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SINGULARITIES AND UNIQUENESS OF CYLINDRICALLY SYMMETRIC SURFACES MOVING BY MEAN CURVATURE

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Introduction

In this paper we study the singularities and the uniqueness properties of cylindrically symmetric hypersurfaces which move by mean curvature in $I\!R^N$, with $N \geq 3$. More precisely, we study the properties of the evolution Γ_t of Γ_0 by mean curvature, where Γ_0 is a smooth cylindrically symmetric surface in $I\!R^N$ parametrized by

$$r = h_0(z), \tag{0.1}$$

where h_0 may be multivalued with the several branches matching together in a smooth way. Throughout this paper we consider torus-like Γ_0 as opposed to bar-bell looking shapes, although our arguments can be modified to apply to the latter case.

Following recent work of Evans and Spruck [ES] and Chen, Giga and Goto [CGG], Γ_t is defined as the zero level set of the solution u of the geometric pde

$$\begin{cases} u_t = \text{ trace } [(I - \dfrac{Du \otimes Du}{|Du|^2})D^2u] \text{ in } I\!R^N \times (0,\infty), \\ \\ u(x,0) = u_0(x), \end{cases} \tag{0.2}$$

where the continuous function $u_0 : \mathbb{R}^N \to \mathbb{R}$ is chosen so that $\Gamma_0 = \{x \in \mathbb{R}^N : u_0(x) = 0\}$ and (0.2) is interpreted in the viscosity sense of Crandall and Lions. (For the precise definitions as well as a general overview of the theory of viscosity solutions we refer to the "user's guide" by Crandall, Ishii and Lions [CIL]). This approach, known as the level set approach, provides Γ_t, which is defined globally in time, but leaves open the question of its regularity. Another intriguing question related to the above definition is whether Γ_t has non-empty interior. Partial regularity results for motion by mean curvature were obtained by Evans and Spruck [ES] and Ilmanen [I], who actually reconciled, under some assumptions, the level set approach with the geometric measure theory approach of Brakke [B]. General, but not necessary, conditions on Γ_0, which guarantee empty interior for Γ_t, were given in Barles, Soner and Souganidis [BSS]. DeGiorgi [DG] has also conjectured several results related to interior of Γ_t.

We continue by formulating the main results. As long as Γ_t is smooth, (since Γ_0 is smooth, Γ_t will be smooth for small time (cf. Evans, Spruck [ES])), it can be parametrized by

$$r = h(z,t), \tag{0.3}$$

where the, possibly multivalued, smooth function h solves the equation

$$\begin{cases} h_t = \dfrac{h_{zz}}{1+h_z^2} - \dfrac{N-2}{h} \\ \\ h(z,0) = h_0(z), \end{cases} \tag{0.4}$$

in some time dependent domains for each branch of h_0, and, therefore, h. It is, of course, immediate that as long as $h \neq 0$, (0.4) has classical solutions, and, therefore, Γ_t is smooth. The only potential singularities of Γ_t may therefore occur when $h = 0$ for the first time. To formulate our first result, let us specify that h vanishes for the first time at $(0,T)$ and that (0.4) holds in

$$\Omega = (-2A, 2A) \times (0,T), \tag{0.5}$$

for some $A > 0$. Throughout this discussion we will be assuming that for all $(z,t) \in (-2A, 2A) \times [0,T]$,

$$h(z,t) = h(-z,t) \text{ and } zh_z(z,t) \geq 0. \tag{0.6}$$

It is immediate that (0.6) yields that $h(\cdot,t)$ has only one minimum at the origin. Although the above assumption will be used extensively throughout the paper we believe that our arguments can be easily modified to allow for more local minima which vary in time.

Theorem 1: *Assume that (0.6) holds. Then*

$$\lim_{t\uparrow T}(T-t)^{-\frac{1}{2}}h(y(T-t)^{\frac{1}{2}},t)=\sqrt{2(N-2)}, \tag{0.7}$$

with the limit uniform for $|y|$ bounded.

The following corollary is an immediate consequence.

Corollary 2: *Up to a parabolic scaling, at the singularity the surface Γ_t converges to a cylinder.*

A similar result was obtained by Huisken [H] when $N=3$ under the condition that Γ_0 has positive mean curvature but without assuming (0.6). Note that if Γ_0 has positive mean curvature, then Γ_t has empty interior for all $t\geq 0$ (see [BSS] or Theorem 3.1, below).

Another consequence of Theorem 1 is that Γ_t does not develop interior at $t=T$ and that it continues as a smooth surface for $t>T$ till the time it becomes extinct. We will prove this result for the special case of a torus. A brief discussion of how to extend this result for other initial data is given before Theorem 3.8.

Consider the evolution $t\mapsto\Gamma_t$ by mean curvature of the torus Γ_0, which is given by

$$\Gamma_0=\{x\in I\!R^N:(r-1)^2+z^2=R^2\}\qquad(0<R<1),$$

where $r^2=\sum_{i=1}^{N-1}x_i^2$.

Proposition 3: *Γ_t never develops interior. Moreover, there exists $R_0\in(0,1)$ such that for all $R\in(0,R_0),\Gamma_t$ shrinks to a circle and then becomes extinct. For $R\in(R_0,1),\Gamma_t$ "focuses" at 0 at some time T_R. It then "opens up" and flows smoothly until it becomes extinct. Finally, for $R=R_0,\Gamma_t$ focuses at exactly the same time it becomes a circle.*

The above result is related to a conjecture of DeGiorgi [DG]. Numerical evidence suggesting the above described behavior were obtained by Paolini and Verdi [PV].

The paper is organized as follows: In Section 1 we prove a curvature bound. Section 2 is devoted to analyzing the behavior of Γ_t at the focusing point. In Section 3 we show that Γ_t does not develop interior and study its behavior after the focusing occurs. Finally, in the Appendix we state a result about the number of zeroe's of solutions of linear parabolic equations, which we will be using repeatedly throughout the paper.

The results of this paper were announced in [SS]. By the time this paper was completed, the authors learned that Altschuler, Angenent and Giga [AAG1,2] also studied the motion by mean curvature of bodies of rotation. Their set up is and their results are formulated for "barbell" type shapes. On the other hand, they do not need assumption (0.6).

1 A curvature bound

In this section we obtain an upper bound for the ratio of the radial and angular components of the curvature of Γ_t near the singularity. This bound plays a fundamental role in the proof of Theorem 1 and its consequences.

More precisely, let

$$\psi = \frac{hh_{zz}}{1+h_z^2} \tag{1.1}$$

and recall that h solves in (0.4) in $(-2A, 2A) \times (0,T)$.

Proposition 1.1: *There exist $\varepsilon > 0$ and $c_0 > 0$ such that*

$$|\psi| \leq c_0^{-1} \ in \ [-\frac{A}{2}, \frac{A}{2}] \times [0,T) \tag{1.2}$$

and

$$\psi \leq (N-2) - c_0 \ in \ [-\varepsilon, \varepsilon] \times [T-\varepsilon, T). \tag{1.3}$$

The meaning of (1.3) is that near the singularity at $(0,T)$, the radial component of the curvature is strictly dominated by the angular one. As a matter of fact, the ratio between the radial and angular components of the curvature is strictly below of the same ratio of the curvature of the *catenoid*, the stationary solution of (0.4). This, in turn, yields (as we will see in Section 2.4) that $0 \notin$ int Γ_T.

The proof of Proposition 1.1 is based on the existence of an one-parameter family of barriers (q_λ), with $\lambda \in [\lambda_0, \lambda_1]$ for some $0 < \lambda_0 < \lambda_1$, which are defined on

$$\tilde{\Omega}_1 = [-B/2, B/2] \times [0,T),$$

with $0 < B \leq A$, and a result of Angenent ([A1] and Lemma A, below) on the number of zeroes of solutions of linear parabolic equations in one space dimension. The idea of using the bariers goes back to Angenent [A2]. Our construction of bariers here is, however, new.

In the sequel we will need the following notation. For $\lambda \in [\lambda_0, \lambda_1]$, $t \in [0,T)$ and $|\theta| \le B$ we define:

$$\Omega_1 = (-B/2, B/2) \times (0,T), \tilde{\Omega} = (-2B, 2B) \times [0,T),$$

$$I(t,\lambda,\theta) = \{|z| \le B/2 \;:\; q_\lambda(z,t) \le h(z-\theta,t)\},$$
$$n(t,\lambda,\theta) = \#\{|z| \le B/2 \;:\; q_\lambda(z,t) = h(z-\theta,t)\}.$$

Next we state two lemmas which assert the existence of the barriers with the necessary properties.

Lemma 1.3: *There exist $B > 0, 0 < \lambda_0 < \lambda_1$, an one-parameter family*

$$q_\lambda : \tilde{\Omega}_1 \to (0,\infty) \;,\quad (\lambda \in [\lambda_0, \lambda_1]),$$

and constants $c, \delta > 0$ such that for every $|\theta| \le B$ and $t \in [0,T)$:

$$n(t,\lambda_0,\theta) = 2, \tag{1.4}$$

$$I(t,\lambda_1,\theta) = \phi, \tag{1.5}$$

$$n(t,\lambda,\theta) \le 2, \;\; \forall \lambda \in [\lambda_0, \lambda_1], \tag{1.6}$$

$$(\lambda, z, t) \to (q_\lambda(z,t), \frac{\partial}{\partial z} q_\lambda(z,t)) \text{ is continuous on } [\lambda_0, \lambda_1] \times \tilde{\Omega}_1, \tag{1.7}$$

$$z q_{\lambda,z}(z,t) \ge 0, \text{ and } q_\lambda(z,t) = q_\lambda(-z,t), \tag{1.8}$$

$$q_\lambda(\pm\frac{B}{2}, t) \ge \sup_{|z| \le 2B} h(z,t), \tag{1.9}$$

and

$$\psi_\lambda \le (N-2) - c, \text{ whenever } q_\lambda \le \delta, \tag{1.10}$$

where

$$\psi_\lambda = \frac{q_\lambda q_{\lambda,zz}}{1 + (q_{\lambda,z})^2}.$$

Lemma 1.4: *For every* $(z_0, t_0) \in \tilde{\Omega}_1$, *there exists* $(\lambda^*, \theta^*) \in [\lambda_0, \lambda_1] \times [-B/2 - z_0, B/2 - z_0]$ *such that*

$$q_{\lambda^*}(z_0 + \theta^*, t_0) = h(z_0, t_0) \text{ and } q_{\lambda^*, z}(z_0 + \theta^*, t_0) = h_z(z_0, t_0). \tag{1.11}$$

We continue with the proof of Proposition 1.1 and then return to Lemmas 1.3 and 1.4.

