Mathematics of Operations Research

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MATHEMATICS OF OPERATIONS RESEARCH
Vol. 17, No. 4, November 1992
Printed in U S A

TURNPIKE SETS AND THEIR ANALYSIS IN STOCHASTIC PRODUCTION PLANNING PROBLEMS*

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This paper considers optimal infinite horizon stochastic production planning problems with capacity and demand to be finite state Markov chains. The existence of the optimal feedback control is shown with the aid of viscosity solutions to the dynamic programming equations. Turnpike set concepts are introduced to characterize the optimal inventory levels. It is proved that the turnpike set is an attractor set for the optimal trajectories provided that the capacity is assumed to be fixed at a level exceeding the maximum possible demand. Conditions under which the optimal trajectories enter the convex closure of the set in finite time are given. The structure of turnpike sets is analyzed. Last but not least, it is shown that the turnpike sets exhibit a monotone property with respect to capacity and demand. It turns out that the monotonicity property helps in solving the optimal production problem numerically, and in some cases, analytically.

1. Introduction. The convex production planning model is an important paradigm in the operations management/operations research literature. Earliest formulation of the model dates back to Modigliani and Hohn [16] in 1955. They were interested in obtaining a production plan over a finite horizon in order to satisfy a deterministic demand and minimize the total discounted convex costs of production and inventory holding. Since then the model has been further studied and extended in both continuous and discrete times by a number of researchers including Johnson [12], Arrow, Karlin and Scarf [3], Veinott [23], Adiri and Ben-Israel [1], Sprzeuzkouski [20], Lieber [15], and Hartl and Sethi [9]. A rigorous formulation of the problem, along with a comprehensive discussion of the relevant literature, appears in Bensoussan, Crouhy and Proth [4].

A major characteristic of the optimal policy in convex production planning with a sufficiently long horizon is that there exists a time-dependent threshold or turnpike level (see Thompson and Sethi [21]) such that production takes place in order to reach the turnpike level if the inventory is below the turnpike level and no production takes place if the inventory is above the level. Once on the turnpike level, only necessary production takes place so as to remain on the turnpike. Such a policy is commonly referred to as the 'order-up-to' production policy. In a finite horizon case, the above policy holds at every instant sufficiently removed from the horizon. In the infinite horizon case with usual assumption of constant demand, the turnpike level will be a constant and the policy will hold everywhere.

Extensions of the convex production planning problem to handle stochastic demand have been analyzed mostly in the discrete-time framework. A rigorous analysis of the stochastic model has been carried out in Bensoussan et al. [4]. Recently, continuous-time versions of the model that incorporate additive white noise terms in

*Received December 26, 1989; revised May 2, 1991.
AMS 1980 subject classification. Primary: 93E20, 90C39, 90B30. Secondary: 35J65, 35K60, 60J60.
IAOR 1973 subject classification. Main: Production. Cross references: Control Processes, Stochastic Processes.
OR/MS Index 1978 subject classification. Primary: 120 Dynamic Programming/Optimal Control. Secondary: 581 Production planning
Key words. Stochastic production planning, turnpike sets, dynamic programming, viscosity solutions.

0364-765X/92/1704/0932/$01.25

the dynamics of the inventory process were analyzed by Sethi and Thompson [17] and Bensoussan, Sethi, Vickson and Derzko [5].

Preceding works that relate most closely to our formulation of the problem include Kimemia and Gershwin [13], Akella and Krumar [2], Fleming, Sethi and Soner [7], and Lehoczky, Sethi, Soner and Taksar [14]. These works incorporate piecewise deterministic processes either in the dynamics or in the constraints of the model. Fleming et al. [7] considered the demand to be a finite state Markov process. In the models of Kimemia an Gershwin [13], Akllea and Kumar [2], and Lehoczky et al. [14], inspired by flexible manufacturing systems, the production capacity rather than the demand for production is modelled as a stochastic process. In particular, the process of machine breakdown and repair is modelled as a birth-death process, thus making the production capacity over time a finite state Markov process.

It is the purpose of this paper to present a general model that considers uncertainties in both production capacity and demand. The concept of turnpike sets is generalized to such a formulation (see (2.6)) and analyzed in a greater detail than in the existing literature.

In this paper we study the elementary properties of the value function v. We show that v is a convex function and that it is strictly convex provided the inventory cost is strictly convex. Moreover, v is shown to be a viscosity solution to a dynamic programming equation and has upper and lower bounds with polynomial growths. An exact definition of the turnpike sets is given in terms of the value function. We prove that the turnpike sets are attractors of the optimal trajectories as well as provide sufficient conditions under which the optimal trajectories enter the convex closure in finite time. Also, we give conditions to ensure that the turnpike sets are nonempty.

A main result in this article is the monotonicity of the turnpike sets in terms of the capacity level or the demand level. By and large, the problem of solving the optimal production planning is equivalent to the problem of locating the turnpike sets. Therefore, the knowledge about the monotone property of the turnpike sets definitely helps to solve the optimal production problem. On the one hand, the monotonicity can be used to solve some optimal control problems in a closed form (see Examples 4.1 and 5.2) which are very difficult to handle in general. On the other, it can greatly reduce the computation needed for numerical approaches for solving the optimal control problem (cf. Sharifnia [18]).

The plan of the paper is as follows. In the next section, we state the production-inventory model under consideration with stochastic demands and unreliable machines. In §3, we develop the convexity and smoothness properties of the value function by using the method of viscosity solution to the dynamic programming equation. Optimal feedback controls are then given in terms of the partial derivatives of the value function. In §4, we deal with the model with random capacity and deterministic demand and show that the turnpike sets possess a monotone property with respect to the capacity. In §5, we consider the model with a fixed capacity and random demand. It is shown that the convex closure of the turnpike set is an attractor set for the optimal inventory trajectories (see Theorems 5.1 and 5.2). Moreover, the monotonicity of the turnpike sets is proved in two particular cases. It is pointed out by a counter-example (Example 5.1) that such a monotone property does not exist in general. In §§4 and 5, we also apply the results obtained to solve two optimal production planning problems explicitly (Examples 4.1 and 5.2). Finally to conclude the paper, we give an example (Example 5.3) to show that the strict convexity of the cost function is not a necessary condition for the strict convexity of the value function.

2. Problem formulation. We are concerned with a one-dimensional continuous-time production-inventory model with stochastic machine capacity and demand.

While the results to be derived in the next section can be extended to a multi-dimensional framework, the monotonicity properties obtained later in §§4 and 5 make sense only in the one-dimensional case. It is for this reason that we have chosen to deal only with the one-dimensional model. It should also be noted that the classical literature on the convex production planning problem is concerned mainly with one-dimensional problems.

Let $y(t)$, $p(t)$, $m(t)$ and $z(t)$ denote, respectively, the inventory level, the production rate, the capacity level, and the demand rate at time $t \in [0, \infty)$. We assume that $y(t) \in R$, $p(t) \in R$, $m(t) \in \mathscr{M}$, and $z(t) \in \mathscr{Z}$, where $\mathscr{M}$ and $\mathscr{Z}$ are finite sets. Moreover, we assume that the capacity $m(t)$ and the demand rate $z(t)$ are finite state continuous-time Markov chains defined on some underlying probability space $\{\Omega, \mathscr{F}, \mathscr{P}\}$. The associated generators L_m and L_z of the Markov chains $m(t)$ and $z(t)$, respectively, have the following forms: For any function ϕ,

$$L_m \phi(m) = \sum_{m \neq m'} q_{mm'} [\phi(m') - \phi(m)], \tag{2.1}$$

$$L_z \phi(z) = \sum_{z \neq z'} \bar{q}_{zz'} [\phi(z') - \phi(z)], \tag{2.2}$$

where $q_{mm'} \geqslant 0$ and $\bar{q}_{zz'} \geqslant 0$.

