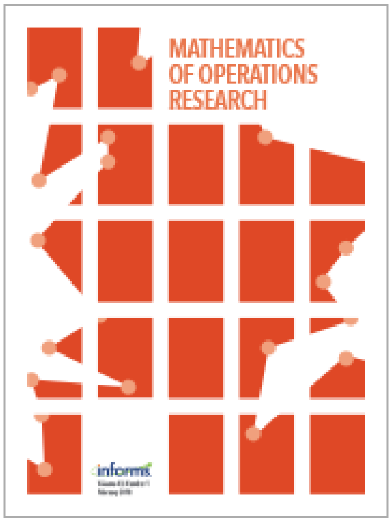

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AN ASYMPTOTIC ANALYSIS OF HIERARCHICAL CONTROL OF MANUFACTURING SYSTEMS UNDER UNCERTAINTY*

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This paper presents an asymptotic analysis of a hierarchical manufacturing system with machines subject to breakdown and repair. The rate of change in machine states is much larger than the rate of fluctuation in demand and the rate of discounting of costs, and this gives rise to a limiting problem in which the stochastic machine availability is replaced by the equilibrium mean availability. The value function for the original problem converges to the value function of the limiting problem. Moreover, the control for the original problem can be constructed from the optimal controls of the limiting problem in a way which guarantees asymptotic optimality of the value function. The limiting problem is computationally more tractable and sometimes has a closed form solution.

Introduction. Most manufacturing systems are large systems characterized by several subsystems such as plants and warehouses, a wide variety of machines and equipment, and a large number of different products. Moreover, these systems are subject to discrete events such as building new facilities, purchasing new equipment and scrapping old ones, machine setups, failures and repairs, new product introductions, etc. These events could be deterministic or stochastic. The management and operation of these systems must recognize and react to these events. Because of the large size of these systems and the presence of these events, exact optimal feedback policies to run these systems may be quite difficult to obtain, both theoretically and computationally.

One way to cope with these complexities is to develop methods of hierarchical control of these systems. The idea is to reduce the overall complex problem into manageable approximate problems or subproblems, each of which is linked by means of a hierarchical integrative system. There are several different, and not mutually exclusive, ways in which the reduction of the complexity might be accomplished. These include decomposition into the problems of the smaller subsystems with a proper coordinating mechanism, aggregation of products along with a disaggregation procedure, replacement of random processes by their averages and possibly other moments, etc. For further details on hierarchical approaches in production planning systems, we refer the reader to a survey of the literature by Bitran and Tirupati [2].

In this paper, we formulate a stochastic production planning problem subject to uncertain machine breakdowns and repairs. The exact optimal solution of such a

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problem is quite complex and difficult to obtain. In order for us to reduce the complexity in the manner of the averaging method mentioned above, we consider the case in which the rate at which the machine breakdown and repair events occur is much larger than the rate of fluctuation in demand and the rate of discounting.

More specifically, we derive a limiting deterministic problem which is simple to solve. This limiting problem is obtained by replacing the stochastic machine availability process by the average total capacity of machines. From its optimal control, we construct an approximate control for the original stochastic problem. As our main result, we show that the cost of the solution obtained with the control thus constructed approaches the optimal cost of the original problem as the rate of machine failure and repair events increase to infinity. In other words, the constructed approximate control is asymptotically optimal.

The significance of this result for the decision-making hierarchy is that the planning level management can ignore the day-to-day fluctuation in machine capacities, or more generally, the details of shop-floor events. The operational level management can then derive approximate optimal policies for running the actual (stochastic) manufacturing system.

It is important to note that the model we have formulated in the paper is sufficiently rich and representative, albeit deliberately simple, to illustrate the idea of asymptotic optimality in the hierarchical control of the stochastic systems, manufacturing or otherwise. Indeed, the main purpose of the paper is to present a formal methodology for handling systems in which some of the exogenous processes, deterministic or stochastic, are changing much faster than the remaining ones. By a fast changing process, we mean a process that is changing so rapidly that from any initial condition, it reaches its stationary distribution in a time period during which there are few, if any, fluctuations in the other processes.

For example, in the case of a fast changing Markov process, the state distribution converges rapidly to a distribution close to its stationary distribution. In the case of a fast changing deterministic process, the time-average of the process reaches a value near its limiting long-run average value. Furthermore, it is possible to associate a *time constant* with each of these processes, namely the reciprocal of the rate of this convergence. It is related to the time it takes the process to cover a specified fraction of the distance between its current value and its equilibrium value, or the time required for the initial distribution to become sufficiently close to the stationary distribution. The concept of a time constant is quite common in the engineering literature. In the special case of exponential radioactive decay, which is related to our exponential discounting process, a familiar measure of the time constant is known as the *half life*. Thus if ρ is the discount rate used in our model, its half life is given by $\log 2/\rho$, which is of the same order as $1/\rho$, and can be taken as the measure of the time constant of the discounting process.