Proof of Proposition 1.1: Let (λ^*, θ^*) be as in Lemma 1.4. We claim that

$$q_{\lambda^*, zz}(z_0 + \theta^*, t_0) \geq h_{zz}(z_0, t_0).$$

Indeed, if not, then by (1.9) and (1.11), the difference $q_{\lambda^*}(\cdot, t_0) - h(\cdot - \theta^*, t_0)$ would have more than two zeroes, which contradicts (1.6).

Therefore

$$\psi(z_0, t_0) \leq \psi_{\lambda^*}(z_0 + \theta^*, t_0).$$

Suppose now that $h(z_0, t_0) \leq \delta$, where δ is the constant appearing in (1.10). Since $h(z_0, t_0) = q_{\lambda^*}(z_0 + \theta^*, t_0)$, (1.10) yields

$$\psi_{\lambda^*}(z_0 + \theta^*, t_0) \leq (N-2) - c.$$

Using (0.4) we conclude that

$$h_t(z_0, t_0) < 0 \text{ and } \psi(z_0, t_0) \leq (N-2) - c.$$

A simple iteration of the above argument then yields that, for all $t \in [t_0, T)$,

$$h(z_0, t) \leq \delta, h_t(z_0, t) < 0 \text{ and } \psi(z_0, t) \leq (N-2) - c.$$

Since $h(0, t) \to 0$ as $t \to T$ and $h(\cdot, t)$ is continuous for every $t \in [0, T)$, there exist $\gamma > 0$ and $t_0 < T$ such that

$$h(\cdot, t_0) \leq \delta \qquad \text{in } [-\gamma, \gamma]$$

and

$$\psi \leq (N-2) - c \text{ in } [-\gamma, \gamma] \times [t_0, T).$$

Hence (1.3) holds with $\varepsilon = \min(\gamma, T - t_0)$.

To prove (1.2), we first observe that a simple but tedious calculation yields that ψ satisfies the equation

$$\psi_t = \frac{\psi_{zz}}{1 + w^2} - \frac{2(1+\psi)w}{h(1+w^2)}\psi_z + \frac{2w^2}{h^2(1+w^2)}(\psi - (N-2))(\psi + 1) \tag{1.12}$$

where to simplify the notation we write

$$w = h_z. \tag{1.13}$$

Since (0.4) holds in $(-2A, 2A) \times (0,T)$ and $\inf\limits_{t\in[0,T)} h(\pm A, t) > 0$, interior elliptic regularity yields that $\sup\limits_{t\in[0,T)} |\psi|(\pm A, t) < \infty$. Finally, applying the maximum principle to (1.12), we see that ψ cannot have an interior minimum smaller than -1. Combining all the above yields

$$\psi \geq \min\left[\inf_{|z|\leq A} \psi(z,0), \inf_{[0,T)} \psi(\pm A, t), -1\right].$$

Arguing in a similar way and also using (1.3), we obtain an upper bound for ψ.

□

We continue with several preliminaries leading to the proof of Lemma 1.3. The construction of the q_λ's is based on catenoids, which are stationary solutions of (0.4), i.e. functions $z \mapsto \lambda q(z/\lambda)$, where

$$\frac{qq_{zz}}{1+q_z^2} = N-2 \text{ and } q(0) = 1.$$

An elementary calculation gives

$$q_z(z) = (q(z)^{2(N-2)} - 1)^{\frac{1}{2}} \text{ and } G(q(z)) = z, \tag{1.14}$$

with the function

$$G(r) = \int_0^r (\rho^{2(N-2)} - 1)^{-\frac{1}{2}} d\rho$$

defined for $r < r_N$ with $r_3 = +\infty$ and $r_N < +\infty$ for $N \geq 4$. Let

$$L = \max\{2T, 2\sqrt{(N-2)T}, 2\|h\|_{L^\infty(\hat{\Omega})} + 2(N-2) + 1\} \tag{1.15}$$

and choose $0 < \lambda^*$ and $B = \min\{r_N, A\}$ such that

$$\begin{cases} (a) & \lambda q(\frac{\mu B}{2\lambda}) \geq L \text{ for every } \mu \in [\frac{1}{2}, 1] \text{ and } \lambda \in (0, \lambda^*], \\ (b) & \lambda^* \leq [\frac{8}{N-2} \sup\{|h_{zz}(z,0)| : |z| \leq A\}]^{-1}, \\ (c) & (\lambda^*)^2 \leq \dfrac{3(N-2)}{4} T. \end{cases} \tag{1.16}$$

The existence of such a λ^* and B follows from the facts that the function $\lambda \to \lambda q(\frac{\mu B}{2\lambda})$ is decreasing for λ near zero and $q(r_N) = q_z(r_N) = +\infty$.

Next we consider an one-parameter family of solutions $p(\cdot, \cdot; \mu)$ ($\mu \in [\frac{1}{2}, 1]$) of

$$\begin{cases} (a) \quad p_t = \dfrac{p_{zz}}{1+(p_z)^2} - \dfrac{N-2}{p} \text{ on } (-B/2, B/2) \times (0, T_\mu), \\ (b) \quad p(z,0;\mu) = \lambda^* q(\dfrac{\mu z}{\lambda^*}) \text{ for } |z| \le B/2, \\ (c) \quad p(\pm B/2, t; \mu) = \lambda^* q(\dfrac{\mu B}{2\lambda^*}) - \alpha(\mu) t \text{ for } t \in [0, T(\mu)), \end{cases}$$

where

$$\alpha(\mu) = \inf_{|z| \le B/2} \left\{ - \left[\frac{p_{zz}}{1+(p_z)^2} - \frac{N-2}{p} \right] (z, 0; \mu) \right\} \ge 0$$

and $T_\mu \le \infty$ is the "focusing" time of $p(\cdot, \cdot, \cdot \mu)$, i.e., the first time such that $p(0, t; \mu) = 0$. A simple calculation yields that there exists $c_0 > 0$ such that

$$\alpha(\mu) = \inf_z \{ \frac{(N-2)}{\lambda^*} \frac{(1-\mu^2)}{q(\mu z/\lambda^*)} \} \ge c_0(1-\mu^2).$$

It is also immediate that the $p(z,t;\mu)$ depends smoothly on μ whenever $p(t,z;\mu) > 0$ and, therefore, the focusing times T_μ's, depend smoothly on μ and that $T_1 = +\infty; p(\cdot, \cdot, 1)$ being a stationary solution of (0.4). Moreover, the maximum principle yields

$$p_t(z,t;\mu) \le -\alpha(\mu) \le -c_0(1-\mu^2).$$

Hence $T_\mu < \infty$ for every $\mu < 1$. Since $q \ge 1, c_0 \ge (N-2)/\lambda^*$. Therefore by (1.16)(c) for every $t \in [0, T_{1/2})$,

$$\begin{aligned} p(0,t;1/2) &= p(0,0;1/2) + \textstyle\int_0^t p_t(0,s;1/2) ds \\ &\le \lambda^* - \frac{3(N-2)}{4\lambda^*} t. \end{aligned}$$

Since $p(0, T_{1/2}; 1/2) = 0$, the above inequality implies that $T_{1/2} \le T$. In view of (1.16)(c), there exists $\mu^* \in (\frac{1}{2}, 1)$ such that

$$T_{\mu^*} = 2T.$$

Set

$$L^* = \lambda^* q(\frac{\mu^* B}{2\lambda^*}) \ge L \tag{1.17}$$

and define, for $\lambda \in (0, \lambda^*]$, the functions

$$q_\lambda : [-\frac{B}{2}, \frac{B}{2}] \times [0, \overline{T}_\lambda] \to [0, \infty)$$

so that

$$(1.18)\qquad \begin{cases} (a) & q_\lambda \text{ solves } (0.4) \text{ on } [-\frac{B}{2},\frac{B}{2}]\times[0,\overline{T}_\lambda), \\ (b) & q_\lambda(z,0)=\min\{L^*,\lambda q(\frac{\mu^* z}{\lambda})\} \text{ on } [-\frac{B}{2},\frac{B}{2}], \\ (c) & q_\lambda(\pm\frac{B}{2},t)=L^*-\alpha t \text{ on } [0,\overline{T}_\lambda), \end{cases}$$

where $\overline{T}_\lambda$ is the focusing time and

$$\alpha=\min\left\{\frac{1}{L^*},\inf_{\substack{\lambda\le\lambda^*\\ |z|\le z^*(\lambda)}}\left\{-\left[\frac{q_{\lambda,zz}}{1+(q_{\lambda,z})^2}-\frac{N-2}{q_\lambda}\right](z,0)\right\}\right\},$$

$z^*(\lambda)$ being the solution of $\lambda q(\frac{\mu^* z}{\lambda})=L^*$ in $[0,B]$. An explicit computation based on (1.14) and the fact that $q_\lambda(z,0)\le L^*$ for $|z|\le z^*(\lambda)$ gives $\alpha>0$. On the other hand, the maximum principle yields

$$(1.19)\qquad q_{\lambda,t}\le-\alpha \text{ in } [-B/2,B/2]\times[0,\overline{T}_\lambda),$$

and, in particular,

$$0\le q_\lambda(0,t)\le q_\lambda(0,0)-\alpha t\le\lambda-\alpha t;$$

hence $\overline{T}_\lambda\le\lambda/\alpha$, i.e. $\overline{T}_\lambda\to0$ as $\lambda\to0$.

Recall that μ^* is chosen so that $T_{\mu^*}=2T$. Moreover, $\alpha\le\alpha(\mu^*)$ and by the maximum principle, $q_{\lambda^*}(z,t)\ge p(z,t,\mu^*)$ and $\overline{T}_{\lambda^*}\ge\overline{T}_{\mu^*}=2T$. Define

$$\lambda_0=\inf\left\{\lambda\le\lambda^* \text{ such that } \overline{T}_\lambda\ge T \text{ for all } \lambda\in[\lambda,\lambda^*]\right\}.$$

It is clear that $\overline{T}_{\lambda_0}=T$; hence all the q_λ's are defined on $\tilde{\Omega}_1$ for $\lambda\in[\lambda_0,\lambda^*]$.

