Viability and Arbitrage under Knightian Uncertainty *

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Abstract

We reconsider the microeconomic foundations of financial economics under Knightian Uncertainty. We remove the (implicit) assumption of a common prior and base our analysis on a common order instead. Economic viability of asset prices and the absence of arbitrage are equivalent. We show how the different versions of the Efficient Market Hypothesis are related to the assumptions one is willing to impose on the common order. We also obtain a version of the Fundamental Theorem of Asset Pricing using the notion of *sublinear* pricing measures. Our approach unifies recent versions of the Fundamental Theorem under a common framework.

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1 Introduction

Recently, a large and increasing body of literature has focused on decisions, markets, and economic interactions under uncertainty. Frank Knight's pioneering

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work (Knight [1921]) distinguishes *risk* – a situation that allows for an objective probabilistic description – from *uncertainty* – a situation that cannot be modelled by a single probability distribution.

In this paper, we discuss the foundations of no–arbitrage pricing and its relation to economic equilibrium under Knightian Uncertainty.

Asset pricing models typically take a basic set of securities as given and determine the range of option prices that is consistent with the absence of arbitrage. From an economic point of view, it is crucial to know if modeling security prices directly is justified; an asset pricing model is called viable if its security prices can be thought of as (endogenous) equilibrium outcomes of a competitive economy.

Under risk, this question has been investigated in Harrison and Kreps' seminal work (Harrison and Kreps [1979]). Their approach is based on a common prior (or reference probability) that determines the null sets, the topology, and the order of the model. The common prior assumption[1] is made in almost all asset pricing models. In recent years, Knightian uncertainty has emerged as a major topic in financial economics. It is widely acknowledged that drift and volatility of asset prices, the term structure of interest rates, and credit risk are important instances in which the probability distribution of the relevant parameters is imprecisely known, if not completely unknown (compare, e.g., Epstein and Ji [2013]).

We replace the common prior with a *common order* with respect to which agents' preferences are monotone. The minimal example we can think of is when agents will prefer a contingent consumption plan over their endowment if the new plan pays off more in every state of the world. This (rather incomplete) pointwise order does not require any probabilistic assumption. Knightian uncertainty is frequently modeled by a set of priors; one might then derive the common order from the set of priors as we discuss below.

Our main result shows that the absence of arbitrage and the (properly defined) economic viability of the model are equivalent. In equilibrium, there are no arbitrage opportunities; conversely, for arbitrage–free asset pricing models, it is possible to construct a heterogeneous agent economy such that the asset prices are equilibrium prices of that economy.

The main result is based on a number of other results that are of independent interest. To start with, in contrast to risk, it is no longer possible to characterize viability through the existence of a single linear pricing measure (or equivalent martingale measure). Instead, it is necessary to use a suitable *nonlinear* pricing expectation, that we call a sublinear martingale expectation. A sublinear expectation has the common properties of an expectation including monotonicity, preservation of constants, and positive homogeneity, yet it is no longer additive. Indeed, sublinear expectations can be represented as the supremum of a class of (linear) expectations, an operation that does not preserve linearity[2]. Nonlinear

[1] Here and in the following, we speak of a *common prior model*, or alternatively a situation of *risk*, whenever there is a reference probability space that serves as a state space for the economy.

[2] In economics, such a representation theorem appears first in Gilboa and Schmeidler [1989].

expectations arise in decision–theoretic models of ambiguity–averse preferences (Gilboa and Schmeidler [1989], Maccheroni et al. [2006]). It is interesting to see that a similar nonlinearity arises here for the *pricing* functional. A general theory of equilibrium with such sublinear prices is developed in Beißner and Riedel [2016][3].

The common order shapes equilibrium asset prices. We study various common orders and how they are related to versions of the Efficient Market Hypothesis (Fama [1970]) in Section 4. The original (strong) version of the Efficient Market Hypothesis posits a common prior and states that properly discounted expected returns of assets are equal to the return of a safe bond. We obtain this conclusion when the common order is based on expected payoffs with respect to the reference measure. When the common order is given by the almost sure order under a common prior, one obtains the weak version of the EMH: under an equivalent pricing measure, expected returns are equal[4]. In situations of Knightian uncertainty, different specifications of the common order can be made. An example is the quasi-sure order induced by a set of priors: a claim dominates quasi– surely another claim if it is almost surely greater or equal under all considered probability measures. Another example is the order induced by smooth ambiguity preferences, as introduced by Klibanoff et al. [2005], where Knightian uncertainty is modeled by a second-order prior over a class of multiple priors. We show that weaker versions of the Efficient Market Hypothesis prevail, depending on the strength of the assumptions we are willing to impose on the common order, and how the related fundamental theorem of asset pricing needs to be suitably adapted.

Further Related Literature

The relation of arbitrage and viability has been discussed in various contexts. Jouini and Kallal [1995b] and Jouini and Kallal [1999] discuss models with transaction cost and other frictions. Werner [1987] and Dana et al. [1999] study the absence of arbitrage in its relation to equilibrium when a finite set of agents is fixed a priori whereas Cassese [2017] characterizes the absence of arbitrage in an order-theoretic framework derived from coherent risk measures. Knightian uncertainty is also closely related to robustness concerns that play an important role in macroeconomic models that deal with the fear of model misspecification (Hansen and Sargent [2001, 2008]). The pointwise order corresponds to the "model-independent" (or rather "probability–free") approach in finance that

Sublinear expectations also arise in Robust Statistics, compare Huber [1981], and they play a fundamental role in theory of risk measures in Finance, see Artzner et al. [1999] and Föllmer and Schied [2011].

[3]Given that we have a nonlinear price system, one might wonder whether agents can generate arbitrage gains by splitting a consumption bundle into two or more plans. The convexity of our price functional excludes such arbitrage opportunities, see Proposition 1 in Beissner and Riedel [2019].

[4]This statement is equivalent to the classic version of the Fundamental Theorem of Asset Pricing Harrison and Kreps [1979], Harrison and Pliska [1981], Duffie and Huang [1985], Dalang et al. [1990], Delbaen and Schachermayer [1998].

has been discussed, e.g., in Riedel [2015], Acciaio et al. [2016], Burzoni et al. [2019] and Bartl et al. [2017]. This literature uses different notions of "relevant payoffs" that our approach allows to unify under a common framework, see Subsection 4.3.4.

The paper is set up as follows. Section 2 describes the model and the two main contributions of this paper in concise form. The assumptions of our model and their relation to previous modeling are discussed in Section 3. Section 4 derives various classic and new forms of the Efficient Market Hypothesis. Section 5 is devoted to the proofs of the main theorems. The appendix contains a detailed study of general discrete time markets when the space of contingent payoffs consists of bounded measurable functions. It also discusses further extensions as, e.g., the equivalence of absence of arbitrage and absence of free lunches with vanishing risk, or the question if an optimal superhedge for a given claim exists.

2 The model and the two main theorems

A non-empty set Ω contains the states of the world; the σ–field $\mathcal{F}$ on Ω collects the possible events.

The commodity space (of contingent claims) $\mathcal{H}$ is a vector space of $\mathcal{F}$-measurable real-valued functions containing all constant functions. We will use the symbol c both for real numbers as well as for constant functions. $\mathcal{H}$ is endowed with a metrizable topology τ and a pre-order $\leq$ that are compatible with the vector space operations.

The abstract vector space model allows to cover the typical models that have been used in financial economics. Under risk, it is common to take a space of suitably integrable functions with respect to a given prior with the usual almost sure order; under Knightian uncertainty, a more general approach is required, in particular, when the common order is induced by a non dominated set of multiple priors (confer the examples in Section 3).

The preorder $\leq$ plays a crucial role in our analysis. A major conclusion of our study is that the strength of the assumptions we are willing to make on the common order (and therefore on the agents populating the economy) shapes the results about market returns as we shall see in detail in Section 4. We assume throughout that the preorder $\leq$ is consistent with the order on the reals for constant functions and with the pointwise order for measurable functions. A consumption plan $Z \in \mathcal{H}$ is negligible if we have $0 \leq Z$ and $Z \leq 0$. $C \in \mathcal{H}$ is nonnegative if $0 \leq C$ and positive if in addition not $C \leq 0$. We denote by $\mathcal{Z}$, $\mathcal{P}$ and $\mathcal{P}^+$ the class of negligible, nonnegative and positive contingent claims, respectively.

We introduce a class of *relevant contingent claims* $\mathcal{R}$, a convex subset of $\mathcal{P}^+$. The relevant claims are used below in two important ways. On the one hand, they signal arbitrage: if a net trade allows to obtain a payoff that dominates a relevant payoff with respect to the common order, we speak of an arbitrage. On the other hand, relevant payoffs identify potentially desirable directions of consumption for our economy. In the spirit of Arrow [1953] and most of the literature, a common choice of the relevant claims is the set of positive, nonzero claims $\mathcal{P}^+$; we invite the reader to make this identification at first reading. However, it might be of interest to consider smaller relevant sets in some economic contexts. The introduction of $\mathcal{R}$ also allows to subsume various notions of arbitrage that were discussed in the literature, compare the discussion in Section 3 and the examples in Section 4.3.4.

The financial market is modeled by the set of *net trades* $\mathcal{I} \subset \mathcal{H}$, a convex cone containing 0. $\mathcal{I}$ is the set of payoffs that the agents can achieve from zero initial wealth by trading in the financial market. In the basic frictionless model of securities, $\mathcal{I}$ contains the payoffs of self-financing strategies with zero initial capital. In a frictionless market, $\mathcal{I}$ is a subspace. In more realistic situations, when short selling constraints, credit line limitations, or transaction costs are imposed, e.g., we are led to a convex cone instead of a subspace, see Example 3.1.

An *agent* in this economy is described by a preference relation (i.e. a complete and transitive binary relation) on $\mathcal{H}$ that is

- *weakly monotone with respect to* $\leq$, i.e. $X \leq Y$ implies $X \preceq Y$ for every $X, Y \in \mathcal{H}$;
- *convex*, i.e. the upper contour sets $\{Z \in \mathcal{H} : Z \succeq X\}$ are convex;
- τ*-lower semi–continuous*, i.e. for every sequence $\{X_n\}_{n=1}^{\infty} \subset \mathcal{H}$ converging to X in τ with $X_n \preceq Y$ for $n \in \mathbb{N}$, we have $X \preceq Y$.

The set of all agents is denoted by $\mathcal{A}$.

In the spirit of Harrison and Kreps [1979], we think of a potentially large set of agents about whom some things are known, without assuming that we know exactly their preferences or their number. We only impose a list of properties on preferences that are standard in economics. In particular, bearing in mind the interpretation of $\leq$ as a *common order*, the preferences are monotone with respect to $\leq$. Moreover, we impose some weak form of continuity with respect to the given topology τ; it is known that, in general, some form of continuity is required for the existence of equilibrium. Convexity reflects a preference for diversification.

A financial market $(\mathcal{H}, \tau, \leq, \mathcal{I}, \mathcal{R})$ is *viable* if there is a family of agents $\{\preceq_a\}_{a \in A} \subset \mathcal{A}$ such that

- 0 is optimal for each agent $a \in A$, i.e.
$$\forall \ell \in \mathcal{I} \quad \ell \preceq_a 0, \tag{2.1}$$
- for every relevant claim $R \in \mathcal{R}$ there exists an agent $a \in A$ such that
$$0 \prec_a R\,. \tag{2.2}$$

We say that $\{\preceq_a\}_{a\in A}$ *supports the financial market* $(\mathcal{H},\tau,\leq,\mathcal{I},\mathcal{R})$.

A market is in equilibrium when agents have no incentive to trade away from their current endowment[5]. In contrast to Harrison and Kreps [1979], we use a definition of equilibrium with heterogeneous agents since, in general, a simple representative agent approach is not feasible under Knightian uncertainty. Condition (2.2) is a form of monotonicity which, in particular, excludes the trivial case of agents who are indifferent between all payoffs. Our adaptation of the concept of viability is dictated by Knightian Uncertainty as we discuss extensively in Section 3 below.

A net trade $\ell \in \mathcal{I}$ is an arbitrage if there exists a relevant claim $R^* \in \mathcal{R}$ such that $\ell \geq R^*$. More generally, a sequence of net trades $\{\ell_n\}_{n=1}^{\infty} \subset \mathcal{I}$ is a *free lunch with vanishing risk* if there exists a relevant claim $R^* \in \mathcal{R}$ and a sequence $\{e_n\}_{n=1}^{\infty} \subset \mathcal{H}$ of nonnegative consumption plans with $e_n \xrightarrow{\tau} 0$ satisfying $e_n + \ell_n \geq R^*$ for all $n \in \mathbb{N}$. We say that the financial market is *strongly free of arbitrage* if there is no free lunch with vanishing risk. In general, the absence of arbitrage is not equivalent to the absence of free lunches with vanishing risk. In Appendix C, we establish the equivalence for finite discrete time financial market.

Our first main theorem establishes the equivalence of viability and absence of arbitrage.

Theorem 2.1. *A financial market is strongly free of arbitrage if and only if it is viable.*

In the standard literature, the model of the economy is constructed on a probability space with a given prior $\mathbb{P}$. In such common prior models, a financial market is viable if and only if there exists a linear pricing measure in the form of a risk-neutral probability measure $\mathbb{P}^*$ that is equivalent to $\mathbb{P}$, as Harrison and Kreps [1979] have shown. In the absence of a common prior, we have to work with a more general, sublinear notion of pricing. A functional

$$\mathcal{E} : \mathcal{H} \to \mathbb{R} \cup \{\infty\}$$

is a *sublinear expectation* if it is monotone with respect to $\leq$, translation-invariant, i.e. $\mathcal{E}(X+c) = \mathcal{E}(X)+c$ for all constant claims $c \in \mathcal{H}$ and $X \in \mathcal{H}$, and sublinear, i.e. for all $X, Y \in \mathcal{H}$ and $\lambda > 0$, we have $\mathcal{E}(X+Y) \leq \mathcal{E}(X) + \mathcal{E}(Y)$ and $\mathcal{E}(\lambda X) = \lambda \mathcal{E}(X)$. $\mathcal{E}$ has *full support* if $\mathcal{E}(R) > 0$ for every $R \in \mathcal{R}$. Last but not least, $\mathcal{E}$ has the *martingale property* if $\mathcal{E}(\ell) \leq 0$ for every $\ell \in \mathcal{I}$. We say in short that $\mathcal{E}$ is a sublinear martingale expectation with full support if all the previous properties are satisfied.

It is well known from decision theory that sublinear expectations can be written as upper expectations over a set of probability measures. In our more abstract framework, probability measures are replaced by suitably normalized

[5]We take the endowment to be zero in our definition; this comes without loss of generality, see also the discussion in Section 3.

functionals. We say that $\varphi \in \mathcal{H}'_+$[6] is a *martingale functional*[7] if it satisfies $\varphi(1) = 1$ (normalization) and $\varphi(\ell) \leq 0$ for all $\ell \in \mathcal{I}$. In the spirit of the probabilistic language, we call a linear functional absolutely continuous if it assigns the value zero to all negligible claims. We denote by $\mathcal{Q}_{ac}$ the set of absolutely continuous martingale functionals.

The notions that we introduced now allow us to state the general version of the fundamental theorem of asset pricing in our order-theoretic context.

Theorem 2.2 (Fundamental Theorem of Asset Pricing)**.** *The financial market is viable if and only if there exists a lower semi–continuous sublinear martingale expectation with full support.*

In this case, the set of absolutely continuous martingale functionals $\mathcal{Q}_{ac}$ is not empty and

$$\mathcal{E}_{\mathcal{Q}_{ac}}(X) := \sup_{\phi \in \mathcal{Q}_{ac}} \phi(X)$$

is a lower semi–continuous sublinear martingale expectation with full support. Moreover, $\mathcal{E}_{\mathcal{Q}_{ac}}$ is maximal, in the sense that any other lower semi–continuous sublinear martingale expectation with full support $\mathcal{E}$ satisfies $\mathcal{E}(X) \leq \mathcal{E}_{\mathcal{Q}_{ac}}(X)$ for all $X \in \mathcal{H}$.

Remark 2.3. Under nonlinear expectations, one has to distinguish martingales from symmetric martingales; a symmetric martingale has the property that the process itself and its negative are martingales. When the set of net trades $\mathcal{I}$ is a linear space as in the case of frictionless markets, a net trade ℓ and its negative $-\ell$ belong to $\mathcal{I}$. In this case, sublinearity and the condition $\mathcal{E}_{\mathcal{Q}_{ac}}(\ell) \leq 0$ for all $\ell \in \mathcal{I}$ imply $\mathcal{E}_{\mathcal{Q}_{ac}}(\ell) = 0$ for all net trades $\ell \in \mathcal{I}$. Thus, the net trades ℓ are symmetric $\mathcal{E}_{\mathcal{Q}_{ac}}$-martingales.

3 Discussion of the model

Common order instead of common prior Preference properties shared by all agents in the market will be reflected in equilibrium prices.

