} \in \mathcal{M}$ with respect to $\mathbb{P}$.

Then

$$\lim_{t \uparrow T} \text{esssup}_{Q \in \mathcal{M}} \mathbb{E}[\varphi(W_T) | \mathcal{F}_t] = \sup_a \varphi(a).$$

**Proof.** For a bounded and predictable process $\delta$ we define the process $Z^{(\delta)}$ by

$$dZ^{(\delta)}_t := -Z^{(\delta)}_t (\lambda_t d\beta_t + (\nu^{(0)}_t + \delta_t) dW_t) \quad Z^{(\delta)}_0 := 1.$$ 

A simple calculation yields the following expression

$$Z^{(\delta)}_T = Z^{(0)}_T \mathcal{E}(-\delta \cdot W^{(0)}),$$

where $W^{(0)}_t := W_t + \int_0^t \nu^{(0)} u \, du$ is a $Q^{(0)}$-Brownian motion. With $\mathbb{E}^{(0)}$ denoting the expectation with respect to $Q^{(0)}$, we have

$$\mathbb{E}[Z^{(\delta)}_T] = \mathbb{E}^{(0)}[\mathcal{E}(-\delta \cdot W^{(0)})] = 1,$$

where the last equality follows from the boundedness of $\delta$. Hence, $Z^{(\delta)}$ is a (true) martingale and can be used as a density of a probability measure $Q^{(\delta)} \in \mathcal{M}$.

To proceed, we fix $t_0 \in (0,T)$ and $a \in \mathbb{R}$ and define

$$\delta^{(a)}_t := \frac{1}{t-t_0} (W_{t_0} 1_{\{|W_{t_0}| \leq 1/(T-t_0)\}} - a) 1_{\{t \geq t_0\}}, \quad t \in [t_0,T],$$

$$W^{(a)}_t := W_t + \int_0^t (\nu^{(0)} u + \delta^{(a)} u) \, du, \quad t \in [0,T].$$

Then we have

$$W_T - a = W^{(a)}_T - W^{(a)}_{t_0} - \int_{t_0}^T \nu^{(0)} u \, du + W_{t_0} 1_{\{|W_{t_0}| > 1/(T-t_0)\}}.$$ 

The process $W^{(a)}$ is a $Q^{(a)}$-Brownian motion, where $Q^{(a)}$ is a short for $Q^{(\delta^{(a)})}$. Therefore, the bound $|\nu^{(0)}| \leq f$ implies that

$$\mathbb{E}^{Q^{(a)}}[|W_T - a| | \mathcal{F}_{t_0}] \leq C(t_0)$$
where
\[ C(t_0) := \sqrt{\frac{2(T-t_0)}{\pi}} + \int_{t_0}^{T} f_u \, du + |W_{t_0}| \mathbf{1}_{\{|W_{t_0}| > 1/(T-t_0)\}}. \]

With \( L_\varphi \) denoting the uniformly Lipschitz constant of \( \varphi \), we have
\[ |\mathbb{E}_Q[\varphi(W_T)|\mathcal{F}_{t_0}] - \varphi(a)| \leq L_\varphi \mathbb{E}_Q[|W_T-a||\mathcal{F}_{t_0}] \leq L_\varphi C(t_0). \]

Therefore,
\[ \limsup_{t_0 \uparrow T} \mathbb{E}_Q[\varphi(W_T)|\mathcal{F}_{t_0}] \geq \limsup_{t_0 \uparrow T} E_Q(\varphi(a)) \geq \limsup_{t_0 \uparrow T} \left( \varphi(a) - L_\varphi C(t_0) \right) = \varphi(a). \]

It remains to note that the left-hand side above does not depend on \( a \) and that \( \sup_a \varphi(a) \) is a trivial upper bound in (6.3).