Finally set $\lambda_1=\lambda^*+1$ and define q_λ, for $\lambda\in[\lambda^*,\lambda_1]$, to be the solution of (1.18)(a),(c) with initial datum

$$(1.20)\qquad q_\lambda(z,0)=(\lambda-\lambda^*)L^*+(\lambda_1-\lambda)q_{\lambda^*}(z,0), \text{ in } [-\frac{B}{2},\frac{B}{2}].$$

Since

$$q_\lambda(z,0)\ge q_{\lambda^*}(t,0) \text{ for } \lambda\in[\lambda^*,\lambda_1],$$

the maximum principle again yields that

$$q_\lambda\ge q_{\lambda^*} \text{ in } \tilde{\Omega}_1.$$

In particular the focusing time $\overline{T}_\lambda$ for q_λ is larger than T, and, hence, q_λ is defined on $\tilde{\Omega}_1$. Moreover,

$$\delta = \inf\{q_\lambda(z,t) : \lambda \in [\lambda^*, \lambda_1], (z,t) \in \tilde{\Omega}_1\} > 0. \tag{1.21}$$

Proof of Lemma 1.3: It is immediate from the construction described above, that (1.7) and (1.8) hold. Also (1.9) follows from (1.15), (1.17), (1.20), (1.21) and the observation that, for all $\lambda \in [\lambda_0, \lambda_1]$,

$$q_\lambda(\pm A/2, t) = L^* - \alpha t \geq L^* - \frac{T}{L^*} \geq L - \frac{T}{L} \geq \|h\|_{L^\infty(\tilde{\Omega})}.$$

To prove (1.10), observe that, for $\lambda \in [\lambda_0, \lambda^*]$,

$$\psi_\lambda = N - 2 + q_\lambda q_{\lambda,t}.$$

Hence, by (1.19), there exists a $c_0 > 0$ such that

$$\max(\psi_\lambda(z,0), \psi_\lambda(\pm\frac{B}{2}, t)) \leq N - 2 - c_0,$$

for every $\lambda \in [\lambda_0, \lambda^*], z \in [-\frac{B}{2}, \frac{B}{2}]$ and $t \in [0, T)$. Applying the maximum principle to the equation satisfied by the ψ_λ's, which similar to (1.12), we conclude that, for all $\lambda \in [\lambda_0, \lambda^*]$,

$$\psi_\lambda(z,t) \leq N - 2 - c_0 \text{ in } \overline{\Omega}_1.$$

The definition of δ in (1.21) completes the proof of (1.10).

Next observe that $q_{\lambda_1}(z,0) \equiv L^*$. We will prove (1.5) by constructing an appropriate subsolution to (0.4). To this end define

$$v(z,t) = L^* - \frac{2(N-2)}{L^*}t, \quad \Big((z,t) \in \tilde{\Omega}_1\Big).$$

Then, by (1.15), (1.17),

$$(L^*)^2 \geq 4(N-2)T.$$

Therefore, for all $(z,t) \in \tilde{\Omega}_1$,

$$v(z,t) \geq L^* - \frac{2(N-2)}{L^*}T = L^* - \frac{L^*}{2}\left(\frac{4T(N-2)}{(L^*)^2}\right) \geq \frac{L^*}{2}.$$

The above inequality implies that v is a subsolution of (0.4). Also the choice of α yields

$$v(\pm B,t) \le q_{\lambda_1}(\pm B,t) \text{ for } t \in [0,T).$$

Using again the maximum principle and (1.15), (1.17) we obtain

$$q_{\lambda^*} \ge v \ge \|h\|_{L^\infty(\tilde{\Omega})} + 1;$$

(1.5) follows from the above inequality.

To verify (1.4) and (1.6), we fix $|\theta| \le B$ and define

$$W_\lambda(z,t) = q_\lambda(z,t) - h(z-\theta,t), \quad \Big((z,t) \in \tilde{\Omega}_1\Big).$$

We now claim that $z \mapsto W_\lambda(z,t)$ has at most two zeroes for each t and exactly two, when $\lambda = \lambda_0$.

Indeed, if $\lambda \ge \lambda^* + \frac{1}{2}$, then (1.20), (1.15) and (1.17) yield

$$W_\lambda(z,0) \ge \frac{L^*}{2} - \|h\|_{L^\infty(\tilde{\Omega})} > 0.$$

If $\lambda \in [\lambda^*, \lambda^* + \frac{1}{2}]$, then

$$W_{\lambda,zz}(z,0) = (\lambda_1 - \lambda)\frac{(\mu^*)^2}{\lambda^*} q_{zz}(\frac{\mu^* z}{\lambda^*}) - h_{zz}(z,0).$$

Since $\mu^* \ge \frac{1}{2}$ and $q_{zz} = (N-2)(1+q_z^2)q^{-1} = (N-2)q^{2N-5} \ge (N-2)$,

$$W_{\lambda,zz}(z,0) \ge \frac{(N-2)}{8\lambda^*} - h_{zz}(z-\theta,0),$$

and, in view of (1.16)(b), $W_{\lambda,zz}(z,0) > 0$. But $W_\lambda(\pm B,0) > 0$. Therefore $W_\lambda(z,0)$ has at most two zeroes.

When $\lambda \in [\lambda_0, \lambda^*]$, (1.18)(b) yields either $q_\lambda(z,0) = L^*$ or

$$W_{\lambda,zz}(z,0) = \frac{(\mu^*)^2}{\lambda} q_{zz}(\frac{\mu^* z}{\lambda}) - h_{zz}(z-\theta,0) \ge \frac{N-2}{4\lambda^*} - h_{zz}(z-\theta,0) > 0,$$

if $q_\lambda(z,0) < L^*$. On the other hand, $W_\lambda(z,0) > 0$, whenever $q_\lambda(z,0) = L^*$ (by the choice of L^*). Hence $W_\lambda(z,0)$ has at most two zeroes. Finally, $W_{\lambda_0}(z,0)$ has exactly two zeroes, if there is a z such that $W_{\lambda_0}(z,0) < 0$. Suppose $W_{\lambda_0}(z,0) \ge 0$ for all $|z| \le B/2$. Since $W_{\lambda_0}(\pm\frac{B}{2},t) > 0$ for all $t \in [0,T)$, the strong maximum principle implies that $W_{\lambda_0}(z,t) > 0$ for all $|z| \le B/2, t \in (0,T]$. Recall, however, that λ_0 is chosen so that $\overline{T}_{\lambda_0} = T$. Hence $\lim_{t\to T} W_{\lambda_0}(0,t) = 0$. Consequently $W_{\lambda_0}(z,0)$ must be strictly negative for some $|z| < B/2$.

Finally, since $W_\lambda(\pm B,t) > 0$, the maximum principle yields that the number of zeroes of W_λ is a nondecreasing function of time and therefore less or equal to two, i.e. (1.8). Arguing as above we conclude that for every $t < T, W_{\lambda_0}(z,t)$ must be strictly negative for some $|z| \leq B/2$.

□

We conclude this section with the proof of Lemma 1.4, which is similar to an argument used by Angenent in [A2, pg 192].

Proof of Lemma 1.4. For fixed $(z_0, t_0) \in \tilde{\Omega}_1$ set

$$A(z_0) = [-\frac{B}{2} - z_0, \frac{B}{2} + z_0]$$

and define $\omega : [\lambda_0, \lambda_1] \times A(z_0) \to I\!R^2$ by

$$\omega(\lambda, \theta) = (q_\lambda(z_0 + \theta), t_0) - h(z_0, t_0), q_{\lambda,z}(z_0 + \theta, t_0) - h_z(z_0, t_0)).$$

We will prove the lemma by showing that the winding number of ω is one. To this end, let $\omega_1(\lambda, \theta), \omega_2(\lambda, \theta)$ denote the first and second components of ω respectively.

First, observe that by the definitions of $I(t, \lambda, \theta)$ and $\omega_1(\lambda, \theta)$,

$$\omega_1(\lambda, \theta) \leq 0 \text{ iff } z_0 + \theta \in I(t_0, \lambda, \theta).$$

Hence (1.5) yields

$$\omega_1(\lambda_1, \theta) > 0 \qquad \text{for all } \theta \in A(z_0). \tag{1.22}$$

Also

$$w_1(\lambda, \pm\frac{B}{2} - z_0) = q_\lambda(\pm\frac{B}{2}, t_0) - h(z_0, t_0),$$

and, by (1.9),

$$w_1(\lambda, \pm\frac{B}{2}, -z_0) > 0, \qquad (\forall \lambda \in [\lambda_0, \lambda_1]). \tag{1.23}$$

On the other hand, since $n(t_0, \lambda_0, \theta) = 2$ for all $|\theta| \leq B$, the equation

$$f(z, \theta) = q_{\lambda_0}(z, t_0) - h(z - \theta, t_0) = 0$$

has exactly two solutions in $[-\frac{B}{2}, \frac{B}{2}]$ for all $|\theta| \leq B$, which we denote by $\mu(\theta)$ and $z(\theta)$, with $\mu(\theta) \leq z(\theta)$. It is immediate that μ and z depend continuously on $\theta \in [-B, B]$. Moreover, since

$$z(B) - B \leq -\frac{B}{2} \text{ and } z(-B) + B \geq \frac{B}{2},$$

there exists $\theta_1 \in [-B, B]$ such that

$$z(\theta_1) - \theta_1 = z_0;$$

in other words

$$\omega_1(\lambda_0, \theta_1) = 0 \text{ and } \theta_1 \in A(z_0). \tag{1.24}$$

Similarly there exists $\theta_2 \in A(z_0)$ such that $\mu(\theta_2) - \theta_2 = z_0$.