A situation of *risk* is described by the fact that agents share a common prior; in the laboratory, a random experiment based on an objective device like a roulette wheel or a coin toss simulates such a market environment. If no such objective device can be invoked, as in the real world, one might still presume the existence of a common subjective belief for all market participants. Such an assumption might be too strong; the Ellsberg experiments show how to create an environment of Knightian uncertainty in the lab. In complex financial markets in which credit risk contracts, options on term structure shapes and volatility dynamics are traded, Knightian uncertainty plays a prominent role

[6] $\mathcal{H}'$ is the topological dual of $\mathcal{H}$ and $\mathcal{H}'_+$ is the set of positive elements in $\mathcal{H}'$.

[7] In this generality the terminology *functional* is more appropriate. When the dual space $\mathcal{H}'$ can be identified with a space of measures, we will use the terminology *martingale measure*. The technical question whether these measures are countably additive is discussed in Appendix D.

because agents lack precise probability estimates of crucial model parameters and they might be wary of potential structural breaks in the data. Moreover, as Epstein and Ji [2013] have shown (see Example 3.3 below), if we model Knightian uncertainty about volatility, it is logically impossible to construct a reference probability measure.

We thus forego any explicit or implicit assumption of a common prior $\mathbb{P}$. Instead, we base our analysis on a *common order* $\leq$, a far weaker assumption that only requires a (typically incomplete) unanimous dominance criterion. A minimal example of a common order is the pointwise order that we discuss in Example 3.2 below. Pointwise dominance is certainly a criterion that we might assume to be unanimously shared in the context of monetary or single good payoffs. The generality of our approach allows to cover a wide variety of situations, including the well-studied case of risk as well as situations of Knightian uncertainty. For example, if payoffs are ordered by their expected value under a common prior $\mathbb{P}$, we obtain the "risk–neutral" world as we show in Subsection 4.1 below. The standard finance case corresponds to the almost sure ordering under $\mathbb{P}$ (Subsection 4.2). In a multiple prior setting, we recall the distinction of objective and subjective rationality that is discussed by Gilboa et al. [2010]; in their paper, the common order is given by Bewley's incomplete expected utility model whereas individual agents have a complete preference relation represented by a multiple prior utility function, see Subsection 4.3.1. The so–called quasi–sure ordering is a choice that is induced by the family of (potentially non–equivalent) priors $\mathcal{M}$; in this case $X \leq Y$ if $\mathbb{P}(X \leq Y) = 1$ for every prior $\mathbb{P}$ in $\mathcal{M}$ (Subsection 4.3.2).

It might be interesting to note an alternative way of setting up the model in which the common order is *derived* from a class of given preference relations. Suppose that no common order is a priori given. Instead, we start with a class of preference relations $\mathcal{A}_0$ on the commodity space $\mathcal{H}$ that are convex and τ-lower semicontinuous. We can then define the *uniform* order derived from the set of preference relations $\mathcal{A}_0$ as follows. Let

$$\mathcal{Z}_{\preceq} := \{Z \in \mathcal{H} : X \preceq Z + X \preceq X, \ \forall \ X \in \mathcal{H}\},$$

be the set of negligible (or null) claims for the preference relation $\preceq \in \mathcal{A}_0$. We call $\mathcal{Z}_{uni} := \bigcap_{\preceq \in \mathcal{A}_0} \mathcal{Z}_{\preceq}$ the set of *unanimously negligible* claims. Let the uniform pre-order $\leq_{uni}$ on $\mathcal{H}$ be given by $X \leq_{uni} Y$ if and only if there exists $Z \in \mathcal{Z}_{uni}$ such that $X(\omega) \leq Y(\omega) + Z(\omega)$ for all $\omega \in \Omega$. Note that we use the pointwise order on the reals and a uniformly negligible payoff to derive the common order from the set of priors $\mathcal{A}_0$. $(\mathcal{H}, \leq_{uni})$ is then a pre-ordered vector space[8], and agents' preferences in $\mathcal{A}_0$ are monotone with respect to the uniform pre-order. One can then derive the (usually larger) class of preference relations $\mathcal{A}$ that satisfy the conditions of our approach, including monotonicity with respect to the derived uniform order, and the analysis goes on from there.

[8] In the same spirit, one could define a pre-order $X \leq'_u Y \Leftrightarrow X \preceq Y$ for all $\preceq \in \mathcal{A}$. In general this will not define a pre-ordered vector space $(\mathcal{H}, \leq')$. The analysis of the paper carries over with minor modifications.

The Financial Market We model the financial market in a rather reduced form with the help of the convex cone $\mathcal{I}$. This abstract approach is sufficient for our purpose of discussing the relation of arbitrage and viability. In the next example, we show how the usual models of static and dynamic trading are embedded.

Example 3.1. We consider four markets with increasing complexity.

1. In a one period setting with finitely many states $\Omega = \{1, \ldots, N\}$, a financial market with $J+1$ securities can be described by its initial prices $x_j \geq 0, j = 0, \ldots, J$ and a $(J+1) \times N$–payoff matrix F, compare LeRoy and Werner [2014]. A portfolio $\bar{H} = (H_0, \ldots, H_J) \in \mathbb{R}^{J+1}$ has the payoff $\bar{H}F = \left(\sum_{j=0}^{J} H_j F_{j\omega}\right)_{\omega=1,\ldots,N}$; its initial cost satisfies $H \cdot x = \sum_{j=0}^{J} H_j x_j$. If the zeroth asset is riskless with a price $x_0 = 1$ and pays off 1 in all states of the world, then a net trade with zero initial cost can be expressed in terms of the portfolio of risky assets $H = (H_1, \ldots, H_J) \in \mathbb{R}^J$ and the return matrix $\hat{F} = (F_{j\omega} - x_j)_{j=1,\ldots,J,\omega=1,\ldots,N}$. $\mathcal{I}$ is given by the image of the $J \times N$ return matrix $\hat{F}$, i.e.

$$\mathcal{I} = \{H\hat{F} : H \in \mathbb{R}^J\}.$$

2. Our model includes the case of finitely many trading periods. Let $\mathbb{F} := (\mathcal{F}_t)_{t=0}^T$ be a filtration on $(\Omega, \mathcal{F})$ and $S = (S_t)_{t=0}^T$ be an adapted stochastic process with values in $\mathbb{R}_+^J$ for some $J \geq 1$; S models the uncertain assets. We assume that a riskless bond with interest rate zero is also given. Then, the set of net trades can be described by the gains from trade processes: $\ell \in \mathcal{H}$ is in $\mathcal{I}$ provided that there exists predictable integrands $H_t \in \left(\mathcal{L}^0(\Omega, \mathcal{F}_{t-1})\right)^J$ for $t = 1, \ldots, T$ such that,

$$\ell = (H \cdot S)_T := \sum_{t=1}^{T} H_t \cdot \Delta S_t, \quad \text{where} \quad \Delta S_t := (S_t - S_{t-1}).$$

In the frictionless case, the set of net trades is a subspace of $\mathcal{H}$. In general, one might impose restrictions on the set of admissible trading strategies. For example, one might exclude short-selling of risky assets, or impose a bound on agents' credit line; in these cases, the marketed subspace $\mathcal{I}$ is a convex cone, compare Luttmer [1996], Jouini and Kallal [1995a], and Araujo et al. [2018], e.g.

3. In Harrison and Kreps [1979], the market is described by a marketed space $M \subset \mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$ and a (continuous) linear functional π on M. In this case, $\mathcal{I}$ is the kernel of the price system, i.e.

$$\mathcal{I} = \{X \in M : \pi(X) = 0\}.$$

4. In continuous time, the set of net trades consists of stochastic integrals of the form

$$\mathcal{I} = \left\{ \int_0^T \theta_u \cdot dS_u \;:\; \theta \in \mathcal{A}_{adm} \right\},$$

for a suitable set of *admissible* strategies $\mathcal{A}_{adm}$. There are several possible choices of such a set. When the stock price process S is a semi-martingale one example of $\mathcal{A}_{adm}$ is the set of all S-integrable, predictable processes whose integral is bounded from below.[9] Other typical choices for $\mathcal{A}_{adm}$ would consist of simple integrands only; when S is a continuous process and $\mathcal{A}_{adm}$ is the set of process with finite variation then the above integral can be defined through integration by parts (see Dolinsky and Soner [2014a, 2015]).

In general, the absence of a common prior poses some non-trivial technical questions about the integrability of contingent claims and net trades. Clearly, it is possible to restrict the commodity space to the class of bounded measurable function (that are integrable with respect to any prior). The condition $\mathcal{I} \subset \mathcal{H}$ could be restrictive in some applications and we provide a way to overcome this difficulty in Appendix B.

Relevant Claims We use the notion of relevant claims to generalize the typical approach to define arbitrage as positive net trades. This approach introduces some additional flexibility and allows to cover potentially important variants of the notion of arbitrage. For example, if some positive claim cannot be liquidated without costs, agents would not consider a net trade that achieves such a payoff as free lunch if the liquidation costs are larger than the potential gains. It is then reasonable to consider as relevant only a restricted class of positive claims, possibly only cash.

Moreover, relevant payoffs identify potentially desirable directions of consumption for our economy. The commodity spaces that are used to model markets with Knightian uncertainty are often large. Thus in some finance applications, it makes sense to work with a set of relevant claims that is smaller than the positive claims, confer also Example 3.3 below.

Viability Knightian uncertainty requires a careful adaptation of the notion of economic viability. In competitive markets under risk, it is possible to construct a representative agent *with strictly monotone preferences* for a given arbitrage-free set of security prices. In Harrison and Kreps [1979], Kreps [1981], viability is equivalent to the existence of certain strictly positive linear functionals that induce (continuous) strictly monotone preferences; by the Riesz representation theorem, they are described by densities that are almost surely strictly positive with respect to the common prior.

Introducing Knightian uncertainty to the standard financial models leads to very large commodity spaces on which no strictly positive linear functionals exist. Consequently, the notion of viability needs to be adapted. Below, we illustrate these issues in Example 3.2 with a simple model in which the unit interval is the state space. Subsequently, Example 3.3 shows that the same issues arise in the ubiquitous Samuelson-Merton-Black-Scholes model when there is Knightian

[9] In continuous time, to avoid doubling strategies a lower bound (maybe more general than above) has to be imposed on the stochastic integrals. In such cases, the set $\mathcal{I}$ is not a linear space.

uncertainty about volatility. If an arbitrage–free model can be supported by a *single agent with strictly monotone preferences* satisfying our standing assumptions, in equilibrium, the marginal utility of that agent would need to be strictly positive for all positive payoffs[10]. Thus, the commodity space necessarily needed to support strictly positive linear functionals. Of course, the notion of economic equilibrium does not require the existence of a representative agent; allowing for a heterogeneous agent economy is a conceptually appealing approach as actual financial markets certainly are populated by a diverse range of agents. We thus relax the strict monotonicity condition of Harrison and Kreps [1979] and Kreps [1981] by allowing for agents with weakly monotone preferences. To exclude the trivial equilibrium case of agents who are indifferent between all claims, we need to introduce some form of strict monotonicity for the market as a whole. Condition (2.2) ensures that the relevant claims are desired by *some* agents in the supporting economy.

Example 3.2. Take $\Omega = [0, 1]$. Let $\mathcal{F}$ be the Borel sets and $\mathcal{H}$ be the set of all bounded, measurable functions on Ω. We consider the quasi-sure common order induced by a set of probability measures $\mathcal{Q}$ on Ω, i.e. $X \geq Y$ ("$\mathcal{Q}$ quasi-surely") provided that $\mathbb{Q}(X \geq Y) \geq 0$ for every $\mathbb{Q} \in \mathcal{Q}$. Let the relevant claims be $\mathcal{R} = \mathcal{P}^+$; a bounded measurable function $R \in \mathcal{H}$ is thus relevant if $R \geq 0$ $\mathcal{Q}$ quasi-surely and $\mathbb{Q}(R > 0) > 0$ for some $\mathbb{Q} \in \mathcal{Q}$.

To illustrate our main points succinctly, we consider the extreme case of Knightian uncertainty when $\mathcal{Q}$ is the set of all probability measures. Then, the common order is given by the pointwise order; call a bounded measurable function relevant if it is non-negative everywhere and is strictly positive for some $\omega \in \Omega$. Consider the Gilboa–Schmeidler utility function

$$U(X) = \inf_{\mathbb{Q} \in \mathcal{Q}} E_{\mathbb{Q}}[u(X)] = \inf_{\omega \in \Omega} u(X(\omega))$$

for some strictly monotone, strictly concave function $u : \mathbb{R} \to \mathbb{R}$. This particular agent weakly prefers the zero trade to any claim whose minimum value is less than zero. In particular, for the relevant claim $R(\omega) = 1_{(0,1]}(\omega)$ we have $U(0) = u(0) \geq U(\lambda R)$ for all $\lambda \in \mathbb{R}$. For positive λ, the agent cares only about the worst state $\omega = 0$ in which the claim λR has a payoff of zero. The agent does not desire negative multiples of the claim either because he would then lose money in each state of the world except at $\omega = 0$.

The commodity space in this example does not carry any strictly positive linear functional (Aliprantis and Border [1999], Section 8.10, also Example 8.21). Consequently, although carefully chosen agents may strictly prefer the relevant contract $1_{(0,1]}(\omega)$ to zero, the abstract argument provided before the example shows that no single agent economy can strictly prefer *all* relevant contracts to zero.

[10]For convex and lower semi-continuous preferences, marginal utility at zero exists by the separation theorem. One can show that it has to be a strictly positive linear functional if the preferences are strictly monotone.

Example 3.3. In the classical Samuelson-Merton-Black-Scholes model, the stock price S_t is a geometric Brownian motion satisfying the stochastic differential equation $dS_t = \sigma S_t dB_t$ where B is a standard Brownian motion and σ is the *volatility*. As in Epstein and Ji [2013], Vorbrink [2014] and Beissner and Denis [2018], we consider a financial market with Knightian uncertainty about the volatility of the price process, modeled by a certain interval $[\underline{\sigma}, \overline{\sigma}]$. Let the maturity $T > 0$ be fixed. Then, any map in the set

$$\Sigma := \Sigma_{\underline{\sigma},\overline{\sigma}} := \{\sigma : [0,T] \to [\underline{\sigma}, \overline{\sigma}] \mid \sigma \text{ is adapted } \} \tag{3.1}$$

is a possible volatility process.

For $\sigma = (\sigma_t)_{t\in[0,T]} \in \Sigma$, let $\mathbb{P}^\sigma$ be the distribution of the stock price process with volatility process[11] σ. We let the commodity space consist of payoffs that are integrable with respect to all priors, i.e., $\mathcal{H} = \bigcap_{\sigma\in\Sigma} \mathcal{L}^1(\Omega, \mathbb{P}^\sigma)$. We use the topology induced by the norm $\|X\|_{\mathcal{H}} := \sup_{\sigma\in\Sigma} \mathbb{E}_{\mathbb{P}^\sigma}|X|$. The common order is the quasi–sure order induced by the family of priors $\{\mathbb{P}^\sigma\}_{\sigma\in\Sigma}$, and we take $\mathcal{R} = \mathcal{P}^+$. As in part 4 of Example 3.1, we let $\mathcal{I}$ be the set of all stochastic integrals with simple integrands that are bounded from below .

1. Let us confirm that the market is viable and supported by a class of Gilboa–Schmeidler agents with preferences $\{\preceq_{a,b}\}_{a,b\in\Sigma}$ represented by

$$U_{a,b}(X) := \inf_{\sigma\in\Sigma_{a,b}} \mathbb{E}_{\mathbb{P}^\sigma}[X],$$

where $\Sigma_{a,b}$ is as in Eqn. (3.1) with possibly state-dependent upper and lower bounds $a, b \in \Sigma$. For $\sigma \in \Sigma$, the stock price and thus every simple stochastic integral $\ell \in \mathcal{I}$ is a $\mathbb{P}^\sigma$-local martingale. Therefore, we have $\mathbb{E}_{\mathbb{P}^\sigma}[\ell] \le 0$, and thus $U_{a,b}(\ell) \le 0$, proving the optimality condition (2.1). By definition of the quasi-sure order, if R is relevant, there exists $\sigma \in \Sigma$ with $\mathbb{P}^\sigma[R \ge 0] = 1$ and $\mathbb{P}^\sigma[R > 0] > 0$, so we have $\mathbb{E}_{\mathbb{P}^\sigma}[R] > 0$ and $U_{\sigma,\sigma}(R) > 0$. This shows that the monotonicity condition (2.2) is satisfied by $\{\preceq_{a,b}\}_{a,b\in\Sigma}$.

2. As in the previous example, agents with τ-lower semicontinuous strictly monotone utility functions do not exist[12]. We can illustrate this issue with a concrete example that is inspired by the highly traded volatility index options. Let

$$R_{a,b,t}(\omega) := 1_{\{RV_t(\omega)\in(a,b)\}}, \quad t \in (0,T], \underline{\sigma} \le a \le b \le \overline{\sigma},$$

be the digital option that pays off 1 when the normalized realized variance of the observed stock price path $RV_t(\omega)$ lies in a certain interval[13]. For

[11] We refer the reader to Soner et al. [2012, 2013] for a formal construction of $\mathbb{P}^\sigma$ and its subtle properties. The class contains mutually singular priors and is not dominated by a common prior.