**Example 6.5.** (Continuation of Example 6.3) We continue Example 6.3 by examining two cases in which Lemma 6.4 applies. In the first one we simply take \( f := 0 \). That can be done if and only if the minimal martingale density \( \mathcal{E}(-\lambda \cdot B) \) defines a martingale which is the case in many popular models including the incomplete models developed in [21] and [27].

In the second case, \( \mathcal{E}(-\lambda \cdot B) \) is a strict local martingale but NFLVR nevertheless still holds. A famous example of a model where this occurs is given in [11]. We present here a time-changed version (using the standard logarithmic time transform \( t \mapsto -\log(1-t) \)), as the original version in Theorem 2.1 in [11] is defined on an infinite horizon. In the notation of Example 6.3 and with \( T := 1 \), we define the local martingales \((\beta'_t)_{t \in [0,1]}\) and \((W'_t)_{t \in [0,1]}\) by
\[ \beta'_t := \int_0^t \frac{1}{\sqrt{1-u}} \, dB_u \quad \text{and} \quad W'_t := \int_0^t \frac{1}{\sqrt{1-u}} \, dW_u, \quad t \in [0,1], \]
as well as the stopping times
\[ \tau := \inf\{t > 0 : \mathcal{E}(\beta') = \frac{1}{2}\} \quad \text{and} \quad \sigma := \inf\{t > 0 : \mathcal{E}(W') = 2\}. \]

With the processes \((\lambda_t)_{t \in [0,1]}\) and \((\nu_0(t))_{t \in [0,1]}\) defined by
\[ \lambda_t := -\frac{1_{\sigma \wedge \tau}(t)}{\sqrt{1-t}}, \quad \nu_0(t) := -\frac{1_{\sigma \wedge \tau}(t)}{\sqrt{1-t}}, \]
it remains to apply Theorem 2.1 in [11] to conclude that the NFLVR condition is satisfied, but that the minimal martingale density \( \mathcal{E}(-\lambda \cdot \beta) \) is a strict local martingale. Our Lemma 6.4 applies because \( |\nu_0(t)| \leq \frac{1}{\sqrt{1-t}} \in \mathbb{L}^1([0,1]) \).

We mention that Examples 6.3 and 6.5 as well as Lemma 6.4 will be used again in the examples in Section 7.

The next example shows that it is quite easy to construct bounded payoffs \( \psi \) which fail to be minimally superreplicable.
Example 6.6. We consider the following one period model with three states:

$$\Delta S := (1, 0, -1)', \quad \psi := (-1, 0, -1)',$$

where $a'$ denotes the transpose of the vector $a$. The set of pairs $(x, \pi)$ for which $x + \pi \Delta S \geq \psi$ is given by $x \geq 0$ and $\pi \in [-1 - x, 1 + x]$. However, the corresponding set of gain outcomes $(x + \pi, x, x - \pi)'$ with $x \geq 0$ and $\pi \in [-1 - x, 1 + x]$ does not contain a smallest element. Indeed, if $(a, b, c)$ is a smallest element, we would have $a \leq -1, b \leq 0, c \leq -1$ but such an element $(a, b, c)$ is not the outcome of any gains process $x + \pi \Delta S$ with $x \geq 0$ and $\pi \in [-1 - x, 1 + x]$.

The main technical result of this section is the following proposition:

**Proposition 6.7.** Under Assumption 5.3, suppose that $-B$ and $-(B + \varepsilon \varphi)$ are minimally superreplicable by $-B$ and $-(B + \varepsilon \varphi)$, respectively, for all $\varepsilon > 0$ in some neighborhood of 0. Then, for each $\hat{\mu} \in \mathcal{D}(B)$, we have

$$\lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \left( \mathcal{U}(B + \varepsilon \varphi) - \mathcal{U}(B) \right) = \mathbb{E}[\hat{Y} \varphi] + \lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \left( \hat{\mu}^{\hat{\varphi}}, B + \varepsilon \varphi - B \right),$$

where $\hat{Y} = \frac{d\hat{\mu}^{\hat{\varphi}}}{d\varphi}$.