Now suppose that there exists $\hat{\theta} \in A(z_0)$ such that $\omega_1(\lambda_0, \hat{\theta}) = 0$. Then either $z(\hat{\theta}) - \hat{\theta} = z_0$ or $\mu(\hat{\theta}) - \hat{\theta} = z_0$. Since (1.9) yields $f(\pm B/2, \hat{\theta}) > 0$ and $f(z(\hat{\theta}), \hat{\theta}) = f(\mu(\hat{\theta}), \hat{\theta}) = 0$, we have

$$f_z(z(\hat{\theta}), \hat{\theta}) \geq 0 \geq f_z(\mu(\hat{\theta}), \hat{\theta}).$$

We have proved the following: if $\omega_1(\lambda_0, \hat{\theta}) = 0$ for some $\hat{\theta} \in A(z_0)$, then

$$\begin{cases} (a) \quad \omega_2(\lambda_0, \hat{\theta}) \geq 0 \quad \text{if} \quad \hat{\theta} + z_0 = z(\hat{\theta}), \\ \\ (b) \quad \omega_2(\lambda_0, \hat{\theta}) \leq 0 \quad \text{if} \quad \hat{\theta} + z_0 = \mu(\hat{\theta}). \end{cases} \tag{1.25}$$

Suppose now that $\omega_1(\lambda_0, \overline{\theta}) = 0$ for some $\overline{\theta} \in A(z_0)$. Then

$$q_{\lambda_0}(z_0 + \hat{\theta}, t_0) = q_{\lambda_0}(z_0 + \overline{\theta}, t_0) = h(z_0, t_0).$$

Using (1.8), we conclude that

$$z_0 + \hat{\theta} = \pm(z_0 + \overline{\theta}).$$

In summary we find that there are at most two θ's such that $\omega_1(\lambda_0, \theta) = 0$. But we also know that there are $\theta_1, \theta_2 \in A(z_0)$ such that

$$z_0 = \mu(\theta_1) - \theta_1 = z(\theta_2) - \theta_2.$$

Hence $\theta_1 \leq \theta_2$ and consequently ω has the following properties,

$$\begin{cases} (a) \quad \omega_1(\lambda_0, \theta) > 0\,, \quad \text{for } \theta \notin (\theta_1, \theta_2), \\ \\ (b) \quad \omega_1(\lambda_0, \theta_1) = 0, \qquad \omega_2(\lambda_0, \theta_1) \leq 0, \\ \\ (c) \quad \omega_1(\lambda_0, \theta_2) = 0\,, \qquad \omega_2(\lambda_0, \theta_2) \geq 0. \end{cases} \tag{1.26}$$

Using (1.22), (1.23) and (1.26), we conclude that the winding number of ω is one.

□

2 Behavior of Γ_t at the focusing point.

In this section we prove Theorem 1 and discuss some of its immediate consequences. As mentioned in the Introduction, a result similar to (0.7) was proved, for $N = 3$, by Huisken [H], under several assumptions, the most important one being a bound on the blow-up rate of the curvature, which was verified in [H] only for barbell-type surfaces, which have strictly positive mean curvature. A similar result was also obtained by Dziuk and Kawohl [DK].

We will organize this section in several subsections where we will explain the basic steps of the proof of Theorem 1.

2.1 Scaling and preliminary estimates.

One of the main estimates in our analysis is

$$\liminf_{t \to T} (T-t)^{-\frac{1}{2}} h(0,t) > 0. \tag{2.1}$$

which is essentially equivalent to the assumption in [H,(2) page 286].

To obtain (2.1) we rewrite (0.4) as

$$(h^2)_t = 2[\psi - (N-2)] \tag{2.2}$$

and observe that (0.6) yields

$$\psi(0,t) \geq 0 \text{ for all } t \in [0,T).$$

Combining this with $h(0,T) = 0$ we obtain an easy upper bound

$$h(0,t) \leq \sqrt{2(N-2)(T-t)}, \qquad (t \in [0,T)). \tag{2.3}$$

The lower bound follows from the nontrivial curvature estimate (1.3), which yields

$$h(0,t) \geq \sqrt{2c_0(T-t)} \qquad (t \in [T-\varepsilon, T)). \tag{2.4}$$

Actually (1.3) yields a more general result than (2.4). Indeed, for $0 < |z| \leq \varepsilon$ and $t \in [T-\varepsilon, T)$ we have

$$\frac{\partial}{\partial z} \log(1 + (h_z(z,t))^2) = 2\psi(z,t)\frac{h_z(z,t)}{h(z,t)} \leq \gamma \frac{z}{|z|}\frac{\partial}{\partial z} \log(h(z,t))^2, \tag{2.5}$$

where

$$\gamma = (N-2) - c_0. \tag{2.6}$$

and, therefore,

$$(2.7) \qquad |h_z(z,t)| \le ((\frac{h(z,t)}{h(0,t)})^{2\gamma} - 1)^{\frac{1}{2}} \text{ in } [-\varepsilon,\varepsilon] \times [T-\varepsilon, T).$$

Next we follow some of the ideas in [H] and the techniques developed by Giga and Kohn [GK] to study the blow-up of the solution of semilinear heat equation.

To this end, we define

$$v(y,s) = (T-t)^{-\frac{1}{2}} h(y\sqrt{T-t}, t) \text{ with } s = -\log(T-t).$$

Observe that $v(y,s)$ is defined for $(ye^{-\frac{s}{2}}, T-e^{-s})$ in $(-2A, 2A) \times (0,T) \subset \Omega$, i.e. v is defined on

$$K(A) = \{(y,s) \ : \ s > -\log T, |y| \le 2Ae^{s/2}\}.$$

We now write

$$v(y,s) = e^{\frac{s}{2}} h(ye^{-\frac{s}{2}}, T-e^{-s})$$

and calculate

$$v_s(y,s) = \frac{1}{2} v(y,s) - yh_z(ye^{-\frac{s}{2}}, T-e^{-s}) - e^{-\frac{s}{2}} h_t(ye^{-\frac{s}{2}}, T-e^{-s}),$$

$$v_y(y,s) = h_z(ye^{-s/2}, T-e^{-s}),$$

$$v_{yy}(y,s) = e^{-s/2} h_{zz}(ye^{-s/2}, T-e^{-s}).$$

Using (2.4) and (2.7) we get

$$(2.8) \qquad v(0,s) \ge \sqrt{2c_0},$$

and

$$(2.9) \qquad |v_y(y,s)| \le ((\frac{v(y,s)}{\sqrt{2c_0}})^{2\gamma} - 1)^{\frac{1}{2}}.$$

To obtain more pointwise estimates for v, v_y, v_{yy} and v_s we use the equation which is satisfied by v, namely

$$(2.10) \qquad v_s = \frac{v_{yy}}{1+(v_y)^2} - \frac{N-2}{v} - \frac{1}{2}(yv_y - v) \text{ in } K(A),$$

and recall that (1.10) holds in $[-\varepsilon,\varepsilon] \times [T-\varepsilon, T)$. Finally, denote $K(\varepsilon) = \{(y,s) \in K(A) \text{ such that } |y| \le \varepsilon e^{s/2} \text{ and } s > -\log\varepsilon\}$.

Proposition 2.1: *There exists $c_i > 0$ $(i = 1,2,3)$ such that for all $(y,s) \in K(\varepsilon)$,*

$$(2.11)\qquad \begin{cases} (a) & \sqrt{2c_0} \le v(y,s) \le c_1(|y|+1), \\ (b) & |v_{yy}(y,s)| \le c_2(|y|^{2\gamma}+1), \\ (c) & |v_s(y,s)| \le c_3(|y|^{2\gamma}+1). \end{cases}$$

Proof: Integrating (2.9) yields that there are $c_0, \varepsilon, \sigma > 0$ satisfying

$$0 < v(y,s) < c_0 \text{ for } (y,s) \text{ such that } |y| \le \varepsilon \text{ and } s \ge \sigma.$$

Next set

$$v^*(y,s) = c|y| \qquad (c > 0)$$

and observe that

$$v_s^* + \frac{(N-2)}{v} - \frac{v_{yy}^*}{1+(v_y^*)^2} + \frac{1}{2}(yv_y^* - v^*) \ge 0 \text{ in } K(A)\backslash\{0\} \times (-\log T, \infty).$$

But

$$v(+2Ae^{s/2}, s) \le e^{s/2}\|h\|_\infty \le v^*(+2Ae^{s/2}, s)$$

if $c \ge \|h\|_\infty/2A$. Applying the maximum principle on $\theta = K(A) \cap \{|y| > \varepsilon, s \ge \sigma\}$ we get $v \le v^*$ on θ provided that $c \ge \|h\|_\infty/2A$ and $c \ge c_0/\varepsilon$. Now (2.11a) follows if c_1 is sufficiently large.

For the second derivative estimate we have

$$|v_{yy}(y,s)| = \frac{1}{v(y,s)}((1+h_z^2)|\psi|)(ye^{-s/2}, T - e^{-s}) \le \frac{1}{c_0 v(0,s)}\left(\frac{v(y,s)}{v(0,s)}\right)^{2\gamma},$$

with the inequality following from (1.2) and (2.7).

Hence, by (2.11)(a),

$$|v_{yy}(y,s)| \le \frac{1}{c_0(\sqrt{2c_0})^{2\gamma}}(c_1(|y|+1))^{2\gamma} \le c_2(|y|^{2\gamma}+1),$$

for some $c_2 > 0$.

Finally,

$$\begin{aligned} v_s(y,s) &= \frac{1}{2}v(y,s) - yv_y(y,s) - e^{-s/2}h_t(ye^{s/2}, T - e^{-s}) \\ &= \frac{1}{2}v(y,s) - yv_y(y,s) - \frac{\psi(ye^{s/2}, T - e^{-s}) - (N-2)}{v(y,s)}. \end{aligned}$$

We now obtain (2.11)(c) using $v(y,s) \geq v(0,s)$, (1.2), (2.7) and (2.11)(a), (b).

□

2.2 A monotonicity formula.

As we will explain in the next subsection, Proposition 2.1 yields the local uniform compactness of v as $s \to \infty$. To show that the whole family converges, it is sufficient (cf. [GK]) to come up with an "energy-type" functional, which will play the role of a Lyaponov function as $\varepsilon \to \infty$. To this end, following [H] we define

$$E(v(\cdot,s)) = \int_{-R(s)}^{R(s)} \rho(y,s)v(y,s)(1+(v_y(y,s))^2)^{\frac{1}{2}}\,dy,$$

where

$$\rho(y,s) = \exp(-\frac{1}{4}(v^2(y,s)+y^2)) \text{ and } R(s) = \varepsilon e^{s/2}.$$

Proposition 2.2:

$$\frac{d}{ds}(E(v(\cdot,s))) = -\int_{-R(s)}^{R(s)} \frac{\rho(y,s)v(y,s)}{[1+(v_y(y,s))^2]^{\frac{1}{2}}} v_s^2(y,s)ds + e(s), \tag{2.12}$$

and

$$\lim_{s\to\infty} e(s) = 0. \tag{2.13}$$

We will prove (2.12) by an elementary computation using (2.10). The error $e(s)$ is due to the boundary $|y| = \varepsilon e^{s/2}$ we are imposing in the formula for E. Finally, we refer the reader to [H,Section 3] for a more elegant proof (using differential geometric arguments).