A control process (production rate) $p(\cdot) = \{p(t, \omega) \in R\colon t \geqslant 0,\ \omega \in \Omega\}$ is called *admissible* if: (i) $p(\cdot)$ is adapted to $\mathscr{F}_t = \sigma\{m(s), z(s)\colon 0 \leqslant s \leqslant t\}$, (ii) $0 \leqslant p(t, \omega) \leqslant m(t, \omega)$ for all $t \geqslant 0$ and $\omega \in \Omega$. The ω-dependence will be suppressed if no confusion arises. Let $\mathscr{A}$ denote the set of admissible control processes. Then for any $p(\cdot) \in \mathscr{A}$, the dynamics of the system have the following form:

$$\dot{y}(t) = p(t) - z(t), \qquad t \geqslant 0. \tag{2.3}$$

Let $h(y)$ and $c(p)$ denote the holding cost and the production cost functions, respectively. For every $p(\cdot) \in \mathscr{A}$, $y = y(0)$, $m = m(0)$, $z = z(0)$, let the objective functional be as follows:

$$J(y, m, z, p(\cdot)) = E \int_0^\infty e^{-\alpha t} [h(y(t)) + c(p(t))]\, dt, \tag{2.4}$$

where $\alpha > 0$ is the given discount rate. The value function is

$$v(y, m, z) = \inf\{J(y, m, z, p(\cdot))\colon p(\cdot) \in \mathscr{A}\}. \tag{2.5}$$

We make the following assumptions on the cost functions $h(y)$ and $c(p)$ throughout the paper.

(A1) $h(y)$ is a nonnegative, convex function with $h(0) = 0$. There are positive constants c_1, c_2, c_3 and $k_0 > 1$, $k_0' > 1$ such that

$$c_1 |y|^{k_0} - c_2 \leqslant h(y) \leqslant c_3 \big(1 + |y|^{k_0'}\big).$$

(A2) $c(p)$ is a nonnegative function, $c(0) = 0$, and $c(p)$ is twice differentiable. Moreover, $c(p)$ is either strictly convex or linear.

(A3) The capacity and the demand processes have the state spaces:

$$\mathscr{M} = \{0, 1, \ldots, M\} \quad \text{and} \quad \mathscr{Z} = \{z_1, z_2, \ldots, z_d\},$$

respectively. Moreover,

$$z_1 < z_2 < \cdots < z_d < M \quad \text{and} \quad \mathcal{Z} \cap \mathcal{M} = \varnothing.$$

REMARK. (i) The representation for $\mathcal{M}$ stands for the case of M identical machines, each with a unit capacity.

(ii) Usually, the demand levels are taken to be nonnegative numbers. The restriction is not required and therefore it is not assumed here.

(iii) The assumption $\mathcal{Z} \cap \mathcal{M} = \varnothing$ is innocuous and is used for the purpose of ensuring the differentiability of the value function v.

Now let us define the turnpike sets.

DEFINITION. We call

(2.6)

$$\mathcal{S}(m, z) = \left\{ y_{m,z} : v(y_{m,z}, m, z) + c'(z) y_{m,z} = \min_{y \in R} \{ v(y, m, z) + c'(z) y \} \right\},$$

the turnpike sets associated with (m, z).

REMARK. (i) Note that for each (m, z) the value function $v(y, m, z)$ is a convex function, so is $v(y, m, z) + c'(z)y$. Thus, the point $y_{m,z}$ in $\mathcal{S}(m, z)$ coincides with the point $y'_{m,z}$ such that $D^- v(y, m, z) + c'(z) = 0$ where $D^- v(y, m, z)$ is the subdifferential of $v(y, m, z)$. In particular, if the value function v is differentiable in y, then the turnpike set becomes (see Lemma 3.1)

$$\mathcal{S}(m, z) = \left\{ y_{m,z} : v_y(y_{m,z}, m, z) + c'(z) = 0 \right\}.$$

Thus, $\mathcal{S}(m, z)$ is a natural generalization of the well-known hedging point concept.

(ii) By Lemma 3.1 in the next section, it is seen that the turnpike sets $\mathcal{S}(m, z)$ is a nonempty set. Another condition is stated in Theorem 5.1(ii).

(iii) An original motivation to define the turnpike sets $\mathcal{S}(m, z)$ is to characterize the equilibrium points of the optimal trajectories. However, if the capacity m is less than the demand z, such an equilibrium point no longer exists. Hence the definition of $\mathcal{S}(m, z)$ in (2.6) does not have any dynamical meaning when $m < z$. It is introduced for every m and z because of notational brevity and completeness.

In later sections (§§4 and 5), we will study the inner structure of the turnpike sets under certain conditions. A major concern is the monotonicity of $\{y_{m,z}\}$ in terms of m and z.

3. Properties of the value function.

In this section, we examine the properties of the value function. We will show that the value function is a convex function and satisfies a dynamic programming equation. Then optimal feedback controls will be given in terms of the partial derivatives of the value function.

LEMMA 3.1. (i) *For each* (m, z), $v(\cdot, m, z)$ *is convex on* R *and* $v(\cdot, m, z)$ *is strictly convex if* $h(\cdot)$ *is so.*

(ii) *There exist positive constants* C_1, C_2 *and* C_3 *such that for each* (m, z)

$$C_1 |y|^{k_0} - C_2 \leqslant v(y, m, z) \leqslant C_3 \left(1 + |y|^{k'_0}\right)$$

where k_0 *and* k'_0 *are the power indices in assumption* (A1).

REMARK. This lemma tells us that for each (m, z), the turnpike set $\mathscr{G}(m, z)$ will not be empty. Moreover, the strict convexity of h implies that $\mathscr{G}(m, z)$ is a singleton due to the strict convexity of v.

PROOF. Under assumptions (A1) and (A2), it is clear that $J(\cdot, m, z, \cdot)$ is jointly convex in (y, p) for each (m, z). As a consequence, the value function $v(\cdot, m, z)$ is convex for each (m, z).

To show the strict convexity of v, let us assume to the contrary. Then there exist m_0, z_0, y_1, y_2, such that $v(y, m_0, z_0)$ is linear on (y_1, y_2), i.e.,

$$v((y_1 + y_2)/2, m_0, z_0) = \tfrac{1}{2}[v(y_1, m_0, z_0) + v(y_2, m_0, z_0)]. \tag{3.1}$$

For $i = 1, 2$, let $p_i^*(\cdot) \in \mathscr{A}$ denote the optimal production rates (the existence of $p_i^*(\cdot)$ is given in [22]) and let $y_i^*(\cdot)$ denote the corresponding optimal inventory level with $y_i^*(0) = y_i$. Then

$$y_i^*(t) = y_i + \int_0^t [p_i^*(t) - z(t)]\, dt, \qquad i = 1, 2, \quad \text{and}$$

$$v(y_i, m_0, z_0) = J(y_i, m_0, z_0, p_i^*(\cdot)).$$

Let

$$y(t) = (y_1 + y_2)/2 + \int_0^t [(p_1^*(t) + p_2^*(t))/2 - z(t)]\, dt.$$

Then (3.1) and the fact that $(p_1^*(\cdot) + p_2^*(\cdot))/2 \in \mathscr{A}$ imply

$$\begin{aligned} &\tfrac{1}{2}[J(y_1, m_0, z_0, p_1^*(\cdot)) + J(y_2, m_0, z_0, p_2^*(\cdot))] \\ &\qquad \leqslant J((y_1 + y_2)/2, m_0, z_0, (p_1^*(\cdot) + p_2^*(\cdot))/2). \end{aligned}$$

The convexity of $h(y)$ and $c(p)$ implies the opposite inequality. Therefore,

$$\begin{aligned} &h(y_1^*(t)) + h(y_2^*(t)) + c(p_1^*(t)) + c(p_2^*(t)) \\ &\qquad = 2h((y_1^*(t) + y_2^*(t))/2) + 2c((p_1^*(t) + p_2^*(t))/2), \end{aligned}$$

a.e. in t and a.s. in ω. Owing to the convexity of c and strict convexity of h, $y_1^*(t) = y_2^*(t)$ a.e. in t and a.s. in ω. But this is not true for t small and, therefore, (i) is proved.