**Proof.** For $\varepsilon > 0$ we let $x_\varepsilon, x_0 \in \mathbb{R}$ and $\pi_\varepsilon, \pi_0 \in \mathcal{A} \cap (-\mathcal{A})$ be such that

$$B + \varepsilon \varphi = \varepsilon x_\varepsilon + \varepsilon (\pi_\varepsilon \cdot S)_T \quad \text{and} \quad B = x_0 + (\pi_0 \cdot S)_T.$$  

Because $B$ is bounded away from zero and $\varphi \in \mathbb{L}^\infty(\mathbb{P})$ we can consider $\varepsilon > 0$ so small that $x_\varepsilon, x_0 > 0$. For $\delta \in \Delta^\varepsilon(\varphi)$ we have $\varepsilon \delta + \hat{\pi} \in \mathcal{A}$ and

$$\varepsilon (\delta \cdot S)_T + (\hat{\pi} \cdot S)_T \geq -\varepsilon \varphi - B.$$  

Therefore, by the minimal superreplicable of $B + \varepsilon \varphi$, we have

$$0 \leq x_\varepsilon + (\delta \cdot S)_T + \frac{1}{\varepsilon} (\hat{\pi} \cdot S)_T + (\pi_\varepsilon \cdot S)_T.$$  

Since $B$ minimally superreplicates $B$ we have $B + (\hat{\pi} \cdot S)_T \geq 0$. Therefore, for any $\hat{\mu} \in \mathcal{D}(B)$, the first part of Equation (4.7) in [6] produces

$$0 \leq \langle \hat{\mu}^{\hat{\varphi}}, B + (\hat{\pi} \cdot S)_T \rangle \leq \langle \hat{\mu}^{\hat{\varphi}}, B + (\hat{\pi} \cdot S)_T \rangle = 0.$$  

The second part of Equation (4.7) in [6] ensures $\langle \hat{\mu}, (\hat{\pi} \cdot S)_T \rangle = 0$ and combining this with $\langle \hat{\mu}^{\hat{\varphi}}, B + (\hat{\pi} \cdot S)_T \rangle = 0$ we see

$$\langle \hat{\mu}^{\hat{\varphi}}, B + (\hat{\pi} \cdot S)_T \rangle = \langle \hat{\mu}, B + (\hat{\pi} \cdot S)_T \rangle = \langle \hat{\mu}, B \rangle.$$  

Because $\hat{Y} = \frac{d\hat{\mu}^{\hat{\varphi}}}{d\varphi}$ we obtain the representation