Proof: We directly calculate

$$\frac{d}{ds}E(v(\cdot,s)) = A(s) + B(s) + C(s) + e(s),$$

where

$$A(s) = \int_{-R(s)}^{R(s)} \rho_s v(1+v_y^2)^{\frac{1}{2}}\,dy,$$

$$B(s) = \int_{-R(s)}^{R(s)} \rho v_s (1+v_y^2)^{\frac{1}{2}} dy,$$

$$C(s) = \int_{-R(s)}^{R(s)} \rho v[(1+v_y^2)^{\frac{1}{2}}] v_{sy} dy,$$

and

$$e_1(s) = Ae^{s/2}\{(\rho v(1+v_y^2)^{\frac{1}{2}})(2Ae^{s/2}, s) - (\rho v(1+v_y^2)^{\frac{1}{2}})(-2Ae^{s/2}, s)\}.$$

Since $\rho(\pm\varepsilon e^{s/2}, s) \leq \exp(-\frac{\varepsilon^2}{4}s)$, (2.11) yields

$$\lim_{s\to\infty} e_1(s) = 0. \tag{2.14}$$

Next we will show that

$$A(s) + B(s) + C(s) = -\int_{-R(s)}^{R(s)} \frac{\rho v}{(1+v_y^2)^{\frac{1}{2}}} v_s^2 + e_2(s) \tag{2.15}$$

with $e_2(s) \to 0$ as $s \to \infty$. This will conclude that proof with $e = e_1 + e_2$.

To obtain (2.15) we calculate

$$\begin{aligned} C(s) &= \int \rho v \frac{w}{(1+w^2)^{\frac{1}{2}}} v_{sy} = -\int (\frac{\rho v w}{(1+w^2)^{\frac{1}{2}}})_y v_s + e_2(s) \\ &= -\int [(\rho v)_y \frac{w}{(1+w^2)^{\frac{1}{2}}} + \frac{\rho v}{(1+w^2)^{\frac{1}{2}}} \frac{v_{yy}}{1+w^2}] v_s + e_2(s) \\ &= -\int \{(\rho v)_y w + \rho v[v_s + \frac{1}{v} + \frac{1}{2}(yw - v)]\} \frac{v_s}{(1+w^2)^{\frac{1}{2}}} + e_2(s), \end{aligned}$$

with

$$e_2(s) = (\frac{\rho v w v_s}{(1+w^2)^{\frac{1}{2}}})(2Ae^{s/2}, s) - (\frac{\rho v w v_s}{(1+w^2)^{\frac{1}{2}}})(-2Ae^{s/2}, s).$$

where all the integrals are over the interval $[-R(s), R(s)], w = v_y$ and we used that

$$\rho_y = -\frac{1}{2}(vw + y)\rho, \rho_s = -\frac{1}{2} vw\rho.$$

Now a straightforward computation gives (2.15). The fact that $e_2 \to 0$ as $\varepsilon \to \infty$, follows again from (2.11) and the form of ρ.

□

2.3 Blow-up

In view of (2.9) and (2.11), there exists $s_j \to \infty$ with $s_{j+1} - s_j \geq 2$ such that

$$v_j(y,s) \equiv v(y, s+s_j) \to v_\infty(y,s) \text{ as } j \to \infty,$$

with the limit uniform on compact subsets of $I\!R^2$ and

$$v_{j,y} \rightharpoonup v_{\infty,y} \text{ in weak}^* \ L^\infty_{\text{loc}}.$$

By passing to a further subsequence, which we again denote by s_j, we also have that

$$v_{j,y} \to v_{\infty,y} \text{ for almost every } (y,s).$$

If for some $s, v_{j,y}(y,s) \to v_\infty(y,s)$ for almost every y, then (2.11)(b) yields that this convergence is uniform for bounded y's. Hence for every integer k, there is a $n_k \in [k, k+1)$ such that

$$v_{j,y}(y, n_k) \to v_{\infty,y}(y, n_k)$$

uniformly for bounded y for each n_k.

Applying the dominated convergence theorem and using the exponential decay of ρ we get, that, for each n_k,

$$\lim_{j\to\infty} E(v_j(\cdot, n_k)) = E(v_\infty(\cdot, n_k)), \tag{2.16}$$

On the other hand, $v_j(\cdot, n_k) = v(\cdot, s_j + n_k)$. Therefore,

$$E(v_{j+1}(\cdot, n_k)) - E(v_j(\cdot, n_k)) = -\int_{n_k}^{n_k+s_{j+1}-s_j} (\ell(j,s) + e(s+s_j))ds,$$

where

$$\ell(j,s) = \int_{-R(s+s_j)}^{R(s+s_j)} [\frac{\rho v_j}{(1+(v_{j,y})^2)^{\frac{1}{2}}} v_{j,s}^2]dy.$$

Finally, in view of (2.7) and (2.11)(a),

$$|v_{j,y}|(y,s) \leq g(y,s) = c(|y|^{2\gamma} + 1)$$

for some $c > 0$. Since $v_j > 0$, this yields

$$\ell(j,s) \geq \int_{-R(s+s_j)}^{R(s+s_j)} \frac{\rho v_j}{1+g^2} (v_{j,s})^2 dy.$$

Using the exponential decay of ρ, the uniform convergence of v_j (2.11) and the fact that by construction $s_{j+1} - s_j \geq 2$, we get, for each n_k,

$$\liminf_{j\to\infty} \int_{n_k}^{n_k+2} \ell(j,s) \geq \int_{n_k}^{n_k+2} \int_{-\infty}^{\infty} \frac{\rho v_\infty}{1+g^2}(v_{\infty,s})^2 dy$$

and

$$\lim_{j\to\infty} \int_{n_k}^{n_k+s_{j+1}-s_j} e(s+s_j)ds = 0.$$

Hence

$$\int_{n_k}^{n_k+2} \int_{-\infty}^{\infty} \frac{\rho v_\infty}{1+g^2}(v_{\infty,s})^2 dy = 0.$$

But, in view of (2.11)(a),

$$v_\infty \geq \sqrt{2c_0};$$

therefore, for almost every y,

$$v_{\infty,s}(y,s) = 0 \text{ for almost every } s,$$

which, by (2.11)(c), yields

$$v_\infty(y,s) \equiv v_\infty(y) \text{ for all } y \in I\!R.$$

Passing to the limit in (2.10) we get that v_∞ solves

$$\frac{v_{\infty,yy}}{1+v_{\infty,y}^2} - \frac{N-2}{v_\infty} - \frac{1}{2}(yv_{\infty,y} - v_\infty) = 0 \text{ in } I\!R. \tag{2.17}$$

As a matter of fact, the estimates of Proposition 2.1 yield that v_∞ is a classical solution of (2.17) and, moreover,

$$\sqrt{2c_0} \leq v_\infty(y) \leq c_1(|y|+1) \text{ and } |v_{\infty,yy}(y)| \leq c_2(|y|^{2\gamma}+1). \tag{2.18}$$

A direct calculation also shows that if

$$\Psi = v_\infty v_{\infty,yy}(1+v_{\infty,y}^2)^{-1},$$

then

$$\frac{(\Psi^2)_{yy}}{1+(v_{\infty,y})^2} - \frac{2(1+\Psi)}{1+(v_{\infty,y})^2 v_\infty}(\Psi^2)_y + \frac{4(v_{\infty,y})^2}{(v_\infty)^2(1+(v_{\infty,y})^2)}\Psi(\Psi-(N-2))(\Psi+1)$$
$$-\frac{1}{2}y(\Psi^2)_y = \frac{2(\Psi_y)^2}{1+(v_{\infty,y})^2}.$$

In view of (1.3), if Ψ^2 has an interior maximum at y_0, then

$$-1 \leq \Psi(y_0) \leq 0.$$

We now claim that $\Psi \le 0$ in $I\!R$. Indeed suppose that there exists $y^* \in I\!R$ such that $\Psi(y^*) > 0$. Without any loss of generality we may assume that $\Psi_y(y^*) > 0$, since, else we consider the point $-y^*$. But then

$$\Psi(y) \ge \Psi(y^*) > 0 \text{ in } (y^*, \infty),$$

since, in view of the discussion above, Ψ cannot have a positive interior maximum. This, however, contradicts (2.18). Hence $\Psi \le 0$ in $I\!R$. Hence v_∞ is concave and, in view of (2.18), constant. Using (2.17) we see that the only constant solution is $\sqrt{2(N-2)}$. Since any limit of $v(y, s+s_j)$ is equal to $\sqrt{2(N-2)}$, we have concluded the proof of Theorem 1.

Corollary 2.3: *For any $\varepsilon > 0$ there exists $\delta > 0$ satisfying*

$$\lim_{t\to T} h(z,t) \le \varepsilon|z| \text{ for } z \in [-\delta, \delta]. \tag{2.19}$$

Note that in view of (1.3) the above limit exists for sufficiently small $|z|$.

Proof: Suppose that for a given ε there is a sequence $z_n \to 0$ such that

$$\lim_{t\to T} h(z_n, t) > \varepsilon|z_n|.$$

Recall that $h_t(t,z) < 0$ for all z near zero and $t \in [0,T]$. Hence for sufficiently large n,

$$h(z_n, t) \ge \varepsilon|z_n|, \qquad \forall t \in [0,T]. \tag{2.20}$$

If

$$s_n = 2\log(\frac{2(N-2)}{\varepsilon|z_n|}),$$

then

$$\begin{aligned} v\left(\frac{2(N-2)z_n}{\varepsilon|z_n|}, s_n\right) &= e^{\frac{s_n}{2}} h(\frac{2(N-2)z_n}{\varepsilon|z_n|} e^{-\frac{s_n}{2}}, T - e^{s_n}) \\ &= \frac{2(N-2)}{\varepsilon|z_n|} h(z_n, T - e^{-s_n}). \end{aligned}$$

Since $z_n \to 0$, $s_n \to \infty$ and, by (2.20),

$$\liminf_{n\to\infty} v(\frac{2(N-2)z_n}{\varepsilon|z_n|}, s_n) \ge 2(N-2) > \sqrt{2(N-2)},$$

which contradicts Theorem 1. □

Corollary 2.4: *As $t \to T, h(\cdot,t)$ converges, uniformly in $[-2A,2A]$, to $h(\cdot,T)$, which is smooth for $z \neq 0$ and differentiable at $z=0$ and $h(\cdot,T)>0$ for all $y \neq 0$.*

Proof. In view of (1.3),

$$h_t(z,t) \leq 0 \text{ for } |z| \leq \varepsilon \text{ and } T-t \leq \varepsilon.$$

Hence $h(z,t)$ converges to a limit which we call $h(z,T)$ for $|z| \leq \varepsilon$. On the other hand, (0.6) implies that $h(\cdot,T)$ is nondecreasing on $[0,\varepsilon)$ and nonincreasing on $(-\varepsilon,0]$.