We now show (ii). The upper bound on v comes from assumption (A1) and the fact that $p(t) \equiv 0$, $t \geqslant 0$, is an admissible control.

Now, let $y^*(t)$ denote the optimal inventory level with $y^*(0) = y$ and let $y_1(t) = y - z_d t$, $y_2(t) = y + (M - z_1)t$. Then

$$y_1(t) \leqslant y^*(t) \leqslant y_2(t) \quad \forall t \geqslant 0.$$

Therefore,

$$|y^*(t)| \geqslant |y_1(t)| \ \text{ if } y_1(t) \geqslant 0 \quad \text{and} \quad |y^*(t)| \geqslant |y_2(t)| \ \text{ if } y_2(t) \leqslant 0. \tag{3.2}$$

Let $V(y,m,z) = E\int_0^\infty e^{-\alpha t}|y^*(t)|^{k_0}\,dt$. By our definition of the value function and the assumption on h in (A1),

$$v(y,m,z) \geqslant E\int_0^\infty e^{-\alpha t}h(y^*(t))\,dt \geqslant c_1V(y,m,z) - c_2/\alpha.$$

We are to show that $V(y,m,z) \geqslant c_1'|y|^{k_0} - c_2'$ for some constants $c_1' > 0$, $c_2' > 0$. For $y^*(0) = y > 0$,

$$\begin{aligned} V(y,m,z) &\geqslant E\int_0^{y/z_1} e^{-\alpha t}|y_1(t)|^{k_0}\,dt \quad (\text{by } (3.2)) \\ &= \int_0^{y/z_1} e^{-\alpha t}[y - z_1t]^{k_0}\,dt \\ &\geqslant \int_0^{y/z_1} e^{-\alpha t}\left[2^{1-k_0}|y|^{k_0} - |z_1t|^{k_0}\right]dt \\ &\geqslant \alpha^{-1}2^{1-k_0}|y|^{k_0} - c_0, \end{aligned}$$

for some constant c_0. Similarly, for $y^*(0) = y < 0$,

$$\begin{aligned} V(y,m,z) &\geqslant E\int_0^{y/(M-z_d)} e^{-\alpha t}|y_2(t)|^{k_0}\,dt \quad (\text{by } (3.2)) \\ &\geqslant \alpha^{-1}2^{1-k_0}|y|^{k_0} - c_0', \end{aligned}$$

for some constant c_0'. Thus we conclude $v(y,m,z) \geqslant C_1|y|^{k_0} - C_2$ for some constants $C_1 > 0, C_2$. $\square$

REMARK. In this lemma we only require that the inventory cost h is strictly convex. So the lemma extends a similar strict convexity result in Bensoussan et al. [5].

Let P_m denote the control points $P_m = \{p\colon p \geqslant 0,\ p \leqslant m\}$ and let $F(m,z,r)$ be the following function of (m,z,r) for $r \in R$:

$$F(m,z,r) = \inf\{(p - z)\cdot r + c(p)\colon p \in P_m\}. \tag{3.3}$$

Then the dynamic programming equation associated with our optimal control problem is written formally as follows:

$$\begin{aligned} \alpha v(y,m,z) &= F\big(m,z,v_y(y,m,z)\big) + h(y) \\ &\quad + L_m v(y,\cdot\,,z)(m) + L_z v(y,m,\cdot)(z), \end{aligned} \tag{3.4}$$

for $y \in R$, $m \in \mathscr{M}$, $z \in \mathscr{D}$, where $v_y(y,m,z)$ is the gradient with respect to y and L_m, L_z are the infinitesimal generators given in (2.1) and (2.2), respectively.

In general, the value function v may not be differentiable. In order to handle the nondifferentiability, we consider the viscosity solution to the dynamic programming equation. We show that the value function (cf. (2.5)) is a viscosity solution to (3.4) and then show that v is continuously differentiable. Therefore, v satisfies the dynamic programming equation.

To start with, we define (as in [7]) the convex subsets $D^{\pm}v(y,m,z)$ of R as follows:

$$D^{+}v(y,m,z) = \left\{ r \in R\colon \limsup_{h\to 0} \Psi(y,m,z,r,h) \leqslant 0 \right\},$$

$$D^{-}v(y,m,z) = \left\{ r \in R\colon \liminf_{h\to 0} \Psi(y,m,z,r,h) \geqslant 0 \right\},$$

where $\Psi(y,m,z,r,h) := (v(y+h,m,z) - v(y,m,z) - r\cdot h)|h|^{-1}$.

We discuss viscosity solutions in the following sense.

DEFINITION. Any continuous function v is a viscosity solution of (3.4) if for all $r \in D^{+}v(y,m,z)$,

$$\alpha v(y,m,z) \leqslant F(m,z,r) + h(y) + L_m v(y,\cdot,z)(m) + L_z v(y,m,\cdot)(z),$$

and for all $r \in D^{-}v(y,m,z)$,

$$\alpha v(y,m,z) \geqslant F(m,z,r) + h(y) + L_m v(y,\cdot,z)(m) + L_z v(y,m,\cdot)(z).$$

LEMMA 3.2. *The value function $v(y,m,z)$ defined in* (2.5) *is a viscosity solution to the dynamic programming equation* (3.4).

PROOF. The proof follows from a straightforward modification of the proof of Theorem 1.1 in [19]. □

In later sections, we will study optimal feedback controls, which are functions of the gradient v_y. Therefore, we are interested in establishing the C^1 property of the value function v.

THEOREM 3.1. *The value function $v(\cdot,m,z)$ is continuously differentiable and satisfies the dynamic programming equation* (3.4).

PROOF. If the value function v is differentiable, then the two differentials $D^{+}v(y,m,z)$ and $D^{-}v(y,m,z)$ are both equal to $\{v_y(y,m,z)\}$. Therefore, the two inequalities in the definition of viscosity solution yield an equality; thus, v satisfies the dynamic programming equation.

To finish the proof, it remains to show that v is continuously differentiable. By assumption (A3), it can be shown that the map $r \mapsto F(m,z,r)$ is not constant on any nontrivial convex subset of R. Then following from the same arguments as in [7, Theorem 2.2], we can show that $v(\cdot,m,z)$ is differentiable with continuous derivatives. □

THEOREM 3.2 (VERIFICATION THEOREM). *If there exists $p^*(\cdot) \in \mathscr{A}$, for which the corresponding $y^*(t)$ satisfies* (2.3) *with $y = y^*(0)$, $r^*(t) = v_y(y^*(t), m(t), z(t))$, and*

$$F(m(t),z(t),r^*(t)) = (p^*(t) - z(t))\cdot r^*(t) + c(p^*(t))$$

a.e. in t with probability one, then $p^*(\cdot)$ is optimal, i.e.,

$$v(y,m,z) = J(y,m,z,p^*(\cdot)).$$

PROOF. The proof is standard; we refer the readers to [6] for details. □

Based on the above verification theorem, the optimal feedback production policies should have the following form:

$$(3.5)\qquad p^*(y,m,z)=\begin{cases}0 & \text{if } v_y(y,m,z)\geqslant 0,\\ (c')^{-1}\big(-v_y(y,m,z)\big) & \text{if } -c'(m)\leqslant v_y(y,m,z)<0,\\ m & \text{if } v_y(y,m,z)<-c'(m),\end{cases}$$

when $c''(p)$ is strictly positive and

$$(3.6)\qquad p^*(y,m,z)=\begin{cases}0 & \text{if } v_y(y,m,z)>-c,\\ \min\{z,m\} & \text{if } v_y(y,m,z)=-c,\\ m & \text{if } v_y(y,m,z)<-c,\end{cases}$$

when $c(p)=cp$ for some constant $c\geqslant 0$. Note that in (3.6), the value of p^* could be set arbitrarily at $y=y_{m,z}$ for which $v_y(y_{m,z},m,z)+c=0$. We take $p^*=\min\{z,m\}$ at $y=y_{m,z}$, only because we want the optimal trajectory y to stay at the level $y_{m,z}$ as long as possible.