$$\mathbb{E}[\hat{Y} (\hat{\pi} \cdot S)_T] = \langle \hat{\mu}^{\hat{\varphi}}, B \rangle.$$
The property $\varepsilon \delta + \hat{\pi} \in \mathcal{A}$ produces $\langle \hat{\mu}, \varepsilon (\delta \cdot S)_T \rangle + (\hat{\pi} \cdot S)_T \rangle \leq 0$ and $\langle \hat{\mu}, (\pi_\varepsilon \cdot S)_T \rangle = 0$, for each $\varepsilon > 0$. Therefore, by (6.5), we find
\[
\mathbb{E}[\hat{Y}(x_\varepsilon + (\pi_\varepsilon \cdot S)_T + (\delta \cdot S)_T + \frac{1}{\varepsilon} (\hat{\pi} \cdot S)_T)] \\
\leq \langle \hat{\mu}, x_\varepsilon + (\pi_\varepsilon \cdot S)_T + (\delta \cdot S)_T + \frac{1}{\varepsilon} (\hat{\pi} \cdot S)_T \rangle \\
\leq \langle \hat{\mu}, x_\varepsilon \rangle.
\]
(6.8)
To show that the upper bound in (6.8) above is attained we pick
\[
\delta_\varepsilon = (\frac{\varepsilon}{x_0} - \frac{1}{\delta})\hat{\pi} + \frac{\varepsilon}{x_0} \pi_\varepsilon - \pi_\varepsilon.
\]
Because $x_\varepsilon > 0$ and $x_0 > 0$ one can check that $\delta_\varepsilon \in \Delta^\varepsilon(\varphi)$. Then we have
\[
\mathbb{E}[\hat{Y}(x_\varepsilon + (\pi_\varepsilon \cdot S)_T + (\delta_\varepsilon \cdot S)_T + \frac{1}{\varepsilon} (\hat{\pi} \cdot S)_T)] \\
= \mathbb{E}[\hat{Y}(x_\varepsilon + \frac{\varepsilon}{x_0} ((\hat{\pi} + \pi_\varepsilon) \cdot S)_T)] \\
= \frac{x_\varepsilon}{x_0} \mathbb{E}[\hat{Y}(\hat{B} + (\hat{\pi} \cdot S)_T)] \\
= \frac{x_\varepsilon}{x_0} \langle \hat{\mu}, \hat{B} + (\hat{\pi} \cdot S)_T \rangle \\
= \langle \hat{\mu}, x_\varepsilon \rangle,
\]
where the last equality follows from $\langle \hat{\mu}, \hat{B} \rangle = \langle \hat{\mu}, x_0 \rangle$ and $\langle \hat{\mu}, (\hat{\pi} \cdot S)_T \rangle = 0$. Therefore, $\delta_\varepsilon$ indeed attains the upper bound of (6.8), and, so,
\[
\sup_{\delta \in \Delta^\varepsilon(\varphi)} \mathbb{E}[\hat{Y}((\delta \cdot S)_T + \varphi)] = \\
= \langle \hat{\mu}, x_\varepsilon \rangle - \frac{1}{\varepsilon} \mathbb{E}[\hat{Y}(\hat{B} + \varepsilon \varphi)] - \frac{1}{\varepsilon} \mathbb{E}[\hat{Y}(\hat{\pi} \cdot S)_T] + \mathbb{E}[\hat{Y} \varphi] \\
= \langle \hat{\mu}^r, \varphi \rangle + \frac{1}{\varepsilon} \langle \hat{\mu}^s, \hat{B} + \varepsilon \varphi - \hat{B} \rangle.
\]
The sets $\Delta^\varepsilon(\varphi)$ monotonically increase to $\Delta(\varphi)$ as $\varepsilon \searrow 0$. This implies
\[
(6.10) \sup_{\delta \in \Delta^\varepsilon(\varphi)} \mathbb{E}[\hat{Y}((\delta \cdot S)_T + \varphi)] \searrow \sup_{\delta \in \Delta(\varphi)} \mathbb{E}[\hat{Y}((\delta \cdot S)_T + \varphi)]
\]
as $\varepsilon \searrow 0$. Because the left-hand-side of (6.10) equals $\langle \hat{\mu}^r, \varphi \rangle + \frac{1}{\varepsilon} \langle \hat{\mu}^s, \hat{B} + \varepsilon \varphi - \hat{B} \rangle$, we see that (6.4) holds by Proposition 5.5.

A first consequence of Proposition 6.7 is that the situation encountered in Example 5.1 cannot happen if $-\hat{B}$ is minimally superreperlicable. Indeed, the primal value function $\hat{\mu}$ is then differentiable in all replicable directions:

**Corollary 6.8.** Suppose that Assumption 5.3 holds, that $-\hat{B}$ is minimally superreperlicable and that $\varphi$ is replicable. Then the following two-sided limit exists
\[
(6.11) \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left( \hat{\mu}(B + \varepsilon \varphi) - \hat{\mu}(B) \right) = \langle \hat{\mu}, \varphi \rangle \text{ for each } \hat{\mu} \in \hat{D}(B).
\]
In particular, there exists a constant $y_B > 0$ such that
\[
y_B = \hat{\mu}(\Omega) \text{ for each } \hat{\mu} \in \hat{D}(B).
\]
Proof. First observe that for replicable \( \varphi \) we have \( B + \varepsilon \varphi = B + \varepsilon \varphi \). Then we can apply Proposition 6.7 to both \( \varphi \) and \( -\varphi \) to see (6.11). The last claim follows by setting \( \varphi = 1 \). \( \square \)

Now that we have identified the circumstances under which all dual minimizers \( \hat{\mu} \in \hat{D}(B) \) have the same total mass, the following result follows directly from Theorem 4.2 and Corollary 6.8 above.