Suppose now that $h(2z_0,T)=0$ for some $2z_0 \in (0,\varepsilon)$. Then $h(z,T) \equiv 0$ for $z \in [0,2z_0]$ and by Dini's Theorem $h(z,t) \to 0$ uniformly on $[0,2z_0]$.

Set

$$\Phi(z,t) = h(z+z_0,t) - h(z,t) \ , \quad (z,t) \in O,$$

where

$$O = (0,z_0) \times (T-\varepsilon,T).$$

It is immediate that Φ satisfies

$$\Phi_t = a\Phi_{zz} + b\Phi_z + c\Phi \text{ in } O, \tag{2.21}$$

where

$$\begin{aligned} a(z,t) &= (1+(h_z(z,t))^2)^{-1}, \\ c(z,t) &= (h(z+z_0,t)h(z,t))^{-1}, \\ b(z,t) &= -h_{zz}(z+z_0,t)[h_z(z+z_0,t)+h_z(z,t)]a(z+z_0,t)a(z,t). \end{aligned}$$

But $0 \leq a \leq 1$ and $|b(z,t)| \leq |h_{zz}(z+z_0,t)| \leq h^{-1}(z+z_0,t)|\psi|(z+z_0,t)(h(z+z_0,t)h^{-1}(0,t))^{2\gamma}$. Using (1.3), we conclude that, for $(z,t) \in O$,

$$\begin{cases} (a) \quad |b(z,t)| \leq c(T-t)^{-\frac{1}{2}} \\ (b) \quad |c(z,t)| \leq c(T-t)^{-1}. \end{cases} \tag{2.22}$$

Moreover, again using (1.3), we see that, for some appropriate constant K,

$$(2c_0(T-t))^{\frac{1}{2}} \leq -h_t \leq (K(T-t))^{-\frac{1}{2}} \text{ in } [-\varepsilon,\varepsilon] \times [T-\varepsilon,t).$$

Integrating we obtain

$$0 \leq \Phi(z,t) \leq K\sqrt{T-t}, \text{ in } 0 \tag{2.23}$$

For $\mu > 0$ consider the auxiliary function

$$\hat{\Phi}(z,t) = \Phi(z,t) + \frac{\mu}{T-t} + \frac{\mu}{2}(z-z_0)^2.$$

A direct calculation yield

$$\hat{\Phi}_t - a\hat{\Phi}_{yy} - b\hat{\Phi}_y = c\Phi + \mu\{\frac{1}{(T-t)^2} - a - b(z - z_0)\}.$$

Using (2.22) and (2.23), we conclude that, there is $t_o < T$ such that for all $\mu \geq 0$,

$$\hat{\Phi}_t - a\hat{\Phi}_{yy} - b\hat{\Phi}_y \geq 0, (z,t) \in (0, z_0) \times (t_0, T).$$

Moreover, for all sufficiently small μ there is $\hat{t} < T$ such that

$$\inf_{t\in[\hat{t},T)} \hat{\Phi}(z_0, t) < \inf\left\{\hat{\Phi}(z,t) : (z,t) \in \{0, 2z_0\} \times [\hat{t}, T) \cup [0, 2z_0] \times \{\hat{t}\}\right\}.$$

Hence $\hat{\Phi}$ has an interior minimum, which contradicts with the fact that $\hat{\Phi}$ is a supersolution to a linear equation.

In summary we have shown that

$$h(z,T) > 0 \text{ in } (0, \varepsilon).$$

Similarly we show that

$$h(z,T) > 0, \text{ in } [-\varepsilon, 0).$$

Using (0.6), we

$$\inf\{h(z,t) \,:\, |z| \geq \varepsilon, t \in [0,T)\} > 0.$$

Finally, equation (0.4) for $|z| > \varepsilon$, we can easily show that $h(z,t)$ has a limit as $t \to T$.

□

2.4 No interior at the focusing point

We conclude this long section with a brief discussion, without any proofs, of why the focusing point 0 cannot be in the interior of $\Gamma_{T+\rho}$ for $\rho > 0$ and very small. This will be a consequence of (1.3).

To this end, we consider the solution u of (0.2) with initial datum g, such that $\{g = 0\} = \{r = h_0(z)\}$. It follows (cf. [ES]) that, for $t \leq T$,

$$\Gamma_t = \{u(\cdot, t) = 0\} = \{r = h(z,t)\},$$

where here, as usual, we denote by h the solutions of (0.4) which correspond to the different branches of h_0. Assume now that $0 \in I\!R^N$ belongs to the interior of $\Gamma_{T+\rho}$ for $\rho > 0$. This implies that there exist $R(\rho) > 0$ such that $B(0, R(\rho)) \subset \Gamma_{T+\rho}$. On the other hand, (1.3) yields that we can bound the part of h which focuses by catenoids as close to $(0,T)$ as we want. Recall that catenoids are stationary solutions to (0.2). Finally, we recall that the distance between two surfaces which move by mean curvature is a nondecreasing function in time (cf. [ES]).

To conclude this heuristic discussion we argue as follows. If $0 \in \Gamma_{T+\rho}$, choose $\varepsilon > 0$ so small that $h(0, T-\varepsilon) = \frac{1}{2}R(\rho)$ and bound h by the catenoid passing through $(\frac{1}{2}R(\rho), 0)$. In view of the above discussion, the set $\{u(\cdot, T+\rho) = 0\}$ cannot touch the catenoid which contradicts the choice of ε. This argument can be made rigorous at the expense of technical arguments. We choose, therefore, to omit the details.

3 No interior - Motion after the focusing

Our goal here is to show that, under certain assumtions, if Γ_t "focuses at $(0,T)$, then, for $t > T, \Gamma_t$: (i) does not develop interior and (ii) "opens" up.

As mentioned in the Introduction, in general, Γ_t will develop interior, (see for example: Soner [S], [ES]). On the other hand, [BSS] gives a fairly general geometric condition on Γ_0, which yields no interior. We next state this result of [BSS] as it applies to the case of cylindrically symmetric surfaces moving by mean curvature.

Theorem 3.1 ([BSS]): *Assume that Γ_0 is C^2 surface and that there exists a constant C such that*

$$x \cdot Dd + C\Delta d \neq 0 \text{ on } \Gamma_0, \tag{3.1}$$

where d is the signed distance to Γ_0. Then Γ_t has empty interior in $(0, +\infty)$.

Condition (3.1) has a geometric meaning, since the left hand side is the generator of dilations and translations in (x,t) evaluated at $t = 0$ on Γ_0.

If Γ_0 is smooth, then Γ_t is smooth for $t \in [0, t_1)(t_1 > 0)$ and, therefore, has empty interior in $(0, t_1)$. As remarked in [BSS], if the solution u of (0.2) which defines Γ_t satisfies, for some $t_0 \in (0, t_1)$,

$$\frac{x \cdot Du + Cu_t}{|Du|} \neq 0 \text{ on } \Gamma_{t_0} \tag{3.2}$$

then Γ_t has empty interior in $(0, \infty)$. Indeed let $\overline{d}(\cdot, t)$ be the signed distance to Γ_t. Then at t_0

$$\overline{d}_t = u_t/|Du|\,,\ D\overline{d} = Du/|Du|.$$

Also

$$\overline{d}_t = \Delta \overline{d}/(N-1),$$

since Γ_t is a classical solution of the mean curvature flow and $\Delta \overline{d}$ is equal to $(N-1)$ times the mean curvature.

Since in this paper we assume that Γ_0 is smooth, the main goal in this section will be to show that, under some additional assumptions on Γ_0, (3.2) holds near the focusing time T, although it may not hold at $t=0$.

Throughout the discussion below we will need to go back and forth to parametrizing Γ_t, for $t \in (0,T)$, in terms of both z and r. More precisely, we will need the existence of positive numbers $z(t), r_i(t) (i=1,2,3)$ with $t \in [0,T)$ and smooth functions $h, H : [-z(t), z(t)] \to [0,\infty)$ and $g : [r_1(t), r_2(t)] \to [0,\infty)$ such that:

$$r_1(t) = h(0,t), r_2(t) = H(0,t) \text{ and } z(t) = g(r_3(t),t), \tag{3.3}$$

$$\begin{aligned} \Gamma_t &= \{r = h(z,t) : |z| \le z(t)\} \cup \{r = H(z,t) : |z| \le z(t)\} \\ &= \{|z| = g(r,t) : r \in [r_1(t), r_3(t)]\}, \end{aligned} \tag{3.4}$$

$$h_t = \frac{h_{zz}}{1+h_z^2} - \frac{N-2}{h} \text{ and } H_t = \frac{H_{zz}}{1+H_z^2} + \frac{N-2}{H} \text{ in } [-z(t), z(t)], \tag{3.5}$$

$$g_t = \frac{g_{rr}}{1+g_r^2} + \frac{(N-2)g_r}{r} \text{ in } [r_1(t), r_3(t)], \tag{3.6}$$

$$\begin{cases} (i) & g(h(z,t),t) = g(H(z,t),t) = z \text{ in } [-z(t), z(t)], \\ (ii) & h(g(r,t),t) = r \text{ in } [r_1(t), r_2(t)], \\ (iii) & H(g(r,t),t) = r \text{ in } [r_2(t), r_3(t)], \end{cases} \tag{3.7}$$

and

$$\begin{cases} (r - r_2(t))g_r(r,t) < 0 \quad \text{in } [r_1(t), r_2(t)] \\ \text{and} \\ g_r(r_2(t),t) = 0. \end{cases} \tag{3.8}$$

It is immediate that if (3.3)-(3.8) hold, then

$$\begin{cases} g_r = h_z^{-1} = H_z^{-1}, g_t = -h_t h_z^{-1} = -H_t H_z^{-1} \\ \text{and} \\ g_{rr} = -h_{zz} h_z^{-3} = -H_{zz} H_z^{-3}. \end{cases} \tag{3.9}$$

The existence of such $z(t), r_i(t) (i = 1,2,3)$ and h, H and g with the above properties follows from the next proposition.