Recall that $v(\cdot,m,z)$ is a convex function. Thus $p^*(y,m,z)$ is nonincreasing in y. From a result on differential equations (see [10, Theorem 6.2]),

$$(3.7)\qquad \dot{y}(t)=p^*(y(t),m(t),z(t))-z(t),\qquad t\geqslant 0$$

has a unique solution $y^*(t)$ for each sample path $(m(t),z(t))$. Hence, we conclude the following theorem.

THEOREM 3.3. *The feedback control functions in* (3.5) *and* (3.6) *are optimal feedback controls.*

In the rest of the paper, we are concerned with the behavior of the optimal trajectories and the monotonicity of the turnpike sets. We shall only consider the systems with constant demand and stochastic capacity or the systems with stochastic demand and constant capacity. This is because our proof does not extend to systems in which both capacity and demand are stochastic (see the remark following the proof of Theorem 4.1).

4. Turnpike sets with unreliable machines. In this section we concentrate on the case with constant demand $z(t)=z_0$, $t\geqslant 0$ and randomly fluctuating capacity $m(t)$. The other case will be dealt with in the following section.

We assume $0<z_0<M$, i.e., demand is not zero and that it can be satisfied by the maximum available capacity. For brevity, we suppress z in the value function, i.e., we use $v(y,m)$ instead of $v(y,m,z_0)$. The dynamic programming equation (3.4) becomes

$$(4.1)\qquad \alpha v(y,m)=F\big(m,v_y(y,m)\big)+h(y)+L_m v(y,\cdot)(m),$$

where $F(m,r):=F(m,z_0,r)$ as in (3.4).

We assume in this section that $m(t)$ is a birth-death process, i.e., the generator L_m of $m(t)$ for any function ϕ has the following form:

$$L_m\phi(i) = \begin{cases} \mu_0(\phi(1) - \phi(0)) & \text{if } i = 0 \\ \mu_i(\phi(i+1) - \phi(i)) + \lambda_i(\phi(i-1) - \phi(i)) & \text{if } 0 < i < M, \\ \lambda_M(\phi(M-1) - \phi(M)) & \text{if } i = M, \end{cases} \tag{4.2}$$

where the machine breakdown rates λ_i, $i = 1, 2, \ldots, M$ and the machine repair rates μ_i, $i = 0, 1, \ldots, M-1$ are nonnegative constants.

REMARK. Our assumption on the process $m(t)$ is based on two reasons: (i) In many practical situations, the probability of two breakdown and/or repair events occurring simultaneously can be assumed to be small. Moreover, the assumption is fairly standard in the relevant literature dealing with unreliable machines.

(ii) The monotonicity of the turnpike sets cannot be expected to hold in general. Intuitively, the capacity process needs to be skip-free. If it were not, then one could imagine, say, in a four machine case to have a transition from $m = 4$ to $m = 2$, and thus skip $m = 3$, with a sufficiently high probability to conceivably have

$$y_{1,z_0} \geqslant y_{2,z_0} \geqslant y_{4,z_0} > y_{3,z_0},$$

provided the turnpike sets are singletons, thus violating the monotonicity property. Indeed, a counterexample could be easily constructed. However, since we will construct a similar example (Example 5.1) in the next section, we will not do so here.

Now the turnpike set $\mathscr{G}(m, z)$ becomes

$$\mathscr{G}(m) := \{y \in R: v_y(y, m) = -c'(z_0)\}. \tag{4.3}$$

Let $y^*(t)$ denote the optimal inventory level with $y^*(0) = y$, $m(0) = m$. Then

$$y^*(t) = y + \int_0^t [p^*(y(s), m(s)) - z_0]\, ds, \qquad t \geqslant 0.$$

Observe that when the capacity state m is absorbing, then $y^*(t)$ converges to the set $\mathscr{G}(m)$ provided $m > z_0$ (this is a particular case of Lemma 5.1).

Recall that $v(\cdot, m)$ is a convex function and is bounded by two functions as shown in Lemma 3.1. Thus, $\mathscr{G}(m)$ is a bounded interval. In particular, if $v(\cdot, m)$ is strictly convex, $\mathscr{G}(m)$ shrinks to a singleton.

Define $i_0 \in \mathscr{M}$ to be such that $i_0 < z_0 < i_0 + 1$. Observe that for $m \leqslant i_0$,

$$\dot{y}^*(t) \leqslant m - z_0 \leqslant i_0 - z_0 < 0.$$

Therefore $y^*(t) \searrow -\infty$ as $t \to \infty$ provided m is absorbing. Hence only those $m \in \mathscr{M}$ for which $m \geqslant i_0 + 1$ are of special interest to us.

As mentioned in §1, one of the major purposes of this paper is to analyze the monotonicity of the turnpike sets. From Lemma 3.1, if h is strictly convex, each turnpike set defined in (4.3) reduces to a singleton. That is, there exists y_m such that $\mathscr{G}(m) = \{y_m\}$, $m \in \mathscr{M}$.

If the production cost is linear, i.e., $c(p) = cp$ for some constant c, then y_m is the threshold inventory level with capacity m. Specifically, if $y > y_m$, $p^*(y, m, z) = 0$ and if $y < y_m$, $p^*(y, m, z) = m$ (full available capacity).

Let us make the following observation. If the capacity $m > z_0$, then the optimal trajectory will move towards the turnpike y_m. Suppose the inventory level is y_m and then capacity increases to $m_1 > m$, it then becomes costly to keep the inventory at level y_m. So the turnpike inventory level y_{m_1} should be smaller than y_m. The remaining section is devoted to proving this intuitive observation.

To begin with, we provide a technical lemma concerning the evaluation of the dynamic programming equation at points in the turnpike sets.

LEMMA 4.1. *Assume that the inventory cost function $h(y)$ is differentiable and strictly convex. Let*

$$K_m = h'(y_m) + L_m v_y(y_m, \cdot)(m) - \alpha v_y(y_m, m), \qquad m \in \mathscr{M}. \tag{4.4}$$

Then

$$K_m \geqslant 0 \quad \textit{for } m \leqslant i_0 \qquad \textit{and} \qquad K_m = 0 \quad \textit{for } m \geqslant i_0 + 1.$$

PROOF. For $m \leqslant i_0$, the dynamic programming equation (4.1) on $(-\infty, y_m)$ becomes

$$\alpha v(y,m) = (m - z_0) v_y(y,m) + h(y) + c(m) + L_m v(y, \cdot)(m).$$

Taking the derivative on both sides of the equation at $y \in (-\infty, y_m)$ and letting $y \uparrow y_m$, we obtain $K_m \geqslant 0$ for $m \leqslant i_0$.