[12] For a detailed analysis of strictly monotone preferences on rich domains, we refer to the recent survey by Hervés-Beloso and del Valle-Inclán Cruces [2019], Section 4.

[13] Compare Soner et al. [2012, 2013] for the exact pointwise definition.

volatility processes $\sigma \in \Sigma$ that leave the interval between time 0 and time t, the event $\{RV_t(\omega) \in (a,b)\}$ is a null set for $\mathbb{P}^\sigma$ and we thus have $\mathbb{E}_{\mathbb{P}^\sigma}[R_{a,b,t}] = 0$. Agents with Gilboa-Schmeidler utility functions

$$U(X) := \inf_{\sigma \in \Sigma} \mathbb{E}_{\mathbb{P}^\sigma}[X],$$

are thus indifferent between the relevant claims $R_{a,b,t}$ and zero.

3. The set of relevant contracts can be used to consider different notions of arbitrage within the same framework. With the choice $\mathcal{R} = \mathcal{P}^+$, $\ell \in \mathcal{I}$ is an arbitrage if $\mathbb{P}^\sigma(\ell \geq 0) = 1$ for every $\sigma \in \Sigma$ and if there exists $\hat{\sigma} \in \Sigma$ such that $\mathbb{P}^{\hat{\sigma}}(\ell > 0) > 0$. Other choices are also plausible; with the choice

$$\mathcal{R}_u := \{R \,:\, \mathbb{P}^\sigma(R > 0) > 0, \quad \forall \sigma \in \Sigma\},$$

ℓ is an arbitrage if it is a arbitrage in the probabilistic sense with common prior $\mathbb{P}^\sigma$.

The reader might note that our notion of equilibrium does not model endowments explicitly as we assume that the zero trade is optimal for each agent. This reduced approach comes without loss of generality in our context. In general, an agent is given by a preference relation $\succeq \in \mathcal{A}$ and an endowment $e \in \mathcal{H}$. Given the set of net trades, the agent chooses $\ell^* \in \mathcal{I}$ such that $e + \ell^* \succeq e + \ell$ for all $\ell \in \mathcal{I}$. By suitably modifying the preference relation, this can be reduced to the optimality of the zero trade at the zero endowment for a suitably modified preference relation. Let $X \succeq' Y$ if and only if $X + e + \ell^* \succeq Y + e + \ell^*$. It is easy to check that $\succeq'$ is also an admissible preference relation. For the new preference relation $\succeq'$, we then have $0 \succeq' \ell$ if and only if $e + \ell^* \succeq e + \ell^* + \ell$. As $\mathcal{I}$ is a cone, $\ell + \ell^* \in \mathcal{I}$, and we conclude that we have indeed $0 \succeq' \ell$ for all $\ell \in \mathcal{I}$.

Sublinear Expectations Our fundamental theorem of asset pricing characterizes the absence of arbitrage with the help of a non–additive expectation $\mathcal{E}$. In decision theory, non–additive probabilities have a long history; Schmeidler [1989] introduces an extension of expected utility theory based on non–additive probabilities. The widely used max-min expected utility model of Gilboa and Schmeidler [1989] is another instance. If we define the subjective expectation of a payoff to be the minimal expected payoff over a class of priors, then the resulting notion of expectation has some of the common properties of an expectation like monotonicity and preservation of constants, but is no longer additive.

In our case, the non–additive expectation has a more objective than subjective flavor because it describes the pricing functional of the market. Whereas an additive probability measure is sufficient to characterize viable markets in models with a common prior, in general, such a construction is no longer feasible. Indeed, Harrison and Kreps [1979] prove that viability implies that the linear market pricing functional can be extended from the marketed subspace to a *strictly positive* linear functional on the whole space of contingent claims[14]. Under Knightian uncertainty, however, strictly positive linear functionals frequently *do not exist* (compare Examples 3.2 and 3.3). We thus rely on a non–additive notion of expectation[15].

The pricing functional assigns a nonpositive value to all net trades; in this sense, net trades have the (super)martingale property under this expectation. If we assume for the sake of the discussion that the set of net trades is a linear subspace, then the pricing functional has to be additive over that subspace. As a consequence, the value of all net trades under the sublinear pricing expectation is zero. For contingent claims that lie outside the marketed subspace, the pricing operation of the market is sub–additive.

The following two examples illustrate the issue. We start with the simple case of complete financial markets within finite state spaces. Here, an additive probability is sufficient to characterise the absence of arbitrage, as is well known.

Example 3.4 (The atom of finance and complete markets)**.** The basic one–step binomial model, that we like to call the atom of finance, consists of two states of the world, $\Omega = \{1, 2\}$. An element $X \in \mathcal{H}$ can be identified with a vector in $\mathbb{R}^2$. Let $\leq$ be the usual partial order of $\mathbb{R}^2$. The relevant claims are the positive ones, $\mathcal{R} = \mathcal{P}^+$.

There is a riskless asset B and a risky asset S. At time zero, both assets have value $B_0 = S_0 = 1$. The riskless asset yields $B_1 = 1 + r$ for an interest rate $r > -1$ at time one, whereas the risky asset takes the values u in state 1 and respectively d in state 2 with $u > d$.

We use the riskless asset B as numéraire. The discounted net return on the risky asset is $\hat{\ell} := S_1/(1 + r) - 1$. $\mathcal{I}$ is the linear space spanned by $\hat{\ell}$. There is no arbitrage if and only if the unique candidate for a martingale probability of state one

$$p^* = \frac{1 + r - d}{u - d}$$

belongs to $(0, 1)$ which is equivalent to $u > 1 + r > d$. p^* induces the unique martingale measure $\mathbb{P}^*$ with expectation

$$\mathbb{E}^*[X] = p^* X(1) + (1 - p^*) X_1(2)\,.$$

$\mathbb{P}^*$ is a linear measure; moreover, it has the full support property since for every $R \in \mathcal{R}$ we have $\mathbb{E}^*[R] > 0$. The market is viable with $A = \{\preceq^*\}$, the preference relation given by the linear expectation $\mathbb{P}^*$, i.e. $X \preceq^* Y$ if and only if $\mathbb{E}^*[X] \leq \mathbb{E}^*[Y]$. Indeed, under this preference $\ell \sim^* 0$ for any $\ell \in \mathcal{I}$ and $X \prec^* X + R$ for any $X \in \mathcal{H}$ and $R \in \mathcal{P}^+$. In particular, any $\ell \in \mathcal{I}$ is an optimal portfolio and the market is viable.

The preceding analysis carries over to all finite Ω and complete financial markets.

[14] Note that the pricing functional has to assign a strictly positive price to all relevant claims as otherwise, there would be some agent who would want to purchase it, violating the equilibrium conditions.

[15] Beissner and Riedel [2019] develop a general equilibrium model based on such non–additive pricing functionals.

We now turn to a somewhat artificial one period model with uncountably many states. It serves well the purpose to illustrate the need for sublinear expectations and thus stands in an exemplary way for more complex models involving continuous time and uncertain violatility, e.g.

Example 3.5 (Highly incomplete one-period models)**.** This example shows that sublinear expectations are necessary to characterize the absence of arbitrage under Knightian uncertainty and with incomplete markets.

Let $\Omega = [0, 1]$ and $\leq$ be the usual pointwise partial order. Payoffs X are bounded Borel measurable functions on Ω. As in the previous example $X \in \mathcal{P}$ if and only if $X(\omega) \geq 0$ for every $\omega \in \Omega$. Let the relevant claims be again $\mathcal{R} = \mathcal{P}^+$. Assume that there is a riskless asset with interest rate $r \geq 0$. Let the risky asset have the price $S_0 = 1$ at time 0 and assume it pays off $S_1(\omega) = 2\omega$ at time 1. As in the previous example, $\mathcal{I}$ is spanned by the net return $\hat{\ell} := S_1/(1 + r) - 1$.

There exist uncountably many martingale measures because any probability measure $\mathbb{Q}$ satisfying $\int_\Omega 2\omega \, \mathbb{Q}(d\omega) = 1 + r$ is a martingale measure. Denote by $\mathcal{Q}_{ac}$ the set of all martingale measures.

No single martingale measure is sufficient to characterize the absence of arbitrage because there is no single linear martingale probability measure $\mathbb{Q}$ with the full support property, i.e. such that $\mathbb{E}_\mathbb{Q}[R] > 0$ for every $R \in \mathcal{R}$. Indeed, as $1_{\{\omega\}} \in \mathcal{R}$ for every $\omega \in \Omega$, such a measure would have to assign a non-zero value to every point, an impossibility for uncountable Ω. Hence, the equivalence "no arbitrage" to "there is a martingale measure with some monotonicity property" does not hold true if one insists on having a linear martingale measure. Instead, one needs to work with the nonlinear expectation

$$\mathcal{E}(X) := \sup_{\mathbb{Q} \in \mathcal{Q}_{ac}} \mathbb{E}_\mathbb{Q}[X]$$

for $X \in \mathcal{H}$.

We claim that $\mathcal{E}$ has full support and characterizes the absence of arbitrage in the sense of Theorem 2.1 and Theorem 2.2. To see that $\mathcal{E}$ has full support, note that $R \in \mathcal{P}^+$ if and only if $R \geq 0$ and there is $\omega^* \in \Omega$ so that $R(\omega^*) > 0$. Define $\mathbb{Q}^*$ by

$$\mathbb{Q}^* := \frac{1}{2} \left(\delta_{\{\omega^*\}} + \delta_{\{1-\omega^*\}} \right).$$

Then $\mathbb{Q}^*$ is a martingale measure; in particular, we have

$$\mathcal{E}(R) \geq \mathbb{E}_{\mathbb{Q}^*}[R] = \frac{1}{2} R(\omega^*) + \frac{1}{2} R(1 - \omega^*) > 0 = \mathcal{E}(0).$$

This example shows that the heterogeneity of agents needed in equilibrium to support an arbitrage-free financial market and the necessity to allow for sublinear expectations are two complementary faces of the same issue.

4 The Efficient Market Hypothesis

The Efficient Market Hypothesis (EMH) plays a fundamental role in the history of Financial Economics. Fama [1970] calls markets informationally efficient if all available information is reflected properly in current asset prices. There are several interpretations of this conjecture; in the early days after its appearance, the efficient market hypothesis was usually interpreted as asset prices being random walks in the sense that (log-) returns be independent from the past and identically distributed with the mean return being equal to the return of a safe bond. Later, the informational efficiency of asset prices was interpreted as a martingale property; (conditional) expected returns of all assets are equal to the return of a safe bond under some probability measure. This conjecture of the financial market's being a "fair game" dates back to Bachelier [1900] and was rediscovered by Paul Samuelson (Samuelson [1965, 1973]). In dynamic settings, market efficiency is thus strongly related to (publicly available) information. Under Knightian uncertainty, the role of information and the martingale property of prices needs to be adapted properly as we shall see in this section[16].

Throughout this section, let us assume that we have a frictionless one–period or discrete–time multiple period financial market as in Example 3.1, 1. and 2. In particular, the set of net trades $\mathcal{I}$ is a subspace of $\mathcal{H}$.

4.1 Strong Efficient Market Hypothesis under Risk

Let $\mathbb{P}$ be a probability measure on $(\Omega, \mathcal{F})$. Set $\mathcal{H} = \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$. Let the common order by given by $X \leq Y$ if and only if the expected payoffs under the common prior $\mathbb{P}$ satisfy

$$\mathbb{E}_{\mathbb{P}}[X] \leq \mathbb{E}_{\mathbb{P}}[Y]. \tag{4.1}$$

In this case, negligible claims coincide with the claims with mean zero under $\mathbb{P}$. Moreover, $X \in \mathcal{P}$ if $\mathbb{E}_{\mathbb{P}}[X] \geq 0$. We take $\mathcal{R} = \mathcal{P}_+$.

Proposition 4.1. *Under the assumptions of this subsection, the financial market is viable if and only if the common prior $\mathbb{P}$ is a martingale measure. In this case, $\mathbb{P}$ is the unique martingale measure.*

Proof. Note that the common order as given by (4.1) is complete. If $\mathbb{P}$ is a martingale measure, the common order $\leq$ itself defines a linear preference relation under which the market is viable with $A = \{\leq\}$.

On the other hand if the market is viable, Theorem 2.2 ensure that there exists a sublinear martingale expectation with full support. By the Riesz duality theorem, a martingale functional $\phi \in \mathcal{Q}_{ac}$ can be identified with a probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$. It is absolutely continuous (in our sense defined above) if and only if it assigns the value 0 to all negligible claims. As a consequence, we have $E_{\mathbb{Q}}[X] = 0$ whenever $E_{\mathbb{P}}[X] = 0$. Then $\mathbb{Q} = \mathbb{P}$ follows[17]. $\square$

[16] We refer to Jarrow and Larsson [2012] for a detailed analysis of the interplay between different information sets and market efficiency under a common prior. In our framework, the information flow is taken as given; it is implicitly encoded in the set of available claims $\mathcal{I}$. We do not consider the issue of private information of insiders.

[17] If $\mathbb{Q} \neq \mathbb{P}$, there is an event $A \in \mathcal{F}$ with $\mathbb{Q}(A) < \mathbb{P}(A)$. Set $X = 1_A - \mathbb{P}(A)$. Then $0 = E_{\mathbb{P}}[X] > \mathbb{Q}(A) - \mathbb{P}(A) = E_{\mathbb{Q}}[X]$.

The only absolutely continuous martingale measure is the common prior itself. As a consequence, all traded assets have zero net expected return under the common prior. A financial market is thus viable if and only if the strong form of the expectations hypothesis holds true.

4.2 Weak Efficient Market Hypothesis under Risk

In its weak form, the efficient market hypothesis states that expected returns are equal under some (pricing) probability measure $\mathbb{P}^*$ that is equivalent to the common prior (or "real world" probability) $\mathbb{P}$.

Let $\mathbb{P}$ be a probability on $(\Omega, \mathcal{F})$ and $\mathcal{H} = \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$. In this example, the common order is given by the almost sure order under the common prior $\mathbb{P}$, i.e.,

$$X \leq Y \quad \Leftrightarrow \quad \mathbb{P}(X \leq Y) = 1.$$

A payoff is negligible if it vanishes $\mathbb{P}$–almost surely and is positive if it is $\mathbb{P}$–almost surely nonnegative. Let the relevant claims $\mathcal{R}$ consist of the $\mathbb{P}$–almost surely nonnegative payoffs that are strictly positive with positive $\mathbb{P}$–probability,

$$\mathcal{R} = \left\{ R \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})_+ \ : \ \mathbb{P}(R > 0) > 0 \right\}.$$

A functional $\phi \in \mathcal{H}'_+$ is an absolutely continuous martingale functional if and only if it can be identified with a probability measure $\mathbb{Q}$ that is absolutely continuous with respect to $\mathbb{P}$ and if all net trades have expectation zero under ϕ. In other words, discounted asset prices are $\mathbb{Q}$-martingales. We thus obtain a version of the Fundamental Theorem of Asset Pricing under risk, similar to Harrison and Kreps [1979] and Dalang et al. [1990].

Proposition 4.2. *Under the assumptions of this subsection, the financial market is viable if and only if there is a martingale measure $\mathbb{Q}$ that has a bounded density with respect to $\mathbb{P}$.*

Proof. If $\mathbb{Q}$ is a martingale measure equivalent to $\mathbb{P}$, define $X \preceq^* Y$ if and only if $\mathbb{E}_{\mathbb{Q}}[X] \leq \mathbb{E}_{\mathbb{Q}}[Y]$. Then the market is viable with $A = \{\preceq^*\}$. Condition (2.2) is satisfied because $\mathbb{Q}$ is equivalent to $\mathbb{P}$.

If the market is viable, Theorem 2.2 ensures that there exists a sublinear martingale expectation with full support. By the Riesz duality theorem, a martingale functional $\phi \in \mathcal{Q}_{ac}$ can be identified with a probability measure $\mathbb{Q}_\phi$ that is absolutely continuous with respect to $\mathbb{P}$, has a bounded density with respect to $\mathbb{P}$, and all net trades have zero expectation zero under $\mathbb{Q}_\phi$. In other words, discounted asset prices are $\mathbb{Q}_\phi$-martingales. From the full support property, the family $\{\mathbb{Q}_\phi\}_{\phi \in \mathcal{Q}_{ac}}$ is equivalent to $\mathbb{P}$, meaning that $\mathbb{Q}_\phi(A) = 0$ for every $\phi \in \mathcal{Q}_{ac}$ if and only if $\mathbb{P}(A) = 0$. By the Halmos-Savage Theorem (Halmos and Savage [1949], **?**Theorem 1.61]FoellmerSchied11), there exists a countable subfamily $\{\mathbb{Q}_{\phi_n}\}_{n \in \mathbb{N}} \subset \{\mathbb{Q}_\phi\}_{\phi \in \mathcal{Q}_{ac}}$ which is equivalent to $\mathbb{P}$. The measure $\mathbb{Q} := \sum_{n=1}^\infty 2^{-n} \mathbb{Q}_{\phi_n}$ is the desired equivalent martingale measure. $\square$

4.3 The EMH under Knightian Uncertainty

We turn our attention to the EMH under Knightian uncertainty. We consider first the case when the common order is derived from a common set of priors, inspired by the multiple prior approach in decision theory (Bewley [2002], Gilboa and Schmeidler [1989]). We then discuss a second-order Bayesian approach that is inspired by the smooth ambiguity model (Klibanoff et al. [2005]).