**Corollary 6.9.** Suppose that Assumption 5.3 holds and that \( -B \) is minimally superreplicable. Then each replicable \( \varphi \in L^\infty(\mathbb{P}) \) has the unique \( B \)-conditional Davis price \( \langle \hat{\mu}, \varphi \rangle / \hat{\mu}(\Omega) \).

When \( B \) is a constant (and more generally, when \( B \) is replicable), it is known that the product of the primal and dual optimizers is a martingale (see, e.g., the discussion on p.911-2 in [23]). When the dual optimizer is only a finitely-additive measure, the following corollary may serve as a surrogate. The result relies on [20] where a positive supermartingale deflator \( \{\hat{Y}_t\}_{t \in [0,T]} \) is constructed from \( \hat{\mu} \in \hat{D}(B) \) (see Equation 2.5 in [20]).

**Corollary 6.10.** Suppose that Assumption 5.3 holds, that \( -B \) is minimally superreplicable by \(-B\), and write \( B = x_0 + (\pi_0 \cdot S)_T \).

Let \( \hat{\pi} \in A \) denote the optimizer for the problem \( \sup_{\pi \in A} \mathbb{E}[U(B + (\pi \cdot S)_T)] \). Then the process
\[
\hat{Y}_t \left( x_0 + ((\pi_0 + \hat{\pi}) \cdot S)_t \right), \quad t \in [0,T],
\]
is a nonnegative martingale where \( \{\hat{Y}_t\}_{t \in [0,T]} \) is the supermartingale deflator corresponding to \( \hat{\mu} \in \hat{D}(B) \).

**Proof.** From Theorem 2.10 in [20] we know that the process in question is a nonnegative supermartingale. Furthermore, also from [20], we have \( \hat{Y}_T = \frac{d\mu^r}{d\mathbb{P}} \) and \( \hat{Y}_0 \leq \hat{\mu}(\Omega) \). To obtain the constant expectation property we use (6.6) to get
\[
\langle \hat{\mu}, x_0 \rangle = \langle \hat{\mu}, (\hat{\pi} \cdot S)_T + B \rangle = \langle \hat{\mu}^r, (\hat{\pi} \cdot S)_T + B \rangle
= \mathbb{E} \left[ \hat{Y}_T((\hat{\pi} \cdot S)_T + B) \right] \leq \hat{Y}_0 x_0 \leq \hat{\mu}(\Omega)x_0,
\]
and the claimed martingale property follows. \( \square \)

7. The Interval of Conditional Davis Prices

This section closes the loop and gives an explicit expression for the interval of conditional Davis prices under the assumption of minimal superreplicability. The superdifferential characterization in Proposition 3.7 together with the properties of minimally superreplicable claims, allows us to compute the interval of conditional Davis prices quite explicitly in many cases of interest:
Theorem 7.1. Suppose that Assumption 5.3 holds and that $-B$ and $-(B + \varepsilon \varphi)$ are minimally superreplicable by $-B$ and $-(B + \varepsilon \varphi)$, respectively, for all $\varepsilon$ in some neighborhood of 0. The interval of $B$-conditional Davis prices of $\varphi$ is given by

$$\frac{1}{y_B} E[\hat{Y} \varphi] + \frac{1}{y_B} \left[ \lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \left( \hat{\mu}^s, B + \varepsilon \varphi - B \right) , \lim_{\varepsilon \nearrow 0} \frac{1}{\varepsilon} \left( \hat{\mu}^s, B + \varepsilon \varphi - B \right) \right],$$

where $y_B$ is the common value of $\hat{\mu}(\Omega)$ for all $\hat{\mu} \in \hat{D}(B)$.