For each $m \geqslant i_0 + 1$, note that $F(m, r)$ is a concave function in r for each m and it reaches its unique maximum at $r = -c'(z_0)$. Note that the strict convexity of $h(y)$ implies the strict convexity of $v(\cdot, m)$ (by Lemma 3.1). Therefore, there exist sequences $0 < \delta_1(k) \leqslant 1/k$ and $0 < \delta_2(k) \leqslant 1/k$, such that

$$F\big(m, v_y(y_m + \delta_1(k), m)\big) = F\big(m, v_y(y_m - \delta_2(k), m)\big).$$

Then, subtracting the two equalities of the dynamic programming equation valued, respectively, at $(y_m + \delta_1(k), m)$ and $(y_m - \delta_2(k), m)$ yields

$$\begin{aligned}
&\alpha\big[v(y_m + \delta_1(k), m) - v(y_m - \delta_2(k), m)\big] \\
&\quad = \big[h(y_m + \delta_1(k), m) - h(y_m - \delta_2(k), m)\big] \\
&\qquad + L_m\big[v(y_m + \delta_1(k), \cdot) - v(y_m - \delta_2(k), \cdot)\big](m).
\end{aligned}$$

Dividing both sides by $\delta_1(k) + \delta_2(k)$ and letting $k \to \infty$, we conclude that $K_m = 0$ for $m \geqslant i_0 + 1$. □

Now we are in a position to show the monotonicity of the turnpike sets.

THEOREM 4.1. *Assume $h(y)$ to be differentiable and strictly convex. Then*

$$y_{i_0} \geqslant y_{i_0+1} \geqslant \cdots \geqslant y_M \geqslant c_{z_0},$$

where $c_{z_0} = (h')^{-1}(-\alpha c'(z_0))$.

PROOF. Let $h_m = h'(y_m) + \alpha c'(z_0)$ and $\beta_{m,l} = v_y(y_m, l) + c'(z_0)$, $\forall m, l \in \mathscr{M}$. By the definition of y_m, we have $\beta_{m,l} = v_y(y_m, l) - v_y(y_l, l)$. Therefore, the strict convexity of $h(y)$ and $v(y,l)$ (by Lemma 3.1) implies the following:

$$y_m < y_l \Leftrightarrow h_m < h_l \Leftrightarrow \beta_{m,l} < 0, \quad \forall m, l \in \mathscr{M}.$$

Rewrite K_m defined in (4.4) in terms of h_m and $\beta_{m,l}$ as follows:

$$K_m = \begin{cases} h_0 + \mu_0 \beta_{0,1} & \text{if } m = 0, \\ h_m + \mu_m \beta_{m,m+1} + \lambda_m \beta_{m,m-1} & \text{if } 0 < m < M, \\ h_M + \lambda_M \beta_{M,M-1} & \text{if } m = M. \end{cases}$$

Suppose $y_M > y_{M-1}$. Then $h_M > h_{M-1}$, and $\beta_{M,M-1} > 0$. $K_M = 0$ implies that $h_M \leqslant 0$. Thus $h_{M-1} < h_M \leqslant 0$. Furthermore, $h_{M-1} < 0$, $\beta_{M-1,M} < 0$ and $K_{M-1} \geqslant 0$ yield $\lambda_{M-1}\beta_{M-1,M-2} > 0$. Thus $h_{M-2} < h_{M-1}$. We can carry on this procedure back to $m = 1$. Then we obtain

$$h_0 < h_1 < \cdots < h_M < 0.$$

But $h_0 < h_1$ implies $\beta_{0,1} < 0$. Hence $K_0 \leqslant h_0 < 0$. This contradicts with $K_0 \geqslant 0$. Thus, we can conclude $y_M \leqslant y_{M-1}$.

Recall the fact that $K_m = 0$ for $m \geqslant i_0 + 1$. We can derive the following induction chain:

$$\begin{aligned} & y_M \leqslant y_{M-1} \\ \Rightarrow\ & h_M \leqslant h_{M-1}, \quad \beta_{M-1,M} \geqslant 0, \quad \beta_{M,M-1} \leqslant 0 \\ \Rightarrow\ & h_{M-1} \geqslant 0 \quad (\text{by } K_M = 0) \\ \Rightarrow\ & \beta_{M-1,M-2} \leqslant 0 \quad (by\ K_{M-1} = 0) \\ \Rightarrow\ & y_{M-1} \leqslant y_{M-2} \\ \Rightarrow\ & 0 \leqslant h_M \leqslant h_{M-1} \leqslant \cdots \leqslant h_{i_0}. \end{aligned} \tag{4.5}$$

Note also that $h_M = h'(y_M) + \alpha c'(z_0) = h'(y_M) - h'(c_{z_0}) \geqslant 0$. This completes the proof. □

REMARK. Observe that if both capacity and demand were to be stochastic processes, then there would be an extra term $L_z v_y$ involved in (4.4) and the argument in (4.5) would not go through. Our proof is based directly on the dynamic programming equation. A study of possible monotonicity property for systems with stochastic capacity and demand processes would require a further study of the properties of the value function. At present, we do not know how to do this.

Now let us consider our problem when some machines are reliable. Let us assume that m_0 of the M machines are reliable. Then the set of capacity levels $\mathscr{M} = \{m_0, m_0 + 1, \ldots, M\}$, and the jumping rates in (2.1) satisfy

$$0 = \lambda_1 = \lambda_2 = \cdots = \lambda_{m_0}, \qquad 0 = \mu_0 = \mu_1 = \cdots = \mu_{m_0 - 1}. \tag{4.6}$$

THEOREM 4.2. *Assume* (4.6) *and assume $h(y)$ to be a differentiable and strictly convex function. We then have the following*:

(i) *if $z_0 > m_0$, then $y_{i_0} \geqslant y_{i_0+1} \geqslant \cdots \geqslant y_M \geqslant c_{z_0}$,*

(ii) *if $z_0 < m_0$, then $y_{m_0} = y_{m_0+1} = \cdots = y_M = c_{z_0}$.*

PROOF. It is similar to the proof of Theorem 4.1, except that $K_{m_0} = 0$ implies the equalities in (ii). □

Before we finish this section, we solve a particular optimal control problem with the aid of the above theorem.

EXAMPLE 4.1. Let the production cost $c(p) = cp$ and the inventory cost $h(y) = y^2/2$. We assume that demand $z_0 < m_0$. Then the turnpike sets are

$$y_{m_0} = y_{m_0+1} = \cdots = y_M = -\alpha c.$$

The optimal production policy is given by the following:

$$p^*(y,m) = \begin{cases} 0 & \text{if } y > -\alpha c, \\ z_0 & \text{if } y = -\alpha c, \\ m & \text{if } y < -\alpha c. \end{cases}$$

For another application of Theorem 4.2, see Jiang and Sethi [11].

5. Turnpike sets with reliable machines. In this section, we consider turnpike sets with M reliable machines and randomly fluctuating demand. We shall study the optimal inventory levels under capacity M and stochastic demand $z(t) \in \mathscr{Z}$ with $z_1 < z_2 < \cdots < z_d < M$. As before in the last section, the dependence of the value function on the capacity level m will be suppressed, i.e., $v(y, z)$ will be used in place of $v(y, m, z)$.

The turnpike set $\mathscr{G}(m, z)$ becomes

$$\mathscr{G}(z) = \{y: v_y(y, z) = -c'(z)\}.$$

Then $\mathscr{G}(z)$ is a bounded closed interval in view of Lemma 3.1. Note that under the strict convexity of v, the turnpike sets $\mathscr{G}(z)$ become singletons, say $\mathscr{G}(z) = \{y_z\}$.