4.3.1 Strong Efficient Market Hypothesis under Knightian Uncertainty

We consider a generalization of the original EMH to Knightian uncertainty that shares a certain analogy with Bewley's incomplete expected utility model (Bewley [2002]) and Gilboa and Schmeidler's maxmin expected utility (Gilboa and Schmeidler [1989])[18].

Let Ω be a metric space and $\mathcal{M}$ be a convex, weak*-closed set of priors on $(\Omega, \mathcal{F})$. Define the semi-norm

$$\|X\|_{\mathcal{M}} := \sup_{\mathbb{P}\in\mathcal{M}} \mathbb{E}_{\mathbb{P}}|X|.$$

Let $\mathcal{L}^1(\Omega, \mathcal{F}, \mathcal{M})$ be the closure of continuous and bounded functions on Ω under the semi-norm $\|\cdot\|_{\mathcal{M}}$. If we identify the functions which are $\mathbb{P}$-almost surely equal for every $\mathbb{P} \in \mathcal{M}$, then $\mathcal{H} = \mathcal{L}^1(\Omega, \mathcal{F}, \mathcal{M})$ is a Banach space. The topological dual of $\mathcal{L}^1(\Omega, \mathcal{F}, \mathcal{M})$ can be identified with probability measures that admit a bounded density with respect to some measure in $\mathcal{M}$ (Bion-Nadal et al. [2012], Beissner and Denis [2018]). Therefore, any absolutely continuous martingale functional $\mathbb{Q} \in \mathcal{Q}_{ac}$ is a probability measure and $\mathcal{M}$ is closed in the weak* topology induced by $\mathcal{L}^1(\Omega, \mathcal{F}, \mathcal{M})$.

Consider the uniform order induced by expectations over $\mathcal{M}$,

$$X \leq Y \quad \Leftrightarrow \quad \forall \mathbb{P} \in \mathcal{M} \quad \mathbb{E}_{\mathbb{P}}[X] \leq \mathbb{E}_{\mathbb{P}}[Y]\,.$$

Then, $Z \in \mathcal{Z}$ if $\mathbb{E}_{\mathbb{P}}[Z] = 0$ for every $\mathbb{P} \in \mathcal{M}$. A claim X is positive if $\mathbb{E}_{\mathbb{P}}[X] \geq 0$ for every $\mathbb{P} \in \mathcal{M}$. Let the relevant claims consist of nonnegative claims with a positive return under some prior belief, i.e.

$$\mathcal{R} = \{R \in \mathcal{H} : 0 \leq \inf_{\mathbb{P}\in\mathcal{M}} \mathbb{E}_{\mathbb{P}}[R] \text{ and } 0 < \sup_{\mathbb{P}\in\mathcal{M}} \mathbb{E}_{\mathbb{P}}[R]\}\,.$$

Proposition 4.3. *Under the assumptions of this subsection, if the financial market is viable, then the set of absolutely continuous martingale functionals $\mathcal{Q}_{ac}$ is a subset of the set of priors $\mathcal{M}$.*

Proof. Set $\mathcal{E}_{\mathcal{M}}(X) := \sup_{\mathbb{P}\in\mathcal{M}} \mathbb{E}_{\mathbb{P}}[X]$. Then, $Y \leq 0$ if and only if $\mathcal{E}_{\mathcal{M}}(Y) \leq 0$. Fix $\mathbb{Q} \in \mathcal{Q}_{ac}$ with the preference relation given by $X \preceq_{\mathbb{Q}} Y$ if $\mathbb{E}_{\mathbb{Q}}[X - Y] \leq 0$.

[18] For the relation between the two approaches, compare also the discussion of objective and subjective ambiguity in Gilboa et al. [2010].

Let us assume that $\mathbb{Q} \notin \mathcal{M}$. Since $\mathcal{M}$ is a weak*-closed and convex subset of the topological dual of $\mathcal{L}^1(\Omega, \mathcal{F}, \mathcal{M})$, there exists $X^* \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathcal{M})$ with $\mathcal{E}_{\mathcal{M}}(X^*) < 0 < \mathbb{E}_{\mathbb{Q}}[X^*]$ by the Hahn-Banach theorem. In particular, $X^* \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathcal{M})$ and $X^* \leq 0$. Since $\preceq_{\mathbb{Q}}$ is weakly monotone with respect to $\leq$, $X^* \preceq_{\mathbb{Q}} 0$. Hence, $\mathbb{E}_{\mathbb{Q}}[X^*] \leq 0$ contradicting the choice of X^*. Therefore, $\mathcal{Q}_{ac} \subset \mathcal{M}$.

□

Expected returns of traded securities are thus not necessarily the same under all $\mathbb{P} \in \mathcal{M}$. However, the set of martingale measures is a subset of $\mathcal{M}$ here, and thus the strong form of the EMH holds true on a subset of the set of priors $\mathcal{M}$.

In general, it is not possible to characterize the set of martingale measures in more detail. However, we can identify a subspace of claims on which expectations under all priors coincide. Let $\mathcal{H}_{\mathcal{M}}$ be the subspace of claims that have no ambiguity in the mean in the sense that $\mathbb{E}_{\mathbb{P}}[X]$ is the same constant for all $\mathbb{P} \in \mathcal{M}$. Consider the submarket $(\mathcal{H}_{\mathcal{M}}, \tau, \leq, \mathcal{I}_{\mathcal{M}}, \mathcal{R}_{\mathcal{M}})$ with $\mathcal{I}_{\mathcal{M}} := \mathcal{I} \cap \mathcal{H}_{\mathcal{M}}$ and $\mathcal{R}_{\mathcal{M}} := \mathcal{R} \cap \mathcal{H}_{\mathcal{M}}$. Restricted to this market, the sets of measures $\mathcal{Q}_{ac}$ and $\mathcal{M}$ are identical and the strong EMH holds true.

The following simple example illustrates these points.

Example 4.4. Let $\Omega = \{0,1\}^2$, $\mathcal{H}$ be all functions on Ω. Then, $\mathcal{H} = \mathbb{R}^4$ and we write $X = (x, y, v, w)$ for any $X \in \mathcal{H}$. Let $\mathcal{I} = \{(x, y, 0, 0) : x + y = 0\}$. Consider the priors given by

$$\mathcal{M} := \left\{ \left(p, \frac{1}{2} - p, \frac{1}{4}, \frac{1}{4}\right) \ : \ p \in \left[\frac{1}{6}, \frac{1}{3}\right] \right\}.$$

There is Knightian uncertainty about the first two states, yet no Knightian uncertainty about the last two states. One directly verifies that $\mathcal{Q}_{ac} = \{\mathbb{Q}^*\} = \{(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})\}$. Notice that $\mathbb{Q}^* \in \mathcal{M}$.

In this case, $\mathcal{H}_{\mathcal{M}} = \{X = (x, y, v, w) \in \mathcal{H} \ : \ x = y\}$. In particular, all priors in $\mathcal{M}$ coincide with $\mathbb{Q}^*$ when restricted to $\mathcal{H}_{\mathcal{M}}$. Hence, for the claims that are mean-ambiguity-free, the strong efficient market hypothesis holds true.

4.3.2 Weak Efficient Market Hypothesis under Knightian Uncertainty

Let $\mathcal{M}$ be a common set of priors on $(\Omega, \mathcal{F})$. Let $\mathcal{H}$ be the space of bounded, measurable functions. Let the common order be given by the quasi-sure ordering under the common set of priors $\mathcal{M}$, i.e.

$$X \leq Y \quad \Leftrightarrow \quad \mathbb{P}(X \leq Y) = 1, \ \ \forall\, \mathbb{P} \in \mathcal{M}.$$

In this case, a claim X is negligible if it vanishes $\mathcal{M}$–*quasi surely*, i.e. with probability one for all $\mathbb{P} \in \mathcal{M}$. An indicator function 1_A is thus negligible if the set A is *polar*, i.e. a null set with respect to every probability in $\mathcal{M}$. Take the set of relevant claims to be[19]

$$\mathcal{R} = \{ R \in \mathcal{P} \ : \ \exists\, \mathbb{P} \in \mathcal{M} \quad \text{such that} \quad \mathbb{P}(R > 0) > 0 \, \} \,.$$

Proposition 4.5. *Under the assumptions of this subsection, the financial market is viable if and only if there is a set of finitely additive martingale measures $\mathcal{Q}$ that has the same polar sets as the common set of priors $\mathcal{M}$.*

Proof. Suppose that the market is viable. We show that the class $\mathcal{Q}_{ac}$ from Theorem 2.2 satisfies the desired properties. The martingale property follows by definition and from the fact that $\mathcal{I}$ is a linear space. Suppose that A is polar. Then, 1_A is negligible and from the absolute continuity property, it follows $\phi(A) = 0$ for any $\phi \in \mathcal{Q}_{ac}$. On the other hand, if A is not polar, $1_A \in \mathcal{R}$ and from the full support property, it follows that there exists $\phi_A \in \mathcal{Q}_{ac}$ such that $\phi_A(A) > 0$. Thus, A is not $\mathcal{Q}_{ac}$-polar. We conclude that $\mathcal{M}$ and $\mathcal{Q}_{ac}$ share the same polar sets. For the converse implication, define $\mathcal{E}(\cdot) := \sup_{\phi \in \mathcal{Q}} \mathbb{E}_\phi[\cdot]$. Using the same argument as above, $\mathcal{E}$ is a sublinear martingale expectation with full support. From Theorem 2.2 the market is viable. $\square$

Under Knightian uncertainty, there can be indeterminacy in arbitrage–free prices as there is frequently a range of economically justifiable arbitrage–free prices. Such indeterminacy has been observed in full general equilibrium analysis as well (Rigotti and Shannon [2005], Dana and Riedel [2013], Beissner and Riedel [2019]). In this sense, Knightian uncertainty shares a similarity with incomplete markets and other frictions like transaction costs, but the economic reason for the indeterminacy is different.

4.3.3 A second-order Bayesian version of the EMH

We now consider a common order $\leq$ obtained by a second–order Bayesian approach, in the spirit of the smooth ambiguity model (Klibanoff et al. [2005]).

Let $\mathcal{F}$ be a sigma algebra on Ω and $\mathfrak{P} = \mathfrak{P}(\Omega)$ the set of all probability measures on $(\Omega, \mathcal{F})$. Let μ be a second order prior, i.e. a probability measure[20] on $\mathfrak{P}$. The common prior in this setting is given by the probability measure $\hat{\mathbb{P}} : \mathcal{F} \to [0,1]$ defined as $\hat{\mathbb{P}}(A) = \int_{\mathfrak{P}} \mathbb{P}(A)\mu(d\mathbb{P})$. Let $\mathcal{H} = \mathcal{L}^1(\Omega, \mathcal{F}, \hat{\mathbb{P}})$.

The common order is given by

$$X \leq Y \quad \Leftrightarrow \quad \mu\left(\{\mathbb{P} \in \mathfrak{P} \ : \ \mathbb{P}(X \leq Y) = 1 \, \}\right) = 1 \,.$$

A claim is positive if it is $\mathbb{P}$–almost surely nonnegative for all priors in the support of the second order prior μ. A claim is relevant if the set of beliefs

[19] These sets of positive and relevant claims can be derived from Gilboa–Schmeidler utilities. Define $X \preceq Y$ if and only if $\mathcal{E}_{\mathcal{M}}[U(X)] := \inf_{\mathbb{P} \in \mathcal{M}} E_{\mathbb{P}}[U(X)] \leq \mathcal{E}_{\mathcal{M}}[U(Y)]$ for all strictly increasing and concave real functions U. The $0 \preceq Y$ is equivalent to Y dominating the zero claim in the sense of second order stochastic dominance under all $\mathbb{P} \in \mathcal{M}$. Hence, Y is nonnegative almost surely for all $\mathbb{P} \in \mathcal{M}$.

[20] From Theorem 15.18 of Aliprantis and Border [1999], the space of probability measure is a Borel space if and only if Ω is a Borel space. This allows to define second order priors.

$\mathbb{P}$ under which the claim is strictly positive with positive probability is not negligible according to the second order prior[21].

Proposition 4.6. *Under the assumptions of this subsection, the financial market is viable if and only if there is a martingale measure $\mathbb{Q}$ that has the form*

$$\mathbb{Q}(A) = \int_{\mathfrak{P}} \int_A D \, d\mathbb{P} \, \mu(d\mathbb{P})$$

for some state price density D.

Proof. The set function $\hat{\mathbb{P}} : \mathcal{F} \to [0,1]$ defined as $\hat{\mathbb{P}}(A) = \int_{\mathfrak{P}} \mathbb{P}(A)\mu(d\mathbb{P})$ is a probability measure on $(\Omega, \mathcal{F})$. The induced $\hat{\mathbb{P}}$-a.s. order coincides with $\leq$ of this subsection. The result thus follows from Proposition 4.2 and the rules of integration with respect to $\hat{\mathbb{P}}$. □

The smooth ambiguity model thus leads to a second–order Bayesian approach for asset returns. All asset returns are equal to the safe return for some second order martingale measure; the expectation is the average expected return corresponding to a risk–neutral second order prior $\mathbb{Q}$.

4.3.4 Probability–Free Models in Mathematical Finance

We conclude this section by relating our work to recent results in Mathematical Finance. Our approach gives a microeconomic foundation to the characterization of absence of arbitrage in "robust" or "model–free" finance.

In this subsection, Ω is a metric space. We let $\leq$ be the pointwise order. In the finance literature, this approach is called *model-independent* as it does not rely on any probability measure. There is still a model, of course, given by Ω and the pointwise order.

A claim is nonnegative, $X \in \mathcal{P}$, if $X(\omega) \geq 0$ for every $\omega \in \Omega$ and $R \in \mathcal{P}^+$ if $R \in \mathcal{P}$ and there exists $\omega_0 \in \Omega$ such that $R(\omega_0) > 0$.

In the literature several different notions of arbitrage have been used. Our framework allows to unify these different approaches under one framework with the help of the notion of relevant claims[22].

We start with the following large set of relevant claims

$$\mathcal{R}_{op} := \mathcal{P}^+ = \{R \in \mathcal{P} \ : \ \exists \omega_0 \in \Omega \text{ such that } R(\omega_0) > 0 \ \}.$$

With this notion of relevance, an investment opportunity ℓ is an arbitrage if $\ell(\omega) \geq 0$ for every ω with a strict inequality for some ω, corresponding to the

[21] The order used in this section can be derived from smooth ambiguity utility functions. Define $X \leq Y$ if and only if

$$\int_{\mathfrak{P}} \psi\left(\mathbb{E}_{\mathbb{P}}[U(X)]\right) \mu(d\mathbb{P}) \leq \int_{\mathfrak{P}} \psi\left(\mathbb{E}_{\mathbb{P}}[U(X)]\right) \mu(d\mathbb{P})$$

for all strictly increasing and concave real functions U and ψ. Recall that ψ reflects uncertainty aversion. The $0 \leq Y$ is equivalent to Y dominating the zero claim in the sense of second order stochastic dominance for μ–almost all $\mathbb{P} \in \mathfrak{P}$.

[22] One might also compare the similar approach in Burzoni et al. [2016].

notion of *one point arbitrage* considered in Riedel [2015]. In this setting, no arbitrage is equivalent to the existence a set of martingale measures $\mathcal{Q}_{op}$ so that for each point there exists $\mathbb{Q} \in \mathcal{Q}_{op}$ putting positive mass to that point.

In a second example, one requires the relevant claims to be continuous, i.e.,

$$\mathcal{R}_{open} := \{ R \in C_b(\Omega) \cap \mathcal{P} \ : \ \exists \omega_0 \in \Omega \text{ such that } R(\omega_0) > 0 \ \} .$$

It is clear that when $R \in \mathcal{R}$ then it is non-zero on an open set. Hence, in this example the empty set is the only small set and the large sets are the ones that contain a non-empty open set.

Then, $\ell \in \mathcal{I}$ is an arbitrage opportunity if it is nonnegative and is strictly positive on an open set, corresponding to the notion of *open arbitrage* that appears in Burzoni et al. [2016], Riedel [2015], Dolinsky and Soner [2014b].