Proposition 3.2: *Assume that there exist positive numbers $z_0, r_{0,i} (i = 1,2,3)$ and smooth functions $h_0, H_0 : [-z_0, z_0] \to [0,\infty)$ and $g_0 : [r_{0,1}, r_{0,3}] \to [0,+\infty)$ such that (3.3), (3.4), (3.7) and (3.8) hold at Γ_0. Then there exist smooth $z, r_i : [0,T) \to (0,+\infty) (i = 1,2,3)$ and $h(\cdot,t), H(\cdot,t) : [-z(t), z(t)] \to (0,\infty), g : [r_1(t), r_3(t)] \to [0,+\infty) (t \in [0,T))$ satisfying (3.3)-(3.8) for all $t \in (0,T)$.*

Proof: Consider the solution u of (0.2) with initial data u_0 satisfying

$$u_0(x) = g_0(r) - |z| \qquad (r \in [r_{0,1}, r_{0,3}]),$$

where, for $x = (x_1, \ldots, x_n) \in I\!R^N, z = x_N$ and $r^2 = \sum_{i=1}^{N-1} x_i^2$. Since the zero level set Γ_0 of u_0 is smooth, the mean curvature flow has a, local in time, smooth solution Γ_t. The resulting smooth Γ_t can be parametrized as in (3.4) where h, H and g solve (3.5) and (3.6) with initial data h_0, H_0 and g_0 respectively. Moreover, u satisfies

$$u(x,t) = g(r,t) - |z| \ (r \in [r_1(t), r_3(t)]). \tag{3.10}$$

On the other hand, (3.5) admits a smooth solution as long as the solution stays positive. This yields that u is smooth as long as Γ_t does not focus i.e. for $t \in (0,T)$. Finally (3.8) follows from analyzing the properties of the number of zeroe's of g_r as in the Appendix.

□

Next we use (3.10) to write the expresion in (3.2) as

$$\frac{x \cdot Du + Cu_t}{|Du|} = \frac{rg_r - g + Cg_t}{\sqrt{1+g_r^2}} \quad \text{on } \Gamma_t. \tag{3.11}$$

The first important result in this section is:

Proposition 3.3: *Assume that $\#\{r \in [r_{01}, r_{03}] : rg_{0r} - g_0 - 2Tg_{0t} = 0\} \le 2$. Then there exists $t_0 \in [0,T)$ and $B > 0$ such that*

$$rg_r - g - 2Bg_t < 0 \qquad \text{in } (r_1(t_0), r_3(t_0)) \times \{t_0\}. \tag{3.12}$$

In particular Γ_t has no interior for all $t \ge 0$.

Before we begin with some preliminaries which will lead to the proof of Proposition 3.3, let us first comment on why (3.12) seems reasonable to hold. Indeed that analysis in Section 2 yields that $h_t(0,t) \to -\infty$ as $t \to T$, which, in turn, suggests, by (3.9), that $g_t(r_0(t),t) \to +\infty$ as $t \to T$. This would yield (3.12), provided one is able to control, away from the singularity, the term $rg_r - g$. Keeping this in mind, we define

$$K : \cup_{t\in[0,T)}((r_1(t), r_3(t)) \times \{t\}) \to I\!R$$

by

$$K(r,t) = rg_r(r,t) - g(r,t) - 2(T-t)g_t(r,t). \tag{3.13}$$

Using (3.6) we obtain

$$K_t = \mathcal{L}(K) \text{ in } \cup_{t\in[0,T)} ((r_1(t), r_3(t)) \times \{t\}), \tag{3.14}$$

where

$$\mathcal{L}\psi(r) = \frac{1}{1+g_r^2}\psi_{rr} - \frac{2g_{rr}g_r}{(1+g_r^2)^2}\psi_r + \frac{N-2}{r}\psi_r.$$

The above equation is derived after observing that

$$K = \frac{\partial g^\lambda}{\partial \lambda}\bigg|_{\lambda=1}, \quad g^\lambda(N,t) = \lambda g(\frac{r}{\lambda}, T - \frac{T-t}{\lambda}).$$

and that $g\lambda$ solves (3.6) for every $\lambda > 0$.

We will also need to define the functions $I, J : \cup_{t\in[0,T)}((-z(t), z(t)) \times \{t\}) \to I\!R$ by

$$\begin{cases} I(z,t) = h(z,t) - zh_z(z,t) + 2(T-t)h_t(z,t), \\ J(z,t) = H(z,t) - zH_z(z,t) + 2(T-t)H_t(z,t). \end{cases} \tag{3.15}$$

It follows from (3.7) and (3.9) that

$$I(g(r,t),t) = g_r^{-1}(r,t)K(r,t) \qquad (r \in [r_1(t), r_2(t)))$$

and

$$J(g(r,t),t) = g_r^{-1}(r,t)K(r,t) \qquad (r \in (r_2(t), r_3(t)]).$$

Hence, (3.8) yields

$$\begin{cases} \text{sign}(K(r,t)) = \text{sign}\,(I(g(r,t),t)) & (r \in (r_1(t), r_2(t))) \\ \text{sign}(K(r,t)) = -\,\text{sign}\,(J(g(r,t),t)) & (r \in (r_2(t), r_3(t))). \end{cases} \tag{3.16}$$

Finally, the same computation used to derive (3.14) gives

$$I_t = \tilde{\mathcal{L}}(h, I) \text{ and } J_t = \tilde{\mathcal{L}}(H, J) \text{ in } \cup_{t\in[0,T)} ((-z(t), z(t)) \times \{t\}), \tag{3.17}$$

where

$$\tilde{\mathcal{L}}(f,\phi) = \frac{1}{1+f_z^2}\phi_{zz} - \frac{2f_z f_{zz}}{(1+f_z^2)^2}\phi_z + \frac{(N-2)}{f^2}\phi \text{ in } \cup_{t\in[0,T)}(-z(t), z(t)) \times \{t\}.$$

To state the next result we define $n : [0,T) \to \mathbb{Z}^+$ by

$$\begin{aligned} n(t) \; &= \#\{r \in (r_1(t)+\delta, r_3(t)-\delta) \; : \; K(r,t) = 0\} \\ &+ \#\{|z| \le g(r_1(t)+\delta) \; : \; I(z,t) = 0\} \\ &+ \#\{|z| \le g(r_3(t)-\delta) \; : \; J(z,t) = 0\}, \end{aligned}$$

where $0 < \delta < \min\{r_3(t) - r_2(t), r_2(t) - r_1(t)\}$ is arbitrary. In view of (3.16) the above definition is independent of δ.

Lemma 3.4: *Assume $n(0)) < \infty$. Then $n(t) \le n(0)$ and $t \mapsto n(t)$ is nonincreasing in $[0,T)$.*

Lemma 3.4 follows by applying the lemma in the Appendix about the number of zeroe's of solutions to linear parabolic equations in one dimension. Of course, special care has to be taken for the fact that, in principle, the boundary $\cup_{t\in[0,T)}(\{r_1(t), r_2(t)\} \times \{t\})$ of the domain where K satisfies (3.14) may generate new zeroe's. This difficulty, however, may be overcome, in view of (3.16), by applying the aforementioned lemma to K, I and J and the equations they satisfy. The proof is long but rather standard, we, therefore, omit it.

Next we will extend the statement of Lemma 3.4 up to T. This is not immediate, since the coefficients of $\mathcal{L}$ and $\tilde{\mathcal{L}}$ are no longer bounded at $t = T$. But Corollary 2.4 asserts that h and, therefore, by (3.7) and Proposition 3.2, H and g are defined in a continuous way up to $t = T$. It follows that K, I and J can be extended up to $t = T$ away from their respective singularities. Finally, Proposition 2.1 and Corollary 2.4 also yield that

$$I(z,T) = h(z,T) - zh_z(z,T) \quad (z \in (-z(T), z(T)))$$

and

$$I(0,T) = 0.$$

Moreover I is continuous on its domain possibly except at $(0,T)$.

The next result asserts that $z \mapsto I(z,T)$ is negative in a neighborhood of $z = 0$.

Lemma 3.5: *If $n(0) < \infty$, then there exists $\varepsilon > 0$ such that*

$$I(z,T) < 0 \text{ in } [-\varepsilon, \varepsilon]\setminus\{0\}.$$

Proof: If $I(\cdot,T) \geq 0$ in $(z_1,z_2) \subset (0,z(T))$, then

$$\frac{\partial}{\partial z}\left(\frac{h(z,T)}{z}\right) = -\frac{1}{z^2}I(z,T) \leq 0.$$

Hence

$$\frac{h(z_2,T)}{z_2} \leq \frac{h(z_1,T)}{z_1}. \tag{3.18}$$

Let

$$z^* = \begin{cases} \inf\{z \in (0,z(T)) : I(z,T) = 0\} \\ z(T)\ \text{,if } I(\cdot,T) \neq 0 \text{ in } (0,z(T)). \end{cases}$$

Since $n(0) < \infty$, Lemma 3.4 yields $z^* > 0$. Hence $I(\cdot,T)$ is either negative or positive in $(0,z^*)$. If the latter holds, then

$$\frac{h(z^*,T)}{z^*} \leq \frac{h(z,T)}{z} \text{ for all } z \in (0,z^*)$$

and, since $h_z(0,T) = 0$,

$$h(z^*,T) = 0$$

which contradicts the positivity of $h(\cdot,T)$ in $(0,z(T))$.

□

Lemma 3.6: *If $n(0) < \infty$, then*

$$n(T) = \#\{z \geq 0 : I(z,T) = 0 \text{ or } J(z,T) = 0\} \leq n(0) \tag{3.19}$$

and

$$n^* = \#\{r > 0 : K(r,T) = 0\} \leq n(0) - 1. \tag{3.20}$$

Proof: In view of (3.16) and $I(0,T) = 0$, (3.19) and (3.20) are equivalent. To prove (3.20), we first claim that, for sufficiently small $\gamma > 0$,

$$n_\gamma(T) \leq \varliminf_{t\uparrow T} n_\gamma(t), \tag{3.21}$$

where, for $t \in [0,T]$,

$$n_\gamma(t) = \#\{r \in (h(\gamma,t), r_3(t)) : K(r,t) = 0\}.$$

To this end, choose $\gamma > 0$ sufficiently small so that

$$I(\gamma,T) < 0.$$

Then, by (3.16), $K(h(\gamma,t),t) < 0$ for t near T, hence (3.21) follows from (3.14) and an application of Lemma A. As in the proof of Lemma 3.4, again we need to apply Lemma A to K, I and J.