Now let us define a set $\mathscr{G}$ to be the convex hull of $\mathscr{G}(z)$:

$$\mathscr{G} = \operatorname{co}\left[\bigcup_{z \in \mathscr{Z}} \mathscr{G}(z)\right]. \tag{5.1}$$

Since the convex closure of finite bounded closed intervals is a bounded closed interval, there exist $y_1^{\mathscr{G}}, y_2^{\mathscr{G}}$, such that $\mathscr{G} = [y_1^{\mathscr{G}}, y_2^{\mathscr{G}}]$.

Observe that the optimal policy $p^*(y, z)$ is monotone in y and equals z for $y \in \mathscr{G}(z)$, which yields

(i) $p^*(y, z) - z < 0$, if $y > y_2^{\mathscr{G}}$;

(ii) $p^*(y, z) - z > 0$, if $y < y_1^{\mathscr{G}}$.

Hence for the stochastic demand process, $\mathscr{G}$ is an attractor set for the optimal inventory trajectories. More precisely, we have the following lemma.

LEMMA 5.1. *$\operatorname{dist}(y^*(t), \mathscr{G}) := \inf_{w \in \mathscr{G}} |y^*(t) - w|$ decreases to zero monotonically as $t \to \infty$. Moreover, if $c(p)$ is linear, then for $\tau = \max\{|y^*(0) - y_2^{\mathscr{G}}|/z_1, |y^*(0) - y_1^{\mathscr{G}}|/(M - z_d)\}$, $\operatorname{dist}(y^*(\tau), \mathscr{G}) = 0$.*

PROOF. To prove the first assertion, it suffices to show that $\operatorname{dist}(y^*(t), \mathscr{G}) \to 0$ as $t \to \infty$. Suppose now that the production cost $c(p)$ is strictly convex. If $y^*(0) < y_1^{\mathscr{G}}$, then $y^*(t)$ increases over time t before it enters $\mathscr{G}$. Let $\bar{y} = \limsup_{t \to \infty} y^*(t)$. It remains to show that $\bar{y} \geq y_1^{\mathscr{G}}$. If $\bar{y} < y_1^{\mathscr{G}}$, then Lemma A.1 in the Appendix applies to $y_0 = (\bar{y} + y_1^{\mathscr{G}})/2 < y_1^{\mathscr{G}}$. Thus, there exists a $\delta > 0$, such that $p^*(y, z) - z \geq \delta$, $\forall y \leq y_0$, $\forall z \in \mathscr{Z}$. So $\dot{y}^*(t) \geq \delta$, $\forall t \geq 0$. This in turn implies that $y^*(t)$ reaches y_0

before time $t = (y_0 - y^*(0))/\delta$. This contradicts with the definition of $\bar{y}$. Hence, $\text{dist}(y^*(t), \mathcal{S}) \to 0$. Similarly, it can be shown that when $y^*(0) > y_2^{\mathcal{S}}$, $\text{dist}(y^*(t), \mathcal{S}) \to 0$ as $t \to \infty$ as well.

If the production cost $c(p)$ is linear, then for $y^*(0) > y_2^{\mathcal{S}}$, $p^*(y, z) = 0$, $\forall z \in \mathcal{Q}$. This implies $\dot{y}^*(t) = -z(t) \leqslant -z_1 < 0$. So, $y^*(t)$ reaches $\mathcal{S}$ prior to $t = (y^*(0) - y_2^{\mathcal{S}})/z_1$. On the other hand, for $y^*(0) < y_1^{\mathcal{S}}, p^*(y, z) = M$ if $y < y_1^{\mathcal{S}}$. So $\dot{y}^*(t) \geqslant M - z_d$. Therefore, $y^*(t)$ enters $\mathcal{S}$ before $t = (y_1^{\mathcal{S}} - y^*(0))/(M - z_d)$. The proof is now completed. □

An analog of Lemma 4.1 for the stochastic demand case is useful to provide.

LEMMA 5.2. *Assume that $h(y)$ is differentiable and strictly convex. Let*

$$K_z = h'(y_z) + L_z v_y(y_z, \cdot)(z) - \alpha v_y(y_z, z), \qquad z \in \mathcal{Q}.$$

Then $K_z = 0$, $\forall z \in \mathcal{Q}$.

PROOF. The proof uses similar arguments to those in the proof of Lemma 4.1. □

REMARK. As an immediate result of this lemma, the turnpike set

$$\mathcal{S}(z) = \{y_z: h'(y_z) = -\alpha c'(z)\} = \{(h')^{-1}(-\alpha c'(z))\},$$

provided that the demand state z is absorbing.

REMARK. For the case of a constant deterministic demand, i.e., $z(t) = z_0$, for all $t \geqslant 0$, it is immediate that the turnpike set $\mathcal{S}(z_0)$ is given by the following:

$$\mathcal{S}(z_0) = \{y \in R: -\alpha c'(z_0) = h'(y)\}.$$

An economic interpretation of this is useful to provide. Let $y = y_0$ be a turnpike point. Then $p^*(t) = z_0$. Let $p(t) = z_0 + \epsilon$, $t \in [0, \delta]$ with $\epsilon, \delta > 0$ and $p(t) = p^*(t) = z_0$, $t \in (\delta, \infty)$. Then, the marginal production cost is $c'(z_0)\epsilon\delta + o(\epsilon\delta)$ and the marginal inventory cost is

$$\int_\delta^\infty e^{-\alpha t} h'(y_0)\epsilon\delta\, dt + o(\epsilon\delta) = \alpha^{-1} h'(y_0)\epsilon\delta + o(\epsilon\delta).$$

Setting the total marginal production cost to zero and dividing through by $\epsilon\delta$ gives the relation $-\alpha c'(z_0) = h'(y_0)$ for y_0. Furthermore, note that $z_0 > 0$ implies $y_0 < 0$. Thus, if the initial inventory $y(0) = 0$, then it pays to produce less than the demand until $y(t) = y_0$. This results in savings in production cost. This is exactly offset by the increased shortage cost along the optimal path. Note that the discounting plays an essential role in this balancing act. In fact, in the absence of discounting, i.e., $\alpha = 0$, the turnpike point $y_0 = 0$.

THEOREM 5.1. (i) *Let $\mathcal{S}$ be defined in* (5.1), *$\mathcal{S} = [y_1^{\mathcal{S}}, y_2^{\mathcal{S}}]$. Then $y_1^{\mathcal{S}} \leqslant 0$.*

(ii) *Let $\Psi(z) = L_z(c'(\cdot))(z) - \alpha c'(z)$, $\forall z \in \mathcal{Q}$. Assume that either there exists z, such that $\Psi(z) > 0$ or there exist z_1, z_2, such that $\Psi(z_1) \neq \Psi(z_2)$. Furthermore, assume that $h(y)$ is differentiable. Then $\mathcal{S}$ is a nondegenerate interval, i.e., $y_1^{\mathcal{S}} < y_2^{\mathcal{S}}$.*

PROOF. It is obvious that there exists $\bar{z} \in \mathcal{Q}$, such that $y_1^{\mathcal{S}} \in \mathcal{S}(\bar{z})$. Now suppose that $y_1^{\mathcal{S}} > 0$. Let $y^*(t), p^*(t)$ be, respectively, the optimal inventory level and the optimal production rate with $y^*(0) = 0$, $z(0) = \bar{z}$.