Proof. By Corollary 6.8, the function $u$ is differentiable in $x$ at $x = 0$, with derivative $y_B$. The interval of $B$-conditional Davis prices, according to Proposition 3.7, is given by

$$\frac{1}{y_B} [\partial_{\varepsilon+} u(0,0), \partial_{\varepsilon-} u(0,0)].$$

This, in turn, coincides with the expression in (7.1) thanks to Proposition 6.7. □

7.1. Two illustrative examples. We conclude by giving two illustrative examples, both in an incomplete Brownian setting, of situations where our results can be applied directly and lead to explicit formulas for the nontrivial interval of conditional Davis prices.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space supporting two independent Brownian motions $(Z,W)$ and we let $\{\mathcal{F}_t\}_{t \in [0,T]}$ be their augmented filtration up to some maturity $T > 0$. In both examples, the stock-price dynamics are given by a one-dimensional Itô process

$$dS_t := S_t \sigma_t (\lambda_t dt + dZ_t), \quad S_0 > 0,$$

with processes $\sigma, \lambda \in L^2$. With more driving Brownian motions than assets, this leads to an incomplete financial model. Both examples will feature (an unspanned) contingent claim paying out $\varphi(W_T)$ at time $T$, where $\varphi : \mathbb{R} \to \mathbb{R}$ is a non-constant, bounded, and uniformly Lipschitz function.

The major difference between the examples is that in the first example the random endowment degenerates ($B := x$ for a constant $x > 0$), while in the second example the random endowment $B$ is non-replicable.

The first example illustrates that even when $B := x > 0$ is constant, our setting differs from that of [24] because the corresponding Davis prices are non-unique, whereas the growth condition placed on the claim’s payoff in Assumption 4 in [24] always produces unique Davis prices (the growth condition used in [24] originates from Theorem 3.1(i) in [17]). In other words, the payoffs considered in the first example are not included in [24].

The second example backs up the claim we made in both the abstract and in the introduction: when the endowment $B$ is non-replicable, the generic case is that Davis prices are non-unique.
Example 7.2. We adopt the setting used in Example 6.3 above which is based on [11]. The endowment is taken to be \( B := x > 0 \) constant. It follows from Example 6.3 that the interval of arbitrage-free prices for \( \varphi(W_T) \) is given by \((\varphi, \overline{\varphi})\) where

\[
\varphi := \inf_{a \in \mathbb{R}} \varphi(a), \quad \overline{\varphi} := \sup_{a \in \mathbb{R}} \varphi(a).
\]

Our Theorem 7.1 with \( B := x > 0 \) constant shows that the interval of log-investor’s Davis’ prices for \( \varphi(W_T) \) is given by \([p, \overline{p}]\) where

\[
p := 1 \hat{Y}_0 \mathbb{E}[\hat{Y}_T (\varphi - \overline{\varphi})] + \varphi, \quad \overline{p} := \varphi - 1 \hat{Y}_0 \mathbb{E}[\hat{Y}_T (\overline{\varphi} - \varphi)].
\]

Therefore, since the function \( \varphi \) is not constant, we have \( p - \overline{p} = (\varphi - \overline{\varphi}) (1 - \mathbb{E}[\hat{Y}_T] / \hat{Y}_0) > 0 \).

Example 7.3. In this example, we consider the Samuelson-model setting used in Section 2 in [29] where the stock price dynamics are given by (7.2) with both \( \sigma_t := \sigma > 0 \) and \( \lambda_t := \lambda > 0 \) being constants. Let \( U(\xi) := \xi^\gamma, \quad \xi > 0, \quad \gamma < 1, \) be an arbitrary utility function in the “power” family, with constant relative risk-aversion parameter (as usual \( \gamma := 0 \) is interpreted as the log investor).