Acciaio et al. [2016] defines a claim to be an arbitrage when it is positive everywhere. In our context, this defines the relevant claims as those that are positive everywhere, i.e.,

$$\mathcal{R}_+ := \{ R \in \mathcal{P} \ : \ \ R(\omega) > 0, \ \forall \ \omega \in \Omega \ \} .$$

Bartl et al. [2017] consider a slightly stronger notion of relevant claims. Their choice is

$$\mathcal{R}_u = \{ R \in \mathcal{P} \ : \ \exists c \in (0, \infty) \text{ such that } R \equiv c \ \} . \tag{4.2}$$

Hence, $\ell \in \mathcal{I}$ is an arbitrage if is uniformly positive, which is sometimes called *uniform arbitrage*. Notice that with the choice $\mathcal{R}_u$, the notions of arbitrage and free lunch with vanishing risk are equivalent.

The no arbitrage condition with $\mathcal{R}_u$ is the weakest while the one with $\mathcal{R}_{op}$ is the strongest. The first one is equivalent to the existence of one sublinear martingale expectation. The latter one is equivalent to the existence of a sublinear expectation that puts positive measure to all points.

In general, the no-arbitrage condition based on $\mathcal{R}_+$ is not equivalent to the absence of uniform arbitrage. However, absence of uniform arbitrage implies the existence of a linear bounded functional that is consistent with the market. In particular, risk neutral functionals are positive on $\mathcal{R}_u$. Moreover, if the set $\mathcal{I}$ is "large" enough then one can show that the risk neutral functionals give rise to countably additive measures. In Acciaio et al. [2016], this conclusion is achieved by using the so-called "power-option" placed in the set $\mathcal{I}$ as a static hedging possibility, compare also Bartl et al. [2017].

5 Proof of the Theorems

Let $(\mathcal{H}, \tau, \leq \mathcal{I}, \mathcal{R})$ be a given financial market. Recall that $(\mathcal{H}, \tau)$ is a metrizable topological vector space; we write $\mathcal{H}'$ for its topological dual. We let $\mathcal{H}'_+$ be the set of all positive functionals, i.e., $\varphi \in \mathcal{H}'_+$ provided that $\varphi(X) \geq 0$ for every $X \geq 0$ and $X \in \mathcal{H}$.

The following functional generalizes the notion of super-replication functional from the probabilistic to our order-theoretic framework. It plays a central role in our analysis. For $X \in \mathcal{H}$, let

$$\begin{aligned}\mathcal{D}(X) := \inf\{\ c \in \mathbb{R}\ :\ \exists \{\ell_n\}_{n=1}^\infty \subset \mathcal{I},\ \{e_n\}_{n=1}^\infty \subset \mathcal{H}_+,\ e_n \xrightarrow{\tau} 0, \\ \text{such that} \quad c + e_n + \ell_n \geq X\}\,.\end{aligned} \tag{5.1}$$

Note that $\mathcal{D}$ is extended real valued. In particular, it takes the value $+\infty$ when there are no super-replicating portfolios. It might also take the value $-\infty$ if there is no lower bound.

We observe first that the absence of free lunches with vanishing risk can be equivalently described by the statement that the super-replication functional $\mathcal{D}$ assigns a strictly positive value to all relevant claims.

Proposition 5.1. *The financial market is strongly free of arbitrage if and only if $\mathcal{D}(R) > 0$ for every $R \in \mathcal{R}$.*

Proof. Suppose $\{\ell_n\}_{n=1}^\infty \subset \mathcal{I}$ is a free lunch with vanishing risk. Then, there is $R^* \in \mathcal{R}$ and $\{e_n\}_{n=1}^\infty \subset \mathcal{H}_+$ with $e_n \xrightarrow{\tau} 0$ so that $e_n + \ell_n \geq R^*$. In view of the definition, we obtain $\mathcal{D}(R^*) \leq 0$.

To prove the converse, suppose that $\mathcal{D}(R^*) \leq 0$ for some $R^* \in \mathcal{R}$. Then, the definition of $\mathcal{D}(R^*)$ implies that there is a sequence of real numbers $\{c_k\}_{k=1}^\infty$ with $c_k \downarrow \mathcal{D}(R^*)$, net trades $\{\ell_{k,n}\}_{n=1}^\infty \subset \mathcal{I}$, and $\{e_{k,n}\}_{n=1}^\infty \subset \mathcal{H}_+$ with $e_{k,n} \xrightarrow{\tau} 0$ for $n \to \infty$ such that

$$c_k + e_{k,n} + \ell_{k,n} \geq R^*, \qquad \forall\ n, k \in \mathbb{N}.$$

Let $B_r(0)$ be the ball with radius r centered at zero with the metric compatible with τ. For every k, choose $n = n(k)$ such that $e_{k,n} \in B_{\frac{1}{k}}(0)$. Set $\tilde{\ell}_k := \ell_{k,n(k)}$ and $\tilde{e}_k := e_{k,n(k)} + (c_k \vee 0)$. Then, $\tilde{e}_k + \tilde{\ell}_k \geq R^*$ for every k. Since $\tilde{e}_k \xrightarrow{\tau} 0$, $\{\tilde{\ell}_k\}_{k=1}^\infty$ is a free lunch with vanishing risk. ☐

It is clear that $\mathcal{D}$ is convex and we now use the tools of convex duality to characterize this functional in more detail. Recall the set of absolutely continuous martingale functionals $\mathcal{Q}_{ac}$ defined in Section 2.

Proposition 5.2. *Assume that the financial market is strongly free of arbitrage. Then, the super-replication functional $\mathcal{D}$ defined in (5.1) is a lower semicontinuous, sublinear martingale expectation with full support. Moreover,*

$$\mathcal{D}(X) = \sup_{\varphi \in \mathcal{Q}_{ac}} \varphi(X), \quad X \in \mathcal{H}.$$

The technical proof of this statement can be found in Appendix A. The important insight is that the super-replication functional can be described by a family of linear functionals. In the probabilistic setup, they correspond to the family of (absolutely continuous) martingale measures. With the help of this duality, we are now able to carry out the proof of our first main theorem.

Proof of Theorem 2.1. Suppose first that the market is viable and for some $R^* \in \mathcal{R}$, there are sequences $\{e_n\}_{n=1}^{\infty} \subset \mathcal{H}_+$ and $\{\ell_n\}_{n=1}^{\infty} \subset \mathcal{I}$ with $e_n \xrightarrow{\tau} 0$, and $e_n + \ell_n \geq R^*$. By viability, there is a family of agents $\{\preceq_a\}_{a \in A} \subset \mathcal{A}$ such that for some $a \in A$ we have $R^* \succ_a 0$. Since $\leq$ is a pre-order compatible with the vector space operations, we have $-e_n + R^* \leq \ell_n$. As $\preceq_a \in \mathcal{A}$ is monotone with respect to $\leq$, we have $-e_n + R^* \preceq_a \ell_n$. By optimality of the zero trade, $\ell_n \preceq_a 0$, and we get $-e_n + R^* \preceq_a 0$. By lower semi–continuity of $\preceq_a$, we conclude that $R^* \preceq_a 0$, a contradiction.

Suppose now that the market is strongly free of arbitrage. By Proposition 5.1, $\mathcal{D}(R) > 0$, for every $R \in \mathcal{R}$. In particular, this implies that the family $\mathcal{Q}_{ac}$ is non-empty, as otherwise the supremum over $\mathcal{Q}_{ac}$ would be $-\infty$. For each $\varphi \in \mathcal{Q}_{ac}$, define $\preceq_\varphi$ by,

$$X \preceq_\varphi Y, \quad \Leftrightarrow \quad \varphi(X) \leq \varphi(Y).$$

One directly verifies that $\preceq_\varphi \in \mathcal{A}$. Moreover, $\varphi(\ell) \leq \varphi(0) = 0$ for any $\ell \in \mathcal{I}$ implies that $\ell^*_\varphi = 0$ is optimal for $\preceq_\varphi$ and (2.1) is satisfied. Finally, Proposition 5.1 and Proposition 5.2 imply that for any $R \in \mathcal{R}$, there exists $\varphi \in \mathcal{Q}_{ac}$ such that $\varphi(R) > 0$; thus, (2.2) is satisfied. We deduce that $\{\preceq_\varphi\}_{\varphi \in \mathcal{Q}_{ac}}$ supports the financial market $(\mathcal{H}, \tau, \leq, \mathcal{I}, \mathcal{R})$. □

The previous arguments also imply our version of the fundamental theorem of asset pricing. In fact, with absence of arbitrage, the super-replication function is a lower semi-continuous sublinear martingale expectation with full support. Convex duality allows to prove the converse.

Proof of Theorem 2.2. Suppose the market is viable. From Theorem 2.1, it is strongly free of arbitrage. From Proposition 5.2, the super-replication functional is the desired lower semi-continuous sublinear martingale expectation with full support.

Suppose now that $\mathcal{E}$ is a lower semi-continuous sublinear martingale expectation with full support. In particular, $\mathcal{E}$ is a convex, lower semi-continuous, proper functional. Then, by the Fenchel-Moreau theorem,

$$\mathcal{E}(X) = \sup_{\varphi \in dom(\mathcal{E}^*)} \varphi(X),$$

where $dom(\mathcal{E}^*) = \{\varphi \in \mathcal{H}' \ : \ \varphi(X) \leq \mathcal{E}(X), \ \forall \ X \in \mathcal{H} \ \}$. We can proceed as in the proof of Theorem 2.1, to verify the viability of $(\mathcal{H}, \tau, \leq, \mathcal{I}, \mathcal{R})$ using the preference relations $\{\preceq_\varphi\}_{\varphi \in dom(\mathcal{E}^*)}$.

We finally show the maximality of $\mathcal{E}_{\mathcal{Q}_{ac}}$. Let $\mathcal{E}$ be a lower semi-continuous sublinear martingale expectation with full support. With the help of the martingale property of $\mathcal{E}$ one can show, as in Lemma A.5, that every $\varphi \in dom(\mathcal{E}^*)$ is a martingale functional. As $\mathcal{E}$ is monotone with respect to $\leq$, we also conclude that φ vanishes for negligible payoffs. Hence, we obtain $dom(\mathcal{E}^*) \subset \mathcal{Q}_{ac}$. From the above dual representation for $\mathcal{E}_{\mathcal{Q}_{ac}}$, $\mathcal{E}(X) \leq \mathcal{E}_{\mathcal{Q}_{ac}}(X)$ for every $X \in \mathcal{H}$ follows. □

6 Conclusion

This paper studies economic viability of a given financial market without assuming a common prior. We show that it is possible to understand the equivalence of economic viability and the absence of arbitrage on the basis solely of a common order; the order (which is typically quite incomplete) is unanimous in the sense that agents' preferences are monotone with respect to it. A given financial market is viable if and only if a *sublinear* pricing functional exists that is consistent with the given asset prices.

The properties of the common order are reflected in expected equilibrium returns. When the common order is given by the expected value under some common prior, expected returns under that prior have to be equal in equilibrium, and thus, Fama's Efficient Market Hypothesis results. If the common order is determined by the almost sure order under some common prior, we obtain the weak form of the efficient market hypothesis that states that expected returns are equal under some (martingale) measure that shares the same null sets as the common prior.

In situations of Knightian uncertainty, it might be too demanding to impose a common prior for all agents. When Knightian uncertainty is described by a class of priors, it is necessary to replace the linear (martingale) expectation by a sublinear expectation. It is then no longer possible to reach the conclusion that expected returns are equal under some probability measure. Knightian uncertainty might thus be an explanation for empirical violations of the Efficient Market Hypothesis. In particular, there is always a range of economically justifiable arbitrage–free prices. In this sense, Knightian uncertainty shares similarities with markets with friction or that are incomplete, but the economic reason for the price indeterminacy is different.

The philosophy of our approach might also be interesting for the foundations of other parts of economics; for example, one might similarly pursue an attempt to discuss the foundations of mechanism design and game theory without assuming a common prior.

A Proof of Proposition 5.2

We separate the proof in several steps. Recall that the super-replication functional $\mathcal{D}$ is defined in (5.1).

Lemma A.1. *Assume that the financial market is strongly free of arbitrage. Then, $\mathcal{D}$ is convex, lower semi–continuous and $\mathcal{D}(X) > -\infty$ for every $X \in \mathcal{H}$.*

Proof. The convexity of $\mathcal{D}$ follows immediately from the definitions. To prove lower semi-continuity, consider a sequence $X_k \xrightarrow{\tau} X$ with $\mathcal{D}(X_k) \leq c$. Then, by definition, for every k there exists a sequence $\{e_{k,n}\}_{n=1}^{\infty} \subset \mathcal{H}_+$ with $e_{k,n} \xrightarrow{\tau} 0$ for $n \to \infty$ and a sequence $\{\ell_{k,n}\}_{n=1}^{\infty} \subset \mathcal{I}$ such that $c + \frac{1}{k} + e_{k,n} + \ell_{k,n} \geq X_k$, for every k, n. Let $B_r(0)$ be the ball of radius r centered around zero in the metric compatible with τ. Choose $n = n(k)$ such that $e_{k,n} \in B_{\frac{1}{k}}$ and set $\tilde{e}_k := e_{k,n(k)}$,

$\tilde{\ell}_k := \ell_{k,n(k)}$. Then, $c + \frac{1}{k} + \tilde{e}_k + (X - X_k) + \tilde{\ell}_k \geq X$ and $\frac{1}{k} + \tilde{e}_k + (X - X_k) \xrightarrow{\mathcal{T}} 0$ as $k \to \infty$. Hence, $\mathcal{D}(X) \leq c$. This proves that $\mathcal{D}$ is lower semi-continuous.

The constant claim 1 is relevant and by Proposition 5.1, $\mathcal{D}(1) \in (0, 1]$; in particular, it is finite. Towards a counter-position, suppose that there exists $X \in \mathcal{H}$ such that $\mathcal{D}(X) = -\infty$. For $\lambda \in [0, 1]$, set $X_\lambda := X + \lambda(1 - X)$. The convexity of $\mathcal{D}$ implies that $\mathcal{D}(X_\lambda) = -\infty$ for every $\lambda \in [0, 1)$. Since $\mathcal{D}$ is lower semi-continuous, $0 < \mathcal{D}(1) \leq \lim_{\lambda \to 1} \mathcal{D}(X_\lambda) = -\infty$, a contradiction. □

Lemma A.2. *Assume that the financial market is strongly free of arbitrage. The super-replication functional $\mathcal{D}$ is a sublinear expectation with full-support. Moreover, $\mathcal{D}(c) = c$ for every $c \in \mathbb{R}$, and*

$$\mathcal{D}(X + \ell) \leq \mathcal{D}(X), \quad \forall\, \ell \in \mathcal{I},\ X \in \mathcal{H}. \tag{A.1}$$

In particular, $\mathcal{D}$ has the martingale property.

Proof. We prove this result in two steps.

Step 1. In this step we prove that $\mathcal{D}$ is a sublinear expectation. Let $X, Y \in \mathcal{H}$ such that $X \leq Y$. Suppose that there are $c \in \mathbb{R}$, $\{\ell_n\}_{n=1}^\infty \subset \mathcal{I}$ and $\{e_n\}_{n=1}^\infty \subset \mathcal{H}_+$ with $e_n \xrightarrow{\mathcal{T}} 0$ satisfying, $Y \leq c + e_n + \ell_n$. Then, from the transitivity of $\leq$, we also have $X \leq c + e_n + \ell_n$. Hence, $\mathcal{D}(X) \leq \mathcal{D}(Y)$ and consequently $\mathcal{D}$ is monotone with respect to $\leq$.

Translation-invariance, $\mathcal{D}(c + g) = c + \mathcal{D}(g)$, follows directly from the definitions.