We will conclude by showing

$$\lim_{t \uparrow T} n_\gamma(t) \leq \lim_{t \uparrow T} n(t) - 1. \tag{3.22}$$

Since $n_\gamma(t) \leq n(t)$, if

$$\lim_{t \uparrow T} n_\gamma(t) = \lim_{t \uparrow T} n(t),$$

there must exist $\alpha > 0$ such that

$$I < 0 \text{ in } [0,\alpha] \times [T-\alpha, T) \cup (0,\alpha] \times \{T\} \text{ and } I(0,T) = 0.$$

Also by (0.6), $I(-z,t) = I(z,t)$. Hence

$$I < 0 \text{ on } Q_{\alpha,T} = [-\alpha,\alpha] \times [T-\alpha, T] \backslash \{(0,T)\}.$$

Since I solves a linear parabolic equation in the interior of $Q_{\alpha,T}$ and $I < 0$ on the parabolic boundary of $Q_{\alpha,T}, I(0,T)$ can not be equal to zero.

□

Lemma 3.7: *If $n^* > 0$, then $n^* \geq 2$. In particular, if $n(0) \leq 2$, then $K(r,T) < 0$ for all $r > 0$.*

Proof: Since $J(0,T) = H(0,T) > 0$, there exists $\bar{\varepsilon} > 0$ such that $J(z,T) > 0$ in $[-\bar{\varepsilon}, \bar{\varepsilon}]$. Combining this with Lemma 3.5 and (3.16) we get

$$K(r,T) < 0 \text{ for } r \in (0,\varepsilon] \cup [r_3(T) - \varepsilon, r_3(T)].$$

Hence, if $K(r,T)$ has any zeroe's for $r > 0$, there have to be at least two.

Finally, (3.20) and $n(0) \leq 2$ yield $n^* \leq 1$, which in view of the previous discussion implies that $n^* = 0$.

□

We are now in a position to prove Proposition 3.3.

Proof: In view of (1.3) and Lemma 3.5, there exists $R_1 \in (0, r_2(T))$ such that, for all $C > 0$,

$$\overline{\lim_{t\uparrow T}} \sup_{|z|\le g(R_1,T)} (I(z,t) + 2Ch_t(z,t)) < 0$$

and, therefore by (3.9) and the formulae before (3.16) we have

$$\overline{\lim_{t\uparrow T}} \sup_{r\in(r_1(t),R_1)} [K(r,t) - 2Cg_t(r,t)] < 0. \tag{3.23}$$

Next, choose $R_2 \in (r_2(T), r_3(T))$ and set

$$k = \sup_{r\in[R_1,R_2]} K(r,T) < 0.$$

Hence, for any $C < k \sup_{r\in[R_1,R_2]} |g_t(r,T)|^{-1}$,

$$\overline{\lim_{t\uparrow T}} \sup_{r\in[R_1,R_2]} [K(r,t) - 2Cg_t(r,t)] < 0. \tag{3.24}$$

Also

$$J(z,T) = H(z,T) - zH_z(z,T) > 0 \text{ in } [-z(T), z(T)].$$

In fact

$$k_1 = \inf_{|z|\le z(T)} J(z,T) > 0$$

and

$$k_2 = \sup_{|z|\le g(R_2,T)} |H_t(z,T)| < \infty.$$

Hence, if $C < k_1/k_2$,

$$\underline{\lim}_{t\uparrow T} \inf_{|z|\le g(R_2,T)} [J(z,t) + 2CH_t(z,t)] > 0,$$

and, consequently,

$$\overline{\lim_{t\uparrow T}} \sup_{r\in(R_2,r_3(T))} [K(r,t) - 2Cg_t(r,t)] < 0. \tag{3.25}$$

Combining (3.23), (3.24) and (3.25) we obtain (3.12).

□

In view of the discussion at this beginning of this section, Proposition 3.3 yields

$$\frac{-x \cdot Du + \overline{B}u_t}{|Du|} > 0 \text{ on } \Gamma_{t_0}$$

with $\overline{B} = 2[(T - t_0) + C]$. By (3.10) and the smoothness of g, there exists a $K > 0$ and $\gamma > 0$ such that

$$-x \cdot Du + \overline{B}u_t + K|u| \geq \gamma \text{ on } I\!R^N \times \{t_0\}. \tag{3.26}$$

The maximum principle and the properties of equation (0.2) then yield

$$-x \cdot Du + 2[(T - t) + C]u_t + K|u| \geq \gamma \text{ on } I\!R^N \times [t_0, \infty), \tag{3.27}$$

in the viscosity sense. Suppose now that u is smooth. Then

$$\frac{d}{ds}\{u(x(s), t(s)) \exp[K(sgnu(x(s), t(s)))s]\} \geq \gamma$$

where, for every (x, t) and $s \geq 0$,

$$\begin{cases} x(s) = xe^{-s} \\ t(s) = T + C - (T - t + C)e^{-2s} \end{cases}$$

Hence

$$u(x(s), t(s)) \geq u(x, t) + \gamma s. \tag{3.28}$$

Although (3.28) was derived under the assumption that u is smooth, it follows that it holds for all s, since (3.27) holds in the viscosity sense. But then, for $x = 0$ and $t = T$, (3.29) yield $x(s) \equiv 0, t(s) = T + C - Ce^{-2s}$. Hence by (3.28), for $\varepsilon > 0$,

$$u(0, T + \varepsilon) > 0$$

i.e. Γ_T "opens up". (Recall that $u(0, t) < 0$ for all $t \in [0, T)$ and $u(0, T) = 0$.)

Finally, in order to show that Γ_t is smooth after the singularity, it suffices to show that the equation

$$G_t = \frac{G_{rr}}{1 + G_r^2} + \frac{(N - 2)G_r}{r}$$

admits a smooth solution for $t > 0$, even if $G(\cdot, 0)$ has a singularity like the one of $g(\cdot, T)$. This can be shown by a number of approximations using a combination of standard parabolic regularity [LSU], the gradient estimate of

Evans-Spruck [ES], Angenent's lemma (Lemma A of the Appendix) and the stability properties of surfaces moving by mean curvature (cf. [S]). (Indeed the details of this argument have been caried out in [AAG].) As a matter of fact, such arguments can be used from the beginning to show that Γ_t never develops interior. This is the approach of [AAG] for "barbell" type domains. In this paper, however, we chose to follow the approach described above since it gives rise to (3.27), which has a very nice geometric interpretation.

We now combine all the above to state the main result of the section.

Theorem 3.8: *Suppose that the assumptions of Proposition 3.2 hold and the $n(0) \le 2$. Then the evolution $t \mapsto \Gamma_t$ never develops interior. Moreover, it "opens up" instanteneously after the focusing time and continuous moving as a smooth surface.*

We continue checking that a torus

$$(r-1)^2 + z^2 = R^2 \qquad (0 < R < 1),$$

whose evolution $t \mapsto \Gamma_t$ by mean curvature focuses at $(0, T)$, satisfies $n(0) \le 2$. This will yield Proposition 3 in the Introduction.

A simple calculation yields

$$K(r,0) = -\frac{1}{r g_0(r)}[r^2 + r(R^2 - 1 - 2T(N-1)) + T(N-1)],$$

where

$$g_0(r) = [R^2 - (r-1)^2]^{\frac{1}{2}}.$$

The above claim is then obvious.

We finally conclude this section with the:

Proof of Proposition 3: Let $\Gamma_t(R)$ be the solution of the mean curvature flow with initial data

$$\Gamma_0(R) = \{x \in I\!R^N : (r-1)^2 + z^2 = R^2\} \qquad (0 < R < 1).$$

Let $T_{\text{ext}}(R)$ be the extinction time of $\Gamma_t(R)$. We have already shown that either $\Gamma_t(R)$ is smooth for all $t < T_{\text{ext}}(R)$ or it focuses at some time $T(R) \le T_{\text{ext}}(R)$. If the solution focuses before the extinction time, we have already argued in the paragraph right before Theorem 3.8 that it evolves smoothly after $t > T(R)$ until the extinction. Now we will show that for exactly one value of the parameter R, the extinction and the focusing times are the same. Set

$$R_0 = \inf\{R \in (0,1) : T(\rho) < T_{\text{ext}}(\rho) \qquad \forall \rho \in [R, 1)\}.$$

It is easy to see that $R_0 \in (0,1)$. Now if $R > R_0$ then we have shown that $\Gamma_t(R)$ "focuses" at zero at $t = T(R)$ and then "opens up" and finally goes extinct smoothly. If $R < R_0$ we claim that $\Gamma_t(R)$ evolves smoothly to a circle at $T_{\text{ext}}(R)$ without focusing. Indeed the properties of the mean curvature flow imply that

$$d(t,R) = \inf\{|x-y| : x \in \Gamma_t(R), y \in \Gamma_t(R_0)\} \geq d(0,R) > 0, \tag{3.29}$$

for all $t > 0$. Moreover $\Gamma_t(R) = \partial\Omega(t,R)$ for all R and t for some closed region $\Omega(t,R) \subset I\!R^N$. Since $\Omega(0,R)$ is included in $\Omega(0,R_0)$ we have

$$\Omega(t,R) \subset \Omega(t,R_0) \qquad \forall t \leq T_{\text{ext}}(R) \wedge T_{\text{ext}}(R_0). \tag{3.30}$$

Now choose $t^* = t^*(R)$ satisfying

$$\Omega(t^*,R_0) \subset \{x \in I\!R^N : |x| \leq d(0,R)\}.$$

Then in view of (3.29) and (3.30), we conclude that $T_{\text{ext}}(R) < t^*$. Using (3.29) and (3.30) it is also easy to show that $0 \notin \Omega(t,R)$ for all $t \leq T_{\text{ext}}(R)$. Hence $\Gamma_t(R)$ is smooth for all $t < T(R)$.

□

Appendix.

In this Appendix we state a result of Angenent for the convenience of the reader. The statement of this lemma is taken from [A2]. However its proof is in [A1].

Lemma A. *Let $u : [x_0,x_1] \times (0,t_0) \to R$ be a continuous classical solution of*

$$u_t = a(x,t)u_{xx} + b(x,t)u_x + c(x,t)u$$

with $u(x_0,t) \neq 0, u(x_1,t) \neq 0$ for all $t \in (0,t_0)$.

Assume that a, b, c satisfy

(i) $\delta \leq a(x,t) \leq \delta^{-1}$ for some $\delta > 0$,

(ii) $a, a_t, a_x, a_{xx}, b, b_t, b_x$ and c are bounded measurable functions of $[x_0,x_1] \times (0,t_0)$.

Then the number of zeroes of $u(\cdot,t)$

$$z(t) = \#\{x \in (x_0,x_1) | u(x,t) = 0\}$$

is finite and non-increasing in t.

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