The investor receives the random endowment of the form \( B(W_T) \) at time \( T > 0 \), where \( B \) is a non-constant, bounded and uniformly Lipschitz function. The payoff \( \varphi \) whose \( B \)-conditional Davis prices we are computing, as well as the quantities \( \varphi \) and \( \overline{\varphi} \) are defined exactly as in Example 1 above. We also define the following quantities

\[
B(\varepsilon) := \inf_{a \in \mathbb{R}} \left( B(a) + \varepsilon \varphi(a) \right), \quad \overline{B}(\varepsilon) := \sup_{a \in \mathbb{R}} \left( \varepsilon \varphi(a) - B(a) \right), \quad \varepsilon \geq 0.
\]

Proposition 2.4 in [29] states that the dual optimizer \( \hat{Q} \in \text{ba}(\mathbb{P}) \) for the utility-maximization problem with the random endowment of the form \( B(W_T) \) has a nontrivial singular part in the Yosida-Hewitt decomposition \( \hat{Q} = \hat{Q}^r + \hat{Q}^s \) after a possible shift of the function \( B \) by a constant. Moreover, such a shift can always be arranged so as to keep the values of \( B \) positive and bounded away from 0. Therefore, we assume, without loss of generality that such a shift has already been performed, so that, in particular, we have \( B(0) > 0 \). This loss-of-mass property for \( \hat{Q}^r \) can be partially quantified as follows: Theorem 3.7 in [13] and Proposition 3.2 in [31] allow us to write

\[
\frac{d\hat{Y}_t}{d\hat{Y}_0} = -\hat{Y}_t \left( \frac{\nu_t}{\hat{Y}_t} dZ_t + \dot{\nu}_t dW_t \right), \quad \hat{Y}_0 > 0,
\]

for some process \( \dot{\nu} \in \mathcal{L}^2 \). The presence of the nontrivial singular part \( \hat{Q}^s \) implies that \( \hat{Y} \) is a strict local martingale, i.e., \( \mathbb{E}[\hat{Y}_T] < \hat{Y}_0 \).

Example 6.3 takes care of the conditions of Theorem 7.1 dealing with minimal superreplicability. Indeed, both \(-B\) and \(-(B + \varepsilon \varphi)\) are of the
form treated there, and are, therefore, minimally superreplicable by \(-B\) and \(-B(\epsilon)\), respectively, for \(\epsilon \geq 0\).

The last step before Theorem 7.1 is applied is to simplify the two \(\epsilon\)-limits appearing in (7.1). That is an easy task thanks to the fact that the random variable \(\frac{1}{\epsilon}(B + \epsilon \varphi - B)\) is constant and equal to \(\frac{1}{\epsilon}(B(\epsilon) - B(0))\). Theorem 7.1 guarantees that this quotient admits a left and a right limit at \(\epsilon = 0\) and we introduce the following notation

\[
B'(0+) := \lim_{\epsilon \searrow 0} \frac{1}{\epsilon}(B(\epsilon) - B(0)) \quad \text{and} \quad B'(0-) := \lim_{\epsilon \nearrow 0} \frac{1}{\epsilon}(B(\epsilon) - B(0)).
\]

The total mass in \(\hat{Q}^s\) is given by \(\hat{Y}_0 - \mathbb{E}[\hat{Y}_T]\), and, so the interval of \(B(W_T)\)-conditional Davis prices for the payoff \(\varphi(W_T)\) is given by \([\underline{p}, \overline{p}]\) where

\[
\underline{p} := \frac{1}{\hat{Y}_0} \mathbb{E}\left[\hat{Y}_T (\varphi(W_T) - B'(0+))\right] + B'(0+),
\]

\[
\overline{p} := B'(0+) - \frac{1}{\hat{Y}_0} \mathbb{E}\left[\hat{Y}_T (B'(0+) - \varphi(W_T))\right].
\]

REFERENCES