We next show that $\mathcal{D}$ is sub-additive. Fix $X, Y \in \mathcal{H}$. Suppose that either $\mathcal{D}(X) = \infty$ or $\mathcal{D}(Y) = \infty$. Then, since by Lemma A.1 $\mathcal{D} > -\infty$, we have $\mathcal{D}(X) + \mathcal{D}(Y) = \infty$ and the sub-additivity follows directly. Now we consider the case $\mathcal{D}(X), \mathcal{D}(Y) < \infty$. Hence, there are $c_X, c_Y \in \mathbb{R}$, $\{\ell_n^X\}_{n=1}^\infty, \{\ell_n^Y\}_{n=1}^\infty \subset \mathcal{I}$ and $\{e_n^X\}_{n=1}^\infty, \{e_n^Y\}_{n=1}^\infty \subset \mathcal{H}_+$ with $e_n^X, e_n^Y \xrightarrow{\mathcal{T}} 0$ satisfying,

$$c_X + \ell_n^X + e_n^X \geq X, \quad c_Y + \ell_n^Y + e_n^Y \geq Y.$$

Set $\bar{c} := c_X + c_Y$, $\bar{\ell}_n := \ell_n^X + \ell_n^Y$, $\bar{e}_n := e_n^X + e_n^Y$. Since $\mathcal{I}$ is a positive cone, $\{\bar{\ell}_n\}_{n=1}^\infty \subset \mathcal{I}$, $\bar{e}_n \xrightarrow{\mathcal{T}} 0$ and

$$\bar{c} + \bar{e}_n + \bar{\ell}_n \geq X + Y \quad \Rightarrow \quad \mathcal{D}(X + Y) \leq \bar{c}.$$

Since this holds for any such c_X, c_Y, we conclude that

$$\mathcal{D}(X + Y) \leq \mathcal{D}(X) + \mathcal{D}(Y).$$

Finally we show that $\mathcal{D}$ is positively homogeneous of degree one. Suppose that $c + e_n + \ell_n \geq X$ for some constant c, $\{\ell_n\}_{n=1}^\infty \subset \mathcal{I}$ and $\{e_n\}_{n=1}^\infty \subset \mathcal{H}_+$ with $e_n \xrightarrow{\mathcal{T}} 0$. Then, for any $\lambda > 0$ and for any $n \in \mathbb{N}$, $\lambda c + \lambda e_n + \lambda \ell_n \geq \lambda X$. Since $\lambda \ell_n \in \mathcal{I}$ and $\lambda e_n \xrightarrow{\mathcal{T}} 0$, this implies that

$$\mathcal{D}(\lambda X) \leq \lambda\, \mathcal{D}(X), \quad \lambda > 0,\ X \in \mathcal{H}. \tag{A.2}$$

Notice that above holds trivially when $\mathcal{D}(X) = +\infty$. Conversely, if $\mathcal{D}(\lambda X) = +\infty$ we are done. Otherwise, we use (A.2) with λX and $1/\lambda$,

$$\mathcal{D}(X) = \mathcal{D}\left(\frac{1}{\lambda}\lambda X\right) \leq \frac{1}{\lambda}\mathcal{D}(\lambda X), \quad \Rightarrow \quad \lambda\mathcal{D}(X) \leq \mathcal{D}(\lambda X).$$

Hence, $\mathcal{D}$ positively homogeneous and it is a sublinear expectation.

Step 2. In this step, we assume that the financial market is strongly free of arbitrages. Since $0 \in \mathcal{I}$, we have $\mathcal{D}(0) \leq 0$. If the inequality is strict we obviously have a free lunch with vanishing risk, hence $\mathcal{D}(0) = 0$ and from translation-invariance the same applies to every $c \in \mathbb{R}$. Moreover, by Proposition 5.1, $\mathcal{D}$ has full support. Thus, we only need to prove (A.1).

Suppose that $X \in \mathcal{H}, \ell \in \mathcal{I}$ and $c + e_n + \ell_n^X \geq X$. Since $\mathcal{I}$ is a convex cone, $\ell_n^X + \ell \in \mathcal{I}$ and $c + e_n + (\ell + \ell_n^X) \geq X + \ell$. Therefore, $\mathcal{D}(X + \ell) \leq c$. Since this holds for all such constants, we conclude that $\mathcal{D}(X + \ell) \leq \mathcal{D}(X)$ for all $X \in \mathcal{H}$. In particular $\mathcal{D}(\ell) \leq 0$ and the martingale property is satisfied. $\square$

Remark A.3. Note that for $\mathcal{H} = (\mathcal{B}_b, \|\cdot\|_\infty)$, the definition of $\mathcal{D}$ reduces to the classical one:

$$\mathcal{D}(X) := \inf \{ \ c \in \mathbb{R} \ : \ \exists \ \ell \in \mathcal{I}, \text{ such that } c + \ell \geq X \ \}. \tag{A.3}$$

Indeed, if $c + \ell \geq X$ for some c and ℓ, one can use the constant sequences $\ell_n \equiv \ell$ and $e_n \equiv 0$ to get that $\mathcal{D}$ in (5.1) is less or equal than the one in (A.3). For the converse inequality observe that if $c + e_n + \ell_n \geq X$ for some c, ℓ_n and e_n with $\|e_n\|_\infty \to 0$, then the infimum in (A.3) is less or equal than c. The thesis follows. Lemma A.1 is in line with the well known fact that the classical super-replication functional in $\mathcal{B}_b$ is Lipschitz continuous with respect to the sup-norm topology.

The results of Lemma A.1 and Lemma A.2 imply that the super-replication functional defined in (5.1) is a proper convex function in the language of convex analysis, compare, e.g., Rockafellar [2015]. By the classical Fenchel-Moreau theorem, we have the following dual representation of $\mathcal{D}$,

$$\mathcal{D}(X) = \sup_{\varphi \in \mathcal{H}'} \{\varphi(X) - \mathcal{D}^*(\varphi)\}, \quad X \in \mathcal{H}, \quad \text{where}$$
$$\mathcal{D}^*(\varphi) = \sup_{Y \in \mathcal{H}} \{\varphi(Y) - \mathcal{D}(Y)\}, \quad \varphi \in \mathcal{H}'.$$

Since $\varphi(0) = \mathcal{D}(0) = 0$, $\mathcal{D}^*(\varphi) \geq \varphi(0) - \mathcal{D}(0) = 0$ for every $\varphi \in \mathcal{H}'$. However, it may take the value plus infinity. Set,

$$dom(\mathcal{D}^*) := \{ \ \varphi \in \mathcal{H}' \ : \ \mathcal{D}^*(\varphi) < \infty \}.$$

Lemma A.4. *We have*

$$dom(\mathcal{D}^*) = \left\{ \ \varphi \in \mathcal{H}'_+ \ : \ \mathcal{D}^*(\varphi) = 0 \right\} = \left\{ \ \varphi \in \mathcal{H}'_+ \ : \ \varphi(X) \leq \mathcal{D}(X), \ \ \forall \ X \in \mathcal{H} \right\}. \tag{A.4}$$

In particular,

$$\mathcal{D}(X) = \sup_{\varphi \in dom(\mathcal{D}^*)} \varphi(X), \quad X \in \mathcal{H}.$$

Furthermore, there are free lunches with vanishing risk in the financial market, whenever $dom(\mathcal{D}^*)$ is empty.

Proof. Clearly the two sets on the right of (A.4) are equal and included in $dom(\mathcal{D}^*)$. The definition of $\mathcal{D}^*$ implies that

$$\varphi(X) \leq \mathcal{D}(X) + \mathcal{D}^*(\varphi), \quad \forall\ X \in \mathcal{H},\ \varphi \in \mathcal{H}'.$$

By homogeneity,

$$\varphi(\lambda X) \leq \mathcal{D}(\lambda X) + \mathcal{D}^*(\varphi), \quad \Rightarrow \quad \varphi(X) \leq \mathcal{D}(X) + \frac{1}{\lambda}\mathcal{D}^*(\varphi),$$

for every $\lambda > 0$ and $X \in \mathcal{H}$. Suppose that $\varphi \in dom(\mathcal{D}^*)$. We then let λ go to infinity to arrive at $\varphi(X) \leq \mathcal{D}(X)$ for all $X \in \mathcal{B}_b$. Hence, $\mathcal{D}^*(\varphi) = 0$.

Fix $X \in \mathcal{H}_+$. Since $\leq$ is monotone with respect to the pointwise order, $-X \leq 0$. Then, by the monotonicity of $\mathcal{D}$, $\varphi(-X) \leq \mathcal{D}(-X) \leq \mathcal{D}(0) \leq 0$. Hence, $\varphi \in \mathcal{H}'_+$.

Now suppose that $dom(\mathcal{D}^*)$ is empty or, equivalently, $\mathcal{D}^* \equiv \infty$. Then, the dual representation implies that $\mathcal{D} \equiv -\infty$. In view of Proposition 5.1, there are free lunches with vanishing risk in the financial market. ☐

We next show that, under the assumption of absence of free lunch with vanishing risk with respect to any $\mathcal{R}$, the set $dom(\mathcal{D}^*)$ is equal to $\mathcal{Q}_{ac}$ defined in Section 2. Since any relevant set $\mathcal{R}$ by hypothesis contains $\mathcal{R}_u$ defined in (4.2), to obtain this conclusion it would be sufficient to assume the absence of free lunch with vanishing risk with respect to any $\mathcal{R}_u$.

Lemma A.5. *Suppose the financial market is strongly free of arbitrage with respect to $\mathcal{R}$. Then, $dom(\mathcal{D}^*)$ is equal to the set of absolutely continuous martingale functionals $\mathcal{Q}_{ac}$.*

Proof. The fact that $dom(\mathcal{D}^*)$ is non-empty follows from Lemma A.2 and Lemma A.4. Fix an arbitrary $\varphi \in dom(\mathcal{D}^*)$. By Lemma A.2, $\mathcal{D}(c) = c$ for every constant $c \in \mathbb{R}$. In view of the dual representation of Lemma A.4,

$$c\varphi(1) = \varphi(c) \leq \mathcal{D}(c) = c, \quad \forall c \in \mathbb{R}.$$

Hence, $\varphi(1) = 1$.

We continue by proving the monotonicity property. Suppose that $X \in \mathcal{P}$. Since $0 \in \mathcal{I}$, we obviously have $\mathcal{D}(-X) \leq 0$. The dual representation implies that $\varphi(-X) \leq \mathcal{D}(-X) \leq 0$. Thus, $\varphi(X) \geq 0$.

We now prove the supermartingale property. Let $\ell \in \mathcal{I}$. Obviously $\mathcal{D}(\ell) \leq 0$. By the dual representation, $\varphi(\ell) \leq \mathcal{D}(\ell) \leq 0$. Hence φ is a martingale functional. The absolute continuity follows as in Lemma E.3. Hence, $\varphi \in \mathcal{Q}_{ac}$.

To prove the converse, fix an arbitrary $\varphi \in \mathcal{Q}_{ac}$. Suppose that $X \in \mathcal{H}$, $c \in \mathbb{R}$, $\{\ell_n\}_{n=1}^{\infty} \subset \mathcal{I}$ and $\{e_n\}_{n=1}^{\infty} \subset \mathcal{H}_+$ with $e_n \xrightarrow{\mathcal{T}} 0$ satisfy, $c + e_n + \ell_n \geq X$. From the properties of φ,

$$0 \leq \varphi(c + e_n + \ell_n - X) = \varphi(c + e_n - X) + \varphi(\ell_n) \leq c - \varphi(X - e_n).$$

Since $e_n \xrightarrow{\mathcal{T}} 0$ and φ is continuous, $\varphi(X) \leq \mathcal{D}(X)$ for every $X \in \mathcal{H}$. Therefore, $\varphi \in dom(\mathcal{D}^*)$. ☐

Proof of Proposition 5.2. It follows directly from Lemma A.4 and Lemma A.5. ☐

We have the following immediate corollary, which states that is the first part of the Fundamental Theorem of Asset Pricing in this context.

Corollary A.6. *The financial market is strongly free of arbitrage if and only $\mathcal{Q}_{ac} \neq \emptyset$ and for any $R \in \mathcal{R}$, there exists $\varphi_R \in \mathcal{Q}_{ac}$ such that $\varphi_R(R) > 0$.*

Proof. By contradiction, suppose that there exists R^* such that $e_n + \ell_n \geq R^*$ with $e_n \xrightarrow{\mathcal{T}} 0$. Take φ_{R^*} such that $\varphi_{R^*}(R^*) > 0$ and observe that $0 < \varphi_{R^*}(R^*) \leq \varphi(e_n + \ell_n) \leq \varphi(e_n)$. Since $\varphi \in \mathcal{H}'_+$, $\varphi(e_n) \to 0$ as $n \to \infty$, which is a contradiction.

In the other direction, assume that the financial market is strongly free of arbitrage. By Lemma A.5, $dom(\mathcal{D}^*) = \mathcal{Q}_{ac}$. Let $R \in \mathcal{R}$ and note that, by Proposition 5.1, $\mathcal{D}(R) > 0$. It follows that there exists $\varphi_R \in dom(\mathcal{D}^*) = \mathcal{Q}_{ac}$ satisfying $\varphi_R(R) > 0$. ☐

Remark A.7. The set of positive functionals $\mathcal{Q}_{ac} \subset \mathcal{H}'_+$ is the analogue of the set of local martingale measures of the classical setting. Indeed, all elements of $\varphi \in \mathcal{Q}_{ac}$ can be regarded as supermartingale "measures", since $\varphi(\ell) \leq 0$ for every $\ell \in \mathcal{I}$. Moreover, the property $\varphi(Z) = 0$ for every $Z \in \mathcal{Z}$ can be regarded as absolute continuity with respect to null sets. The full support property is our analog to the converse absolute continuity. However, the full-support property cannot be achieved by a single element of $\mathcal{Q}_{ac}$.

Bouchard and Nutz [2015] study arbitrage for a set of priors $\mathcal{M}$. The absolute continuity and the full support properties then translate to the statement that "$\mathcal{M}$ and $\mathcal{Q}$ have the same polar sets". In the paper by Burzoni et al. [2016], a class of relevant sets $\mathcal{S}$ is given and the two properties can summarised by the statement "the set $\mathcal{S}$ is not contained in the polar sets of $\mathcal{Q}$".

Also, when $\mathcal{H} = \mathcal{B}_b$, $\mathcal{H}'$ is the class of bounded additive measures ba. It is a classical question whether one can restrict $\mathcal{Q}$ to the set of countable additive measures $ca_r(\Omega)$. In several of the examples described in Section 3 and 4 this is proved. However, there are examples for which this is not true.

B Linearly Growing Claims

Let $\mathcal{B}(\Omega, \mathcal{F})$ be the set of all $\mathcal{F}$ measurable real-valued functions on Ω. Any Banach space contained in $\mathcal{B}(\Omega, \mathcal{F})$ satisfies the requirements for $\mathcal{H}$. In our examples, we used the spaces $\mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$, $\mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$, $\mathcal{L}^1(\Omega, \mathcal{F}, \mathcal{M})$ (defined in the subsection 4.3.1) and $\mathcal{B}_b(\Omega, \mathcal{F})$, the set of all bounded functions in $\mathcal{B}(\Omega, \mathcal{F})$, with the supremum norm. In the latter case, the super-hedging functional enjoys several properties as discussed in Remark A.3.

Since we require that $\mathcal{I} \subset \mathcal{H}$ (see Section 2), in the case of $\mathcal{H} = \mathcal{B}_b(\Omega, \mathcal{F})$ this means that all the trading instruments are bounded. This could be restrictive in some applications and we now provide another example that overcomes this difficulty. To define this set, fix $L^* \in \mathcal{B}(\Omega, \mathcal{F})$ with $L^*(\omega) \geq 1$ for every $\omega \in \Omega$. Consider the linear space

$$\mathcal{B}_\ell := \left\{X \in \mathcal{B}(\Omega, \mathcal{F}) \ : \ \exists\ \alpha \in \mathbb{R}^+ \text{ such that } |X(\omega)| \leq \alpha L^*(\omega)\ \forall \omega \in \Omega\right\}$$

equipped with the norm,

$$\|X\|_\ell := \inf\{\alpha \in \mathbb{R}^+ \ : \ |X(\omega)| \leq \alpha L^*(\omega)\ \forall \omega \in \Omega\} = \left\| \frac{X}{L^*} \right\|_\infty .$$

We denote the topology induced by this norm by τ_ℓ. Then, $\mathcal{B}_\ell(\Omega, \mathcal{F})$ with τ_ℓ is a Banach space and satisfies our assumptions. Note that if $L^* = 1$, then $\mathcal{B}_\ell(\Omega, \mathcal{F}) = \mathcal{B}_b(\Omega, \mathcal{F})$.

Now, suppose that

$$L^*(\omega) := c^* + \hat{\ell}(\omega), \qquad \omega \in \Omega, \tag{B.1}$$

for some $c^* > 0$, $\hat{\ell} \in \mathcal{I}$. Then, one can define the super-replication functional as in (A.3).

C No Arbitrage versus No Free-Lunch-with-Vanishing-Risk

Let $(\mathcal{H}, \tau, \leq, \mathcal{I}, \mathcal{R})$ be a financial market. An arbitrage opportunity is always a free lunch with vanishing risk (refer to Section 2 for the definition of the concepts). The purpose of this section is to investigate when these two notions are equivalent.

C.1 Attainment

Definition C.1. We say that a financial market has the *attainment property*, if for every $X \in \mathcal{H}$ there exists a minimizer in (5.1), i.e., there exists $\ell_X \in \mathcal{I}$ satisfying,

$$\mathcal{D}(X) + \ell_X \geq X.$$

Proposition C.2. *Suppose that a financial market has the attainment property. Then, it is strongly free of arbitrage if and only if it has no arbitrages.*

Proof. Let $R^* \in \mathcal{R}$. By hypothesis, there exist $\ell \in \mathcal{I}^*$ so that $\mathcal{D}(R^*) + \ell^* \geq R^*$. If the market has no arbitrage, then we conclude that $\mathcal{D}(R^*) > 0$. In view of Proposition 5.1, this proves that the financial market is also strongly free of arbitrage. Since no arbitrage is weaker condition, they are equivalent. ☐

C.2 Finite discrete time markets

In this subsection and in the next section, we restrict ourselves to arbitrage considerations in finite discrete-time markets.