Our methodology, therefore, applies to stochastic control problems, where the objective function is to minimize average long-run cost or long-run discounted cost with a sufficiently small discount rate, so that its reciprocal is much larger than the time constants of some of the exogenous processes that are involved. In this case, it seems reasonable to replace the fast changing processes by their ergodic versions of their long-run averages in order to simplify the problem. Moreover, an asymptotically optimal solution to the original problem can be derived from the optimal solution of the simplified limiting problem.

Our methodology has recently been applied by Jiang and Sethi [9] to a manufacturing system consisting of machine states modelled by a Markov process with weak and strong interactions. In particular, Jiang and Sethi [9] have been able to reduce a problem with one fast-changing machine and one slowly-changing machine into a single-machine problem, which can then be provided with an explicit solution obtained by Akella and Kumar [1] for single unreliable machine problems.

Earlier, in the case of systems with discrete events that occur at different rates, Gershwin [7], inspired by methods of singular perturbations (see [11]), suggested an alternative hierarchical framework. In his paper, Gershwin conjectured that the "hierarchical decomposition is asymptotically optimal as the time scales separate," and cited this as an outstanding research problem.

In another context, Dempster et al. [5] studied a two-level problem, in which level 1 must decide the optimal number of machines to buy and level 2 must schedule a given number n of jobs on the purchased machines to minimize the makespan. The objective function is to minimize the weighted total cost of machines and the length of the makespan. Dempster et al. [5] solved the simplified level 1 problem that suppresses the combinatorial fine structure of the level 2 problem by replacing the makespan with a given number of machines by a known lower bound. This allowed them to obtain the number of machines to be purchased as the optimal solution of the simplified problems. With these machines, level 2 solves the scheduling problem. They showed that if the processing times of the jobs have independent and identical distributions with finite second moment, then their approximation is asymptotically optimal as the number of jobs approaches infinity; see also Bitran and Tirupati [2] for related references.

The plan of the paper is as follows. In §1, we state the precise stochastic problem under consideration. §2 derives the Bellman equation and studies the properties of the value function. In §3, we prove the main result of the paper. An example is solved explicitly in §4 to illustrate the result and to conclude the paper.

1. Formulation. In the production control model, we assume there are n distinct part types produced by m identical machines. The machines are subject to breakdown and repair. We assume, as in Akella and Kumar [1] and Gershwin [7], that breakdowns occur independently of whether or not machines are being used. Let $\{\alpha_t^\epsilon, t \geqslant 0\}$ represent the process of machine availability. Here $\alpha_t^\epsilon \in M = \{0, 1, \ldots, m\}$ represents the number of machines available for production at time t. The α_t^ϵ process is modelled by a continuous time Markov chain defined on $\{\Omega, \mathscr{F}, P\}$ with infinitesimal generator $\mathbf{Q}^\epsilon = (1/\epsilon)\mathbf{Q}$, $\epsilon > 0$. Here $\mathbf{Q} = \{q_{ij}\}$ is an $(m+1) \times (m+1)$ irreducible stochastic matrix with $\sum_j q_{ij} = 0$, $j = 0, \ldots, m$, and has equilibrium distribution ν which satisfies $\nu\mathbf{Q} = 0$, $\nu \cdot \mathbf{1} = 1$, where $\mathbf{1} = (1, 1, \ldots, 1) \in \mathbf{R}^{m+1}$. Let $\mathscr{F}_t^\epsilon = \sigma(\alpha_s^\epsilon, 0 \leqslant s \leqslant t)$.

The machine availability at time t, α_t^ϵ, determines the set of feasible production rates $u_t \in K(\alpha_t^\epsilon)$ with

$$K(i) = \{u \in \mathbf{R}^n, u \geqslant 0, \gamma \cdot u \leqslant i\}. \tag{1.1}$$

Here $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$ are positive constants which represent the fraction of a single machine needed to produce part type i at rate 1 and $\cdot$ denotes inner product. Let $x_t \in \mathbf{R}^n$ denote the inventory/backorder at time t of each part type. For a given production rate u_t, the inventory is given by the equation

$$\frac{d}{dt} x_t = u_t - d, \tag{1.2}$$

where $d \in \mathbf{R}^n$ is the constant demand for the parts.

The cost for a given control policy $u.= \{u_t,\ t \geqslant 0\}$ is given by

$$J^\epsilon(x,i,u.) = E_{x,i}\int_0^\infty e^{-\rho t}G(x_t,u_t)\,dt, \tag{1.3}$$

where $\rho > 0$ is the fixed discount rate and the function G, which represents the cost of production and inventory holding (or shortage) costs, is positive, convex and satisfies

(A1) $|G(x,u)| \leqslant k(1+|x|^a)$ and

(A2) $|G(x,u) - G(y,u)| \leqslant k(1+G(x,u))|x-y|$ if $|x-y| < b$

for suitable constants $k, a, b \geqslant 0$. Assumptions (A1) and (A2) are usual assumptions on the growth rate of functions to ensure the existence of a solution to the Bellman equation to be considered in §2. Note that $G(x,u)$ is required in (A1) to be bounded only by a function of x. This is sufficient in our problem as u takes values in $K(\alpha_t^\epsilon)$ and α_t