Construct $p(\cdot)$ as follows:

$$p(t) = z(t)\chi_{[0,\tau)}(t) + p^*(t)\chi_{[\tau,\infty)}(t),$$

where χ_A is the indicator function of any given set A and τ is the stopping time defined by $\tau = \inf\{t \geqslant 0: y^*(t) \geqslant y_1^{\mathscr{G}}/2\}$. Then by Lemma 5.1, τ is finite a.s. and

$$y^*(t) \geqslant y^*(\tau) = y_1^{\mathscr{G}}/2 > 0 \quad \forall t \geqslant \tau,$$

and $p(t) \leqslant p^*(t)$, $\forall t \geqslant 0$. Therefore,

$$\begin{aligned} y^*(t) - y^*(\tau) &= \int_\tau^t (p^*(s) - z(s))\, ds \\ &= \int_\tau^t (p(s) - z(s))\, ds \geqslant 0 \quad \forall t \geqslant \tau. \end{aligned} \tag{5.2}$$

Let $y(t)$ be the inventory trajectory corresponding to $p(\cdot)$ with $y(0) = 0$, $z(0) = \bar{z}$. Then

$$y(t) = \begin{cases} 0 & \text{if } t \in [0,\tau), \\ \int_\tau^t (p(s) - z(s))\, ds & \text{if } t \in [\tau,\infty), \end{cases}$$

and (5.2) implies $y(t) \geqslant 0$, $\forall t \geqslant 0$.

Observe that $p(t) < p^*(t)$ for $t < \tau$. Therefore, $y(t) \leqslant y^*(t)$, $\forall t \geqslant 0$. Thus, we obtain

$$J(0, \bar{z}, p(\cdot)) < J(0, \bar{z}, p^*(\cdot)) = v(0, \bar{z}).$$

This contradicts with the optimality of p^*, which completes the proof of (i).

To see (ii), let us assume to the contrary that $y_1^{\mathscr{G}} = y_2^{\mathscr{G}} = y_0$ is the only element of $\mathscr{G}$. Then for all $z \in \mathscr{Z}$, $v_y(y, z) = -c'(z)$. By Lemma 5.2, one obtains $\Psi(z) = h'(y_0)$, $\forall z \in \mathscr{Z}$. Recalling (i) and assumption (A1), $\Psi(z)$ is a nonpositive constant on $\mathscr{Z}$. This contradicts with the assumption on Ψ, thus completing the proof. □

By Lemma 5.1, the optimal inventory $y^*(t)$ goes to $\mathscr{G}$ monotonically over time. The next theorem tells us that under certain conditions, $y^*(t)$ enters $\mathscr{G}$ in a finite time.

THEOREM 5.2. *Let*

$$A_{z_0} = \{\omega \in \Omega: \textit{Lebesgue measure of } \{t: z(t) = z_0\} = \infty\}.$$

Suppose that there exists $z_0 \in \mathscr{Z}$, such that $v_y(y_1^{\mathscr{G}}, z_0) < -c'(z_0)$. Then on A_{z_0}, $y^*(t)$ enters $\mathscr{G}$ in a finite time. In particular, if $z(t)$ is ergodic, then $y^*(t)$ enters $\mathscr{G}$ in a finite time a.s.

PROOF. In view of the proof of Lemma 5.1, it suffices to show that for each $y^*(0) < y_1^{\mathscr{G}}$, $y^*(t)$ enters $\mathscr{G}$ in a finite time.

Let $y_0 = \min \mathscr{G}(z_0)$. Then $y_0 > y_1^{\mathscr{G}}$ by our assumption. Take $y_1 = (y_0 + y_1^{\mathscr{G}})/2 > y_1^{\mathscr{G}}$. Then Lemma A.1 implies that there exists a positive constant δ, such that

$$p^*(y, z_0) - z_0 > \delta \quad \forall y \leqslant y_1.$$

Therefore, $y^*(t)$ increases to y_1 with at least the rate δ. Consequently, $y^*(t)$ reaches $y_1^{\mathscr{G}}$ in a finite time on A_{z_0}. □

Combining Theorem 5.1(ii) and Theorem 5.2, we have the following.

COROLLARY 5.1. *Suppose the conditions of Theorem* 5.1(ii) *hold. Assume that* $z(t)$ *is ergodic. Then* $y^*(t)$ *enters* $\mathscr{G}$ *in a finite time a.s.*

Let $z(t)$ be given by the following infinitesimal generator L_z:

$$L_z\phi(z_i) = \begin{cases} \kappa_1(\phi(z_2) - \phi(z_1)) & \text{if } i = 1, \\ \kappa_i(\phi(z_{i+1}) - \phi(z_i)) + \nu_i(\phi(z_{i-1}) - \phi(z_i)) & \text{if } 1 < i < d, \\ \nu_d(\phi(z_{d-1}) - \phi(z_d)) & \text{if } i = d. \end{cases} \tag{5.3}$$

THEOREM 5.3. *Assume that the condition of Theorem* 4.1 *and* (5.3) *hold. Then*
(i) *if* $c(p) = cp$, *then* $y_{z_1} = y_{z_2} = \cdots = y_{z_d} = -\alpha c$;
(ii) *if* $d = 2$, *then* $y_{z_1} \geqslant y_{z_2}$.

PROOF. The proof of (i) is the same as that of Theorem 4.1 on account of Lemma 5.2. Now let us show (ii). By Lemma 5.2,

$$\begin{aligned} h'(y_{z_1}) &= -(\alpha + \kappa_1)c'(z_1) - \kappa_1 v_y(y_{z_1}, z_2), \\ h'(y_{z_2}) &= -(\alpha + \nu_2)c'(z_2) - \nu_2 v_y(y_{z_2}, z_1). \end{aligned} \tag{5.4}$$

If $y_{z_1} < y_{z_2}$, then

$$\begin{cases} h'(y_{z_1}) < h'(y_{z_2}), \\ v_y(y_{z_1}, z_2) \leqslant -c'(z_2), \\ v_y(y_{z_2}, z_1) \geqslant -c'(z_1). \end{cases} \tag{5.5}$$

Combine (5.4) and (5.5) to conclude $(\alpha + \kappa_1 + \nu_2)c'(z_1) > (\alpha + \kappa_1 + \nu_2)c'(z_2)$. But this contradicts with the convexity of $c(p)$, thereby proving (ii). □

The following example shows that the monotonicity of the turnpike sets does not hold in general.

EXAMPLE 5.1. Take $c(p) = cp$, $h(y) = y^2/2$, $\mathscr{Q} = \{1, 2, 3\}$, $\alpha = 1$, and

$$L_z\phi(z) = \begin{cases} 0 & \text{if } z = 1, 2, \\ \gamma(\phi(1) - \phi(3)) & \text{if } z = 3. \end{cases}$$

Let $v^\gamma(y, z)$ be the corresponding value function. We shall show that for certain values of γ, the monotonicity of the turnpike sets breaks down.

First observe that $v^\gamma(y, 1)$ and $v^\gamma(y, 2)$ are independent of γ, so they will be denoted by $v(y, 1)$ and $v(y, 2)$, respectively. Also we have the following:

$$\lim_{\gamma \to \infty} v^\gamma(y, 3) = v(y, 1).$$

Note that Lemma 3.1 implies that v^γ is strictly convex. Consequently, the turnpike sets $\mathscr{G}(z)$ are singletons. Let $y(1)$, $y(2)$, and $y^\gamma(3)$ denote the points of the corresponding turnpike sets $\mathscr{G}(1)$, $\mathscr{G}(2)$, and $\mathscr{G}^\gamma(3)$. We also write $\mathscr{G}^0(z) = \{y^0(z)\}$ and $\mathscr{G}^\infty(z) = \{y^\infty(z)\}$ to be the turnpike sets with $\gamma = 0$ and $\gamma = \infty$, respectively, for each $z \in \mathcal{G}$. Then the uniform convergence of v^{γ} implies

$$y^{\infty}(3) = \lim_{\gamma\to\infty} y^{\gamma}(3) = y(1), \qquad y^{0}(3) = \lim_{\gamma\to 0} y^{\gamma}(3).$$

As a result of the remark to Lemma 5.2,

$$\mathscr{G}(1) = \{-1\}, \quad \mathscr{G}(2) = \{-2\}, \quad \mathscr{G}^{0}(3) = \{-3\}, \quad \mathscr{G}^{\infty}(3) = \{-1\}.$$

Since $y^{\gamma}(3)$ is continuous in γ, there exists a constant γ such that $y^{\gamma}(3) = -3/2$, i.e., $\mathscr{G}^{\gamma}(3) = \{-3/2\}$. Hence for this γ, the monotonicity of $\mathscr{G}(z)$ breaks down.