We start by introducing a discrete filtration $\mathbb{F} := (\mathcal{F}_t)_{t=0}^T$ on subsets of Ω. Let $S = (S_t)_{t=0}^T$ be an adapted stochastic process[23,24] with values in $\mathbb{R}_+^M$ for some M. For every $\ell \in \mathcal{I}$ there exist predictable integrands $H_t \in \mathcal{B}_b(\Omega, \mathcal{F}_{t-1})$ for all $t = 1, \ldots, T$ such that,

$$\ell = (H \cdot S)_T := \sum_{t=1}^T H_t \cdot \Delta S_t, \quad \text{where} \quad \Delta S_t := (S_t - S_{t-1}).$$

Denote by $\ell_t := (H \cdot S)_t$ for $t \in \mathcal{I}$ and $\ell := \ell_T$.

Set $\hat{\ell} = \sum_{k,i} S_k^i - S_0^i$. Then, one can directly show that with an appropriate c^*, we have $L^* := c^* + \hat{\ell} \geq 1$. Define $\mathcal{B}_\ell$ using $\hat{\ell}$, set $\mathcal{H} = \mathcal{B}_\ell$ and denote by $\mathcal{I}_\ell$ the subset of $\mathcal{I}$ with H_t *bounded* for every $t = 1, \ldots, T$.

We next prescribe the equivalence relation and the relevant sets. Our starting point is the set of negligible sets $\mathcal{Z}$ which we assume is given. We also make the following structural assumption.

Assumption C.3. Assume that the trading is allowed only at finite time points labeled through $1, 2, \ldots, T$. Let $\mathcal{I}$ be given as above and let $\mathcal{Z}$ be a lattice which is closed with respect to pointwise convergence.

We also assume that $\mathcal{R} = \mathcal{P}^+$ and the pre-order is given by,

$$X \leq Y \quad \Leftrightarrow \quad \exists Z \in \mathcal{Z} \quad \text{such that} \quad X \leq_\Omega Y + Z,$$

where $\leq_\Omega$ denotes the pointwise order of functions. In particular, $X \in \mathcal{P}$ if and only if there exists $Z \in \mathcal{Z}$ such that $Z \leq_\Omega X$.

An example of the above structure is the Example 4.3.2. In that example, $\mathcal{Z}$ is polar sets of a given class $\mathcal{M}$ of probabilities. Then, in this context all inequalities should be understood as $\mathcal{M}$ quasi-surely. Also note also that the

[23] When working with N stocks, a canonical choice for Ω would be

$$\Omega = \{\omega = (\omega_0, \ldots, \omega_T) \ : \ \omega_i \in [0, \infty)^N, \ i = 0, \ldots, T \ \}.$$

Then, one may take $S_t(\omega) = \omega_t$ and $\mathbb{F}$ to be the filtration generated by S.

[24] Note that we do not specify any probability measure.

assumptions on $\mathcal{Z}$ are trivially satisfied when $\mathcal{Z} = \{0\}$. In this latter case, inequalities are pointwise.

Observe that in view of the definition of $\leq$ and the fact $\mathcal{R} = \mathcal{P}^+$, $\ell \in \mathcal{I}$ is an arbitrage if and only if there is $R^* \in \mathcal{P}^+$ and $Z^* \in \mathcal{Z}$, so that $\ell \geq_\Omega R^* + Z^*$. Hence, $\ell \in \mathcal{I}$ is an arbitrage if and only if $\ell \in \mathcal{P}^+$. We continue by showing the equivalence of the existence of an arbitrage to the existence of a one-step arbitrage.

Lemma C.4. *Suppose that Assumption C.3 holds. Then, there exists arbitrage if and only if there exists $t \in \{1, \dots, T\}$, $h \in \mathcal{B}_b(\Omega, \mathcal{F}_{t-1})$ such that $\ell := h \cdot \Delta S_t$ is an arbitrage.*

Proof. The sufficiency is clear. To prove the necessity, suppose that $\ell \in \mathcal{I}$ is an arbitrage. Then, there is a predictable process H so that $\ell = (H \cdot S)_T$. Also $\ell \in \mathcal{P}^+$, hence, $\ell \notin \mathcal{Z}$ and there exists $Z \in \mathcal{Z}$ such that $\ell \geq Z$. Define

$$\hat{t} := \min\{t \in \{1, \dots, T\} \ : \ (H \cdot S)_t \in \mathcal{P}^+ \ \} \leq T.$$

First we study the case where $\ell_{\hat{t}-1} \in \mathcal{Z}$. Define

$$\ell^* := H_{\hat{t}} \cdot \Delta S_{\hat{t}},$$

and observe that $\ell_{\hat{t}} = \ell_{\hat{t}-1} + \ell^*$. Since $\ell_{\hat{t}-1} \in \mathcal{Z}$, we have that $\ell^* \in \mathcal{P}^+$ iff $\ell_{\hat{t}} \in \mathcal{P}^+$ and consequently the lemma is proved.

Suppose now $\ell_{\hat{t}-1} \notin \mathcal{Z}$. If $\ell_{\hat{t}-1} \geq_\Omega 0$, then $\ell_{\hat{t}-1} \in \mathcal{P}$ and, thus, also in $\mathcal{P}^+$, which is not possible from the minimality of $\hat{t}$. Hence the set $A := \{\ell_{\hat{t}-1} <_\Omega 0\}$ is non empty and $\mathcal{F}_{\hat{t}-1}$-measurable. Define, $h := H_{\hat{t}} \chi_A$ and $\ell^* := h \cdot \Delta S_{\hat{t}}$. Note that,

$$\ell^* = \chi_A(\ell_{\hat{t}} - \ell_{\hat{t}-1}) \geq_\Omega \chi_A \ell_{\hat{t}} \geq_\Omega \chi_A Z \in \mathcal{Z}.$$

This implies $\ell^* \in \mathcal{P}$. Towards a contradiction, suppose that $\ell^* \in \mathcal{Z}$. Then,

$$\ell_{\hat{t}-1} \geq_\Omega \chi_A \ell_{\hat{t}-1} \geq \chi_A (Z - \ell^*) \in \mathcal{Z},$$

Since, by assumption, $\ell_{\hat{t}-1} \notin \mathcal{Z}$ we have $\ell_{\hat{t}-1} \in \mathcal{P}^+$ from which $\hat{t}$ is not minimal. ☐

The following is the main result of this section. For the proof we follow the approach of Kabanov and Stricker [2001] which is also used in Bouchard and Nutz [2015]. We consider the financial market $\Theta_* = (\mathcal{B}_\ell, \|\cdot\|_\ell, \leq_\Omega, \mathcal{I}, \mathcal{P}^+)$ described above.

Theorem C.5. *In a finite discrete time financial market satifying the Assumption C.3, the following are equivalent:*

1. *The financial market Θ_* has no arbitrages.*

2. *The attainment property holds and Θ_* is free of arbitrage.*

3. *The financial market Θ_* is strongly free of arbitrages.*

Proof. In view of Proposition C.2 we only need to prove the implication 1 ⇒ 2.

For $X \in \mathcal{H}$ such that $\mathcal{D}(X)$ is finite we have that

$$c_n + \mathcal{D}(H) + \ell_n \geq_\Omega X + Z_n,$$

for some $c_n \downarrow 0$, $\ell_n \in \mathcal{I}$ and $Z_n \in \mathcal{Z}$. Note that since $\mathcal{Z}$ is a lattice we assume, without loss of generality, that $Z_n = (Z_n)^-$ and denote by $\mathcal{Z}^- := \{Z^- \mid Z \in \mathcal{Z}\}$.

We show that $\mathcal{C} := \mathcal{I} - (\mathcal{L}^0_+(\Omega, \mathcal{F}) + \mathcal{Z}^-)$ is closed under pointwise convergence where $\mathcal{L}^0_+(\Omega, \mathcal{F})$ denotes the class of pointwise nonnegative random variables. Once this result is shown, by observing that $X - c_n - \mathcal{D}(X) = W_n \in \mathcal{C}$ converges pointwise to $X - \mathcal{D}(X)$ we obtain the attainment property.

We proceed by induction on the number of time steps. Suppose first $T = 1$. Let

$$W_n = \ell_n - K_n - Z_n \to W, \tag{C.1}$$

where $\ell_n \in \mathcal{I}$, $K_n \geq_\Omega 0$ and $Z_n \in \mathcal{Z}^-$. We need to show $W \in \mathcal{C}$. Note that any ℓ_n can be represented as $\ell_n = H_1^n \cdot \Delta S_1$ with $H_1^n \in \mathcal{L}^0(\Omega, \mathcal{F}_0)$.

Let $\Omega_1 := \{\omega \in \Omega \mid \liminf |H_1^n| < \infty\}$. From Lemma 2 in Kabanov and Stricker [2001] there exist a sequence $\{\tilde{H}_1^k\}$ such that $\{\tilde{H}_1^k(\omega)\}$ is a convergent subsequence of $\{H_1^k(\omega)\}$ for every $\omega \in \Omega_1$. Let $H_1 := \liminf H_1^n\ \chi_{\Omega^1}$ and $\ell := H_1 \cdot \Delta S_1$.

Note now that $Z_n \leq_\Omega 0$, hence, if $\liminf |Z_n| = \infty$ we have $\liminf Z_n = -\infty$. We show that we can choose $\tilde{Z}_n \in \mathcal{Z}^-$, $\tilde{K}_n \geq_\Omega 0$ such that $\tilde{W}_n := \ell_n - \tilde{K}_n - \tilde{Z}_n \to W$ and $\liminf \tilde{Z}_n$ is finite on Ω_1. On $\{\ell_n \geq_\Omega W\}$ set $\tilde{Z}_n = 0$ and $\tilde{K}_n = \ell_n - W$. On $\{\ell_n <_\Omega W\}$ set

$$\tilde{Z}_n = Z_n \vee (\ell_n - W), \quad \tilde{K}_n = K_n \chi_{\{Z_n = \tilde{Z}_n\}}.$$

It is clear that $Z_n \leq_\Omega \tilde{Z}_n \leq_\Omega 0$. From Lemma E.1 we have $\tilde{Z}_n \in \mathcal{Z}$. Moreover, it is easily checked that $\tilde{W}_n := \ell_n - \tilde{K}_n - \tilde{Z}_n \to W$. Nevertheless, from the convergence of ℓ_n on Ω_1 and $\tilde{Z}_n \geq_\Omega -(W - \ell_n)^+$, we obtain $\{\omega \in \Omega_1 \mid \liminf \tilde{Z}_n > -\infty\} = \Omega_1$. As a consequence also $\liminf \tilde{K}_n$ is finite on Ω_1, otherwise we could not have that $\tilde{W}_n \to W$. Thus, by setting $\tilde{Z} := \liminf \tilde{Z}_n$ and $\tilde{K} := \liminf \tilde{K}_n$, we have $W = \ell - \tilde{K} - \tilde{Z} \in \mathcal{C}$.

On Ω_1^C we may take $G_1^n := H_1^n / |H_1^n|$ and let $G_1 := \liminf G_1^n \chi_{\Omega_1^C}$. Define, $\ell_G := G_1 \cdot \Delta S_1$. We now observe that,

$$\{\omega \in \Omega_1^C \mid \ell_G(\omega) \leq 0\} \subseteq \{\omega \in \Omega_1^C \mid \liminf Z_n(\omega) = -\infty\}.$$

Indeed, if $\omega \in \Omega_1^C$ is such that $\liminf Z_n(\omega) > -\infty$, applying again Lemma 2 in Kabanov and Stricker [2001], we have that

$$\liminf_{n \to \infty} \frac{X(\omega) + Z_n(\omega)}{|H_1^n(\omega)|} = 0,$$

implying $\ell_G(\omega)$ is nonnegative. Set now

$$\tilde{Z}_n := Z_n \vee -(\ell_G)^-.$$

From $Z_n \leq_\Omega \tilde{Z}_n \leq_\Omega 0$, again by Lemma E.1, $\tilde{Z}_n \in \mathcal{Z}$. By taking the limit for $n \to \infty$ we obtain $(\ell_G)^- \in \mathcal{Z}$ and thus, $\ell_G \in \mathcal{P}$. Since the financial market has no arbitrages $G_1 \cdot \Delta S_1 = Z \in \mathcal{Z}$ and hence one asset is redundant. Consider a partition Ω_2^i of Ω_1^C on which $G_1^i \neq 0$. Since $\mathcal{Z}$ is stable under multiplication (Lemma E.2), for any $\ell^* \in \mathcal{I}$, there exists $Z^* \in \mathcal{Z}$ and $H^* \in \mathcal{L}^0(\Omega_2^i, \mathcal{F}_0)$ with $(H^*)^i = 0$, such that $\ell^* = H^* \cdot \Delta S_1 + Z^*$ on Ω_2^i. Therefore, the term ℓ_n in (C.1) is composed of trading strategies involving only $d-1$ assets. Iterating the procedure up to d-steps we have the conclusion.

Assuming now that C.1 holds for markets with $T-1$ periods, with the same argument we show that we can extend to markets with T periods. Set again $\Omega_1 := \{\omega \in \Omega \mid \liminf |H_1^n| < \infty\}$. Since on Ω_1 we have that,

$$W_n - H_1^n \cdot \Delta S_1 = \sum_{t=2}^{T} H_t^n \cdot \Delta S_t - K_n - Z_n \to W - H_1 \cdot \Delta S_1.$$

The induction hypothesis allows to conclude that $W - H_1 \cdot S_1 \in \mathcal{C}$ and therefore $W \in \mathcal{C}$. On Ω_1^C we may take $G_1^n := H_1^n / |H_1^n|$ and let $G_1 := \liminf G_1^n \chi_{\Omega_1^C}$. Note that $W_n / |H_1^n| \to 0$ and hence

$$\sum_{t=2}^{T} \frac{H_t^n}{|H_1^n|} \cdot \Delta S_t - \frac{K_n}{|H_1^n|} - \frac{Z_n}{|H_1^n|} \to -G_1 \cdot \Delta S_1.$$

Since $\mathcal{Z}$ is stable under multiplication $\frac{Z_n}{|H_1^n|} \in \mathcal{Z}$ and hence, by inductive hypothesis, there exists $\tilde{H}_t$ for $t = 2, \dots, T$ and $\tilde{Z} \in \mathcal{Z}$ such that

$$\tilde{\ell} := G_1 \cdot \Delta S_1 + \sum_{t=2}^{T} \tilde{H}_t \cdot \Delta S_t \geq_\Omega \tilde{Z} \in \mathcal{Z}.$$

The No Arbitrage condition implies that $\tilde{\ell} \in \mathcal{Z}$. Once again, this means that one asset is redundant and, by considering a partition Ω_2^i of Ω_1^C on which $G_1^i \neq 0$, we can rewrite the term ℓ_n in (C.1) with $d-1$ assets. Iterating the procedure up to d-steps we have the conclusion. $\square$

The above result is consistent with the fact that in classical "probabilistic" model for finite discrete-time markets only the no-arbitrage condition and not the no-free lunch condition has been utilized.

D Countably Additive Measures

In this section, we show that in general finite discrete time markets, it is possible to characterize viability through countably additive functionals. Also in this section, $\leq_\Omega$ denotes the pointwise order for functions. We prove this result by combining some results from Burzoni et al. [2019] which we collect in Appendix E.2. We refer to that paper for the precise technical requirements for $(\Omega, \mathbb{F}, S)$, we only point out that, in addition to the previous setting, Ω needs to be a Polish space.

We let $\mathcal{Q}^{ca}$ be the set of countably additive positive probability measures $\mathbb{Q}$, with finite support, such that S is a $\mathbb{Q}$-martingale and $\mathcal{Z}^- := \{-Z^- \mid Z \in \mathcal{Z}\}$. For $X \in \mathcal{H}$, set

$$\mathcal{Z}(X) := \left\{ Z \in \mathcal{Z}^- \ : \ \exists \ell \in \mathcal{I} \quad \text{such that} \quad \mathcal{D}(X) + \ell \geq_\Omega X + Z \ \right\},$$

which is always non-empty when $\mathcal{D}(X)$, e.g. $\forall X \in \mathcal{B}_b$. By the lattice property of $\mathcal{Z}$, if $\mathcal{D}(X) + \ell \geq_\Omega X + Z$ the same is true if we take $Z = Z^-$. From Theorem C.5 we know that, under no arbitrage, the attainment property holds and, hence, $\mathcal{Z}(X)$ is non-empty for every $X \in \mathcal{H}$. For $A \in \mathcal{F}$, we define

$$\begin{aligned} \mathcal{D}_A(X) &:= \inf \{ c \in \mathbb{R} \ : \exists \ell \in \mathcal{I} \quad \text{such that} \quad c + \ell(\omega) \geq X(\omega), \ \forall \omega \in A \ \} \\ \mathcal{Q}^{ca}_A &:= \{ \mathbb{Q} \in \mathcal{Q}^{ca} \ : \ \mathbb{Q}(A) = 1 \ \}. \end{aligned}$$

We need the following technical result in the proof of the main Theorem.

