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On the Propagation of Singularities of Semi-convex Functions

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0. - Introduction

In a recent paper [1], *upper* bounds on the dimension of singular sets of semi-convex functions were derived by measure-theoretic arguments.

To briefly describe these upper bounds, let $u : \mathbb{R}^n \to \mathbb{R}$ be a semi-convex function (Definition 1.2 below). Define

$$S^k(u) = \{x \in \mathbb{R}^n : \dim(\partial u(x)) = k\},$$

where $k \in [0, n]$ is an integer and $\partial u(x)$ denotes, as usual, the subdifferential of u. Clearly, $\{S^k(u)\}_{k=0}^n$ is a partition of $\mathbb{R}^n$ and $S^0(u)$ is the set of all points of differentiability of u. Since we are interested in first-order singularities, we call a point x singular for u if $x \in S^k(u)$ for some $k \geq 1$.

In [1] it is proved that $S^k(u)$ is countably $\mathcal{H}^{n-k}$-rectifiable. In particular,

$$\mathcal{H} - \dim(S^k(u)) \leq n - k,$$

where $\mathcal{H} - \dim$ is the Hausdorff dimension.

The purpose of the present work is to obtain *lower* bounds on the dimension of $S^k(u)$. More precisely, we will describe the structure of $S^k(u)$ in a neighborhood of x, knowing the geometry of $\partial u(x)$.

A motivating application of these results concerns the analysis of singularities of solutions to the Hamilton-Jacobi-Bellman equation

$$H(x, u, \nabla u) = 0. \tag{1}$$

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In fact, if the data are smooth, viscosity solutions of such PDE's (and, in particular, the solutions that are relevant to optimal control) enjoy well-known semi-concavity properties (see for instance [12], [13], [15] on [16]).

The present work is related to [4] and [5], in which viscosity solutions of (1) are shown not to have any isolated singularity if H is strictly convex with respect to p. In [4] and [5], however, no attention is paid to the dimension of ∂u at such singular points, and no attempt is made to estimate the Hausdorff measure of the singular sets.

Different approaches to the analysis of singularities of Hamilton-Jacobi equations are obtained for the one-dimensional case in [14] and, using characteristics, in [21].

Semi-convexity was the only property used in [1] to prove upper bounds on singular sets. On the contrary, to obtain lower bounds we need additional information. This fact is the essential difference between [1] and the present paper. In order to understand the nature of the additional information, let us consider the set of reachable subgradients

$$\nabla_* u(x) = \big\{ \lim_{h\to+\infty} \nabla u(x_h) : x_h \in S^0(u)\backslash\{x\},\ \ x_h \to x \big\}.$$

where ∇u denotes the gradient of u.

The above set is a set of generators of $\partial u(x)$ in the sense of convex analysis. Then, we show that the strict inclusion

$$\nabla_* u(x) \subsetneq \partial u(x) \tag{2}$$

is a sufficient condition for the propagation of any singularity $x \in S^k(u)$, $1 \le k < n$ (see Example 2.1 below). Inclusion (2) is satisfied by any viscosity solution of (1) with a strictly convex Hamiltonian, as $\nabla_* u(x)$ is contained in the zero-level set of $H(x, u(x), \cdot)$.

Moreover, if x is an isolated singularity, by adapting a variational argument of Tonelli (see the proof of the implicit function theorem in [20]), we show that $\nabla_* u(x)$ coincides with $\partial u(x)$, see Theorem 2.1 below.

Furthermore, inserting non-smooth analysis into this procedure, we obtain a more detailed description of the singular sets. In Theorem 2.2 we prove that singularities propagate along directions related to the geometry of $\partial u(x)$. These directions are orthogonal to the exposed faces of $\partial u(x)$. In Theorem 2.3 we give a lower bound on the maximum integer $m \le k$ such that x is a cluster point of

$$\Sigma^m(u) = \bigcup_{i=m}^{n} S^i(u),$$

and in (2.7) we estimate from below the Hausdorff $(n-k)$-dimensional measure of $\Sigma^m(u)$. Roughly speaking, the computation of m takes into account how many vectors in $\nabla_* u(x)$ are necessary to generate $\partial u(x)$.

We conclude with an outline of the paper. The first section contains preliminary material on Hausdorff measures, semi-convex functions, and the