Let us give another example, for which the optimal control problem could be solved with the aid of the monotonicity of turnpike sets.

EXAMPLE 5.2. Let $z(t)$ be the demand process governed by L_z given in (5.3). Take $c(p) = cp$, $h(y) = y^2/2$, $\alpha = 1$. Then the optimal control policy is given by the following:

$$p^*(y,z) = \begin{cases} 0 & \text{if } y > -\alpha c, \\ z & \text{if } y = -\alpha c, \\ M & \text{if } y < -\alpha c. \end{cases}$$

Finally, we give an example, which shows that it is possible to have a strictly convex value function, even if the inventory cost is not strictly convex.

EXAMPLE 5.3. We give an example with a constant demand z_0, a quadratic production cost $c(p) = p^2$, and a nonstrictly convex piecewise differentiable inventory cost

$$h(y) = \begin{cases} -Ky & \text{if } y \leqslant 0, \\ 0 & \text{if } y > 0. \end{cases}$$

It is easy to show that the turnpike set

$$\mathscr{G}(z_0) = \{y \in R \colon -\alpha c'(z_0) \in D^- h(y)\},$$

where D^- is defined in §3.

Case (a). Low shortage cost: $K \leqslant 2\alpha z_0$.

For $K \leqslant 2\alpha z_0$, the value function

$$v(y, z_0) = \begin{cases} -\alpha^{-1}Ky + \alpha^{-2}K\left[z_0 - (4\alpha)^{-1}K\right] & \text{if } y \leqslant 0, \\ (2\alpha^2)^{-1}K\left[2e^{-\alpha T(y)}z_0 - (2\alpha)^{-1}Ke^{-2\alpha T(y)}\right] & \text{if } y > 0, \end{cases}$$

where $T(y)$ is the first time at which the optimal inventory $y^*(T(y)) = 0$, given the initial level $y \geqslant 0$. Moreover, $T(y)$ is the unique positive solution of

$$(2\alpha^2)^{-1}Ke^{-\alpha T(y)} + z_0 T(y) = y + (2\alpha^2)^{-1}, \qquad y > 0.$$

Furthermore,

$$\mathscr{G}(z_0) = \begin{cases} \varnothing & \text{for } K < 2\alpha z_0, \\ (-\infty, 0] & \text{for } K \geqslant 2\alpha z_0. \end{cases}$$

We note that for $K < 2\alpha z_0$, $v_y \geqslant -\alpha^{-1}K > -2z_0$ with the consequence that there exists no turnpike set, i.e., $\mathscr{S} = \varnothing$. Although, we can think of $\{-\infty\}$ to be the turnpike point in the extended sense, as the optimal inventory level in this case approaches $-\infty$ as $t \to \infty$.

The case $K = 2\alpha z_0$ is the critical case. In this case,

$$v(y, z_0) = \begin{cases} -2z_0 y + \alpha^{-1} z_0^2 & \text{if } y \leqslant 0, \\ \alpha^{-1} z_0^2 [2e^{-\alpha T(y)} - e^{-2\alpha T(y)}] & \text{if } y > 0, \end{cases}$$

where $T(y)$ is the unique positive solution of

$$\alpha^{-1} e^{-\alpha T(y)} + T(y) = y/z_0 + \alpha^{-1}, \qquad y > 0.$$

Since $v_y = -2z_0$ for $y \in (-\infty, 0]$, the turnpike set $\mathscr{S}(z_0) = (-\infty, 0]$. For $y > 0$, the optimal inventory level reaches 0 at $t = T(y)$ and then it stays there. For $y \leqslant 0$, the optimal production $p^*(t) = z_0$ and $y^*(t) = y$ for all $t \in [0, \infty)$.

Case (b). High shortage cost: $K > 2\alpha z_0$.

For $K > 2\alpha z_0$, the value function

$$v(y, z_0) = \begin{cases} -\alpha^{-1} K y - (4\alpha^3)^{-1} K^2 + \alpha^{-2} K z_0 & \\ \quad + \left(\alpha^{-1} z_0^2 + (4\alpha^3)^{-1} K^2 - \alpha^{-2} K z_0\right)[2e^{-\alpha\theta(y)} - e^{-2\alpha\theta(y)}] & \text{if } y \leqslant 0, \\ \alpha^{-1}[2e^{-\alpha T(y)} - e^{-2\alpha T(y)}] & \text{if } y > 0, \end{cases}$$

where $T(y)$ is as defined in the critical case above and $\theta(y)$ is the first time the optimal inventory level $y^*(\theta(y)) = 0$ from a given initial inventory level $y \leqslant 0$. Moreover, $\theta(y)$ is given by the unique positive solution of

$$\alpha^{-1} e^{-\alpha\theta(y)} + \theta(y) = -y/\left((2\alpha)^{-1} K - z_0\right) + \alpha^{-1}, \qquad y \leqslant 0.$$

In this case, the turnpike set $\mathscr{S}(z_0) = \{0\}$.

For $K > 2\alpha z_0$, the value function is strictly convex and passes through the point $(0, \alpha^{-1} z_0^2)$, which in turn tells us that the strict convexity of the cost function $h(y)$ is not necessary for the strict convexity of the value function.

Acknowledgements. This work was partly supported by the NSERC Grant A4619, Manufacturing Research Corporation of Ontario, and Canadian International Development Agency. Helpful suggestions from Wendell Fleming are greatly appreciated.

Appendix. We provide a technical lemma which is used in §5.

LEMMA A.1. *Let* $y_z = \min \mathscr{S}(z)$. *Then for each* $y_0 < y_z$, *there exists* $\delta > 0$, *such that*

$$p^*(y, z) - z > \delta \quad \forall y < y_0.$$

Moreover, if $y_0 < \min \mathscr{S}$, *then there exist* $\delta > 0$, *such that*

$$p^*(y, z) - z > \delta \quad \forall z \in \mathscr{D}, \quad \forall y \leqslant y_0.$$

PROOF. Suppose that $c(p)$ is strictly convex. Let $y_0 < y_z$. Then there exists $\delta'_z > 0$, such that

$$v_y(y, z) < -c'(z) - \delta'_z \quad \forall y \leqslant y_0.$$

So there exists $\delta_z^0 > 0$, such that

$$(c')^{-1}\big(-v_y(y, z)\big) > z + \delta_z^0 \quad \forall y \leqslant y_0.$$

These imply that

$$p^*(y, z) = \begin{cases} M & \text{if } v_y(y, z) \leqslant -c'(M), \\ (c')^{-1}\big(-v_y(y, z)\big) & \text{if } v_y(y, z) > -c'(M). \end{cases}$$

Therefore,

$$\begin{aligned} p^*(y, z) - z &\geqslant \min\Big\{M - z, (c')^{-1}\big(-v_y(y, z)\big) - z\Big\} \\ &\geqslant \min\big\{M - z, \delta_z^0\big\} \\ &:= \delta_z > 0. \end{aligned}$$

If $y_0 < \min \mathscr{S}$, take $\delta = \min\{\delta_z\colon z \in \mathscr{D}\} > 0$, then $p^*(y, z) - z > \delta$.
If the production cost $c(p)$ is linear, then for all $y \leqslant y_z$,

$$p^*(y, z) - z = M - z := \delta > 0. \quad \square$$

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