Proposition D.1. *Suppose Assumption C.3 holds and the financial market has no arbitrages. Then, for every $X \in \mathcal{H}$ and $Z \in \mathcal{Z}(X)$, there exists $A_{X,Z}$ such that*

$$A_{X,Z} \subset \{ \ \omega \in \Omega \ : \ Z(\omega) = 0 \ \}, \tag{D.1}$$

and

$$\mathcal{D}(X) = \mathcal{D}_{A_{X,Z}}(X) = \sup_{\mathbb{Q} \in \mathcal{Q}^{ca}_{A_{X,Z}}} \mathbb{E}_{\mathbb{Q}}[X].$$

Before proving this result, we state the main result of this section.

Theorem D.2. *Suppose Assumption C.3 holds. Then, the financial market has no arbitrages if and only if for every $(Z, R) \in \mathcal{Z}^- \times \mathcal{P}^+$ there exists $\mathbb{Q}_{Z,R} \in \mathcal{Q}^{ca}$ satisfying*

$$\mathbb{E}_{\mathbb{Q}_{Z,R}}[R] > 0 \quad \textit{and} \quad \mathbb{E}_{\mathbb{Q}_{Z,R}}[Z] = 0. \tag{D.2}$$

Proof. Suppose that the financial market has no arbitrages. Fix $(Z, R) \in \mathcal{Z}^- \times \mathcal{P}^+$ and $Z_R \in \mathcal{Z}(R)$. Set $Z^* := Z_R + Z \in \mathcal{Z}(R)$. By Proposition D.1, there exists $A_* := A_{R,Z^*}$ satisfying the properties listed there. In particular,

$$0 < \mathcal{D}(R) = \sup_{\mathbb{Q} \in \mathcal{Q}^{ca}_{A_*}} \mathbb{E}_{\mathbb{Q}}[R].$$

Hence, there is $\mathbb{Q}^* \in \mathcal{Q}^{ca}_{A_*}$ so that $\mathbb{E}_{\mathbb{Q}^*}[R] > 0$. Moreover, since $Z_R, Z \in \mathcal{Z}^-$,

$$A_* \subset \{Z^* = 0\} = \{Z_R = 0\} \cap \{Z = 0\}.$$

In particular, $\mathbb{E}_{\mathbb{Q}^*}[Z] = 0$.

To prove the opposite implication, suppose that there exists $R \in \mathcal{P}^+$, $\ell \in \mathcal{I}$ and $Z \in \mathcal{Z}$ such that $\ell \geq_\Omega R + Z$. Then, it is clear that $\ell \geq_\Omega R - Z^-$. Let $\mathbb{Q}^* := \mathbb{Q}_{-Z^-,R} \in \mathcal{Q}^{ca}$ satisfying (D.2). By integrating both sides against $\mathbb{Q}^*$, we obtain

$$0 = \mathbb{E}_{\mathbb{Q}^*}[\ell] \geq \mathbb{E}_{\mathbb{Q}^*}[R - Z^-] = \mathbb{E}_{\mathbb{Q}^*}[R] > 0.$$

which is a contradiction. Thus, there are no arbitrages. □

We continue with the proof of Proposition D.1.

Proof of Proposition D.1. Since there are no arbitrages, by Theorem C.5 we have the attainment property. Hence, for a given $X \in \mathcal{H}$, the set $\mathcal{Z}(X)$ is non-empty.

Step 1. We show that, for any $Z \in \mathcal{Z}(X)$, $\mathcal{D}(X) = \mathcal{D}_{\{Z=0\}}(X)$.
Note that, since $\mathcal{D}(X)+\ell \geq_\Omega X+Z$, for some $\ell \in \mathcal{I}$, the inequality $\mathcal{D}_{\{Z=0\}}(X) \leq \mathcal{D}(X)$ is always true. Towards a contradiction, suppose that the inequality is strict, namely, there exist $c < \mathcal{D}(X)$ and $\tilde{\ell} \in \mathcal{I}$ such that $c + \tilde{\ell}(\omega) \geq X(\omega)$ for any $\omega \in \{Z = 0\}$. We show that

$$\tilde{Z} := (c + \tilde{\ell} - X)^{-} \chi_{\{Z<0\}} \in \mathcal{Z}.$$

This together with $c + \tilde{\ell} \geq_\Omega X + \tilde{Z}$ yields a contradiction. Recall that $\mathcal{Z}$ is a linear space so that $nZ \in \mathcal{Z}$ for any $n \in \mathbb{N}$. From $nZ \leq_\Omega \tilde{Z} \vee (nZ) \leq_\Omega 0$, we also have $\tilde{Z}_n := \tilde{Z} \vee (nZ) \in \mathcal{Z}$, by Lemma E.1. By noting that $\{\tilde{Z} < 0\} \subset \{Z < 0\}$ we have that $\tilde{Z}_n(\omega) \to \tilde{Z}(\omega)$ for every $\omega \in \Omega$. From the closure of $\mathcal{Z}$ under pointwise convergence, we conclude that $\tilde{Z} \in \mathcal{Z}$.

Step 2. For a given set $A \in \mathcal{F}_T$, we let $A^* \subset A$ be the set of scenarios visited by martingale measures (see (E.2) in the Appendix for more details). We show that, for any $Z \in \mathcal{Z}(X)$, $\mathcal{D}(X) = \mathcal{D}_{\{Z=0\}^*}(X)$.

Suppose that $\{Z = 0\}^*$ is a proper subset of $\{Z = 0\}$ otherwise, from Step 1, there is nothing to show. From Lemma E.6 there is a strategy $\tilde{\ell} \in \mathcal{I}$ such that $\tilde{\ell} \geq 0$ on $\{Z = 0\}$[25]. Lemma E.5 (and in particular (E.4)) yields a finite number of strategies $\ell^t_1, \dots \ell^t_{\beta_t}$ with $t = 1, \dots T$, such that

$$\{\hat{Z} = 0\} = \{Z = 0\}^* \qquad \text{where} \qquad \hat{Z} := Z - \sum_{t=1}^{T} \sum_{i=1}^{\beta_t} \chi_{\{Z=0\}} (\ell^t_i)^+ \ . \tag{D.3}$$

Moreover, for any $\omega \in \{Z = 0\} \setminus \{Z = 0\}^*$, there exists (i, t) such that $\ell^t_i(\omega) > 0$. We are going to show that, under the no arbitrage hypothesis, $\ell^t_i \in \mathcal{Z}$ for any $i = 1, \dots \beta_t$, $t = 1, \dots T$. In particular, from the lattice property of the linear space $\mathcal{Z}$, we have $\hat{Z} \in \mathcal{Z}$.

We illustrate the reason for $t = T$, by repeating the same argument up to $t = 1$ we have the thesis. We proceed by induction on i. Start with $i = 1$. From Lemma E.5 we have that $\ell^T_i \geq 0$ on $\{Z = 0\}$ and, therefore, $\{\ell^T_1 < 0\} \subseteq \{Z < 0\}$. Define $\tilde{Z} := -(\ell^T_1)^- \leq_\Omega 0$. By using the same argument as in Step 1, we observe that $nZ \leq_\Omega \tilde{Z} \vee (nZ) \leq_\Omega 0$ with $nZ \in \mathcal{Z}$ for any $n \in \mathbb{N}$. From $\{\ell^T_1 < 0\} \subseteq \{Z < 0\}$ and the closure of $\mathcal{Z}$ under pointwise convergence, we conclude that $\tilde{Z} \in \mathcal{Z}$. From no arbitrage, we must have $\ell^T_1 \in \mathcal{Z}$.
Suppose now that $\ell^T_j \in \mathcal{Z}$ for every $1 \leq j \leq i-1$. From Lemma E.5, we have that $\ell^T_i \geq 0$ on $\{Z - \sum_{j=1}^{i-1} \ell^T_i = 0\}$ and, therefore, $\{\ell^T_i < 0\} \subseteq \{Z - \sum_{j=1}^{i-1} \ell^T_i < 0\}$.

[25] Note that restricted to $\{Z = 0\}$ this strategy yields no risk and possibly positive gains, in other words, this is a good candidate for being an arbitrage.

The argument of Step 1 allows to conclude that $\ell_i^T \in \mathcal{Z}$.

We are now able to show the claim. The inequality $\mathcal{D}_{\{Z=0\}^*}(X) \leq \mathcal{D}_{\{Z=0\}}(X) = \mathcal{D}(X)$ is always true. Towards a contradiction, suppose that the inequality is strict, namely, there exist $c < \mathcal{D}(X)$ and $\tilde{\ell} \in \mathcal{I}$ such that $c + \tilde{\ell}(\omega) \geq X(\omega)$ for any $\omega \in \{Z = 0\}^*$. We show that

$$\tilde{Z} := (c + \tilde{\ell} - X)^- \chi_{\Omega \setminus \{Z=0\}^*} \in \mathcal{Z}.$$

This together with $c + \tilde{\ell} \geq_\Omega X + \tilde{Z}$, yields a contradiction. To see this recall that, from the above argument, $\hat{Z} \in \mathcal{Z}$ with $\hat{Z}$ as in (D.3). Moreover, again by (D.3), we have $\{\tilde{Z} < 0\} \subset \{\hat{Z} < 0\}$. The argument of Step 1 allows to conclude that $\tilde{Z} \in \mathcal{Z}$.

Step 3. We are now able to conclude the proof. Fix $Z \in \mathcal{Z}(X)$ and set $A_{X,Z} := \{Z = 0\}^*$. Then,

$$\mathcal{D}(X) = \mathcal{D}_{\{Z=0\}}(X) = \mathcal{D}_{(A_{X,Z})^*}(X) = \sup_{Q \in \mathcal{Q}^{ca}_{A_{X,Z}}} \mathbb{E}_Q[X],$$

where the first two equalities follow from Step 1 and Step 2 and the last equality follows from Proposition E.7. $\square$

E Some technical tools

E.1 Preferences

We start with a simple but a useful condition for negligibility.

Lemma E.1. *Consider two negligible claims $\hat{Z}, \tilde{Z} \in \mathcal{Z}$. Then, any claim $Z \in \mathcal{H}$ satisfying $\hat{Z} \leq Z \leq \tilde{Z}$ is negligible as well.*

Proof. By definitions, we have,

$$X \preceq X + \hat{Z} \preceq X + Z \preceq X + \tilde{Z} \preceq X \quad \Rightarrow \quad X \sim X + Z.$$

Thus, $Z \in \mathcal{Z}$. $\square$

Lemma E.2. *Suppose that $\mathcal{Z}$ is closed under pointwise convergence. Then, $\mathcal{Z}$ is stable under multiplication, i.e., $ZH \in \mathcal{Z}$ for any $H \in \mathcal{H}$.*

Proof. Note first that $Z_n := Z((H \wedge n) \vee -n) \in \mathcal{Z}$. This follows from by Lemma E.1 and the fact that $\mathcal{Z}$ is a cone. By taking the limit for $n \to \infty$, the result follows. $\square$

We next prove that $\mathcal{E}(Z) = 0$ for every $Z \in \mathcal{Z}$.

Lemma E.3. *Let $\mathcal{E}$ be a sublinear expectation. Then,*

$$\begin{aligned} \mathcal{E}(c + \lambda[X + Y]) &= c + \mathcal{E}(\lambda[X + Y]) = c + \lambda\mathcal{E}(X + Y) \\ &\leq c + \lambda\,[\,-(-\mathcal{E}(X) - \mathcal{E}(Y))]\,, \end{aligned} \tag{E.1}$$

for every $c \in$, $\lambda \geq 0$, $X, Y \in \mathcal{H}$. *In particular,*

$$\mathcal{E}(Z) = 0, \quad \forall\ Z \in \mathcal{Z}.$$

Proof. Let $X, Y \in \mathcal{H}$. The sub-additivity of $U_{\mathcal{E}}$ implies that

$$U_{\mathcal{E}}(X') + U_{\mathcal{E}}(Y') \leq U_{\mathcal{E}}(X' + Y'), \quad \forall\ X', Y' \in \mathcal{H},$$

even when they take values $\pm\infty$. The definition of $U_{\mathcal{E}}$ now yields,

$$\mathcal{E}(X+Y) = -U_{\mathcal{E}}(-X-Y) \leq -\left[U_{\mathcal{E}}(-X) + U_{\mathcal{E}}(-Y)\right] = -\left(-\mathcal{E}(X) - \mathcal{E}(Y)\right).$$

Then, (E.1) follows directly from the definitions.

Let $Z \in \mathcal{Z}$. Then, $-Z, Z \in \mathcal{P}$ and $\mathcal{E}(Z), \mathcal{E}(-Z) \geq 0$. Since $-Z \in \mathcal{P}$, the monotonicity of $\mathcal{E}$ implies that $\mathcal{E}(X-Z) \geq \mathcal{E}(X)$ for any $X \in \mathcal{H}$. Choose $X = Z$ to arrive at

$$0 = \mathcal{E}(0) = \mathcal{E}(Z-Z) \geq \mathcal{E}(Z) \geq 0.$$

Hence, $\mathcal{E}(Z)$ is equal to zero. □

E.2 Finite Time Markets

We here recall some results from Burzoni et al. [2019] (see Section 2 therein for the precise specification of the framework). We are given a filtered space $(\Omega, \mathbb{F}, \mathcal{F})$ with Ω a Polish space and $\mathbb{F}$ containing the filtration generated by a Borel-measurable process S. We denote by $\mathcal{Q}$ the set of martingale measures for the process S, whose support is a finite number of points. For a given set $A \in \mathcal{F}$, $\mathcal{Q}_A = \{Q \in \mathcal{Q} \mid Q(A) = 1\}$. We define the set of scenarios charged by martingale measures as

$$A^* := \{\omega \in \Omega \mid \exists Q \in \mathcal{Q}_A \text{ s.t. } Q(\omega) > 0\} = \bigcup_{Q \in \mathcal{Q}_A} supp(Q). \tag{E.2}$$

Definition E.4. We say that $\ell \in \mathcal{I}$ is a one-step strategy if $\ell = H_t \cdot (S_t - S_{t-1})$ with $H_t \in \mathcal{L}(X, \mathcal{F}_{t-1})$ for some $t \in \{1, \ldots, T\}$. We say that $a \in \mathcal{I}$ is a one-point Arbitrage on A iff $a(\omega) \geq 0$ $\forall \omega \in A$ and $a(\omega) > 0$ for some $\omega \in A$.

The following Lemma is crucial for the characterization of the set A^* in terms of arbitrage considerations.

Lemma E.5. *Fix any* $t \in \{1, \ldots, T\}$ *and* $\Gamma \in \mathcal{F}$. *There exist an index* $\beta \in \{0, \ldots, d\}$, *one-step strategies* $\ell^1, \ldots, \ell^\beta \in \mathcal{I}$ *and* $B^0, ..., B^\beta$, *a partition of* Γ, *satisfying:*

1. *if* $\beta = 0$ *then* $B^0 = \Gamma$ *and there are No one-point Arbitrages, i.e.,*

$$\ell(\omega) \geq 0\ \forall \omega \in B^0 \Rightarrow \ell(\omega) = 0\ \forall \omega \in B^0.$$

2. *if* $\beta > 0$ *and* $i = 1, \ldots, \beta$ *then:*

▷ $B^i \neq \emptyset$,

▷ $\ell^i(\omega) > 0$ *for all* $\omega \in B^i$,

▷ $\ell^i(\omega) \geq 0$ *for all* $\omega \in \cup_{j=i}^{\beta} B^j \cup B^0$.

We are now using the previous result, which is for some fixed t, to identify A^*. Define

$$\begin{aligned} A_T &:= \quad A \\ A_{t-1} &:= \quad A_t \setminus \bigcup_{i=1}^{\beta_t} B_t^i, \quad t \in \{1, \ldots, T\}, \end{aligned} \tag{E.3}$$

where $B_t^i := B_t^{i,\Gamma}$, $\beta_t := \beta_t^\Gamma$ are the sets and index constructed in Lemma E.5 with $\Gamma = A_t$, for $1 \leq t \leq T$. Note that, for the corresponding strategies ℓ_i^t we have

$$A_0 = \bigcap_{t=1}^{T} \bigcap_{i=1}^{\beta_t} \{\ell_i^t = 0\}. \tag{E.4}$$

Lemma E.6. *A_0 as constructed in (E.3) satisfies $A_0 = A^*$. Moreover, No one-point Arbitrage on $A \Leftrightarrow A^* = A$.*

Proposition E.7. *Let $A \in \mathcal{F}$. We have that for any $\mathcal{F}$-measurable random variable g,*

$$\pi_{A^*}(g) = \sup_{Q \in \mathcal{Q}_A} \mathbb{E}_Q[g]. \tag{E.5}$$

with $\pi_{A^*}(g) = \inf\{x \in \mathbb{R} \mid \exists a \in \mathcal{I} \text{ such that } x + a_T(\omega) \geq g(\omega)\ \forall \omega \in A^*\}$. In particular, the left hand side of (E.5) is attained by some strategy $a \in \mathcal{I}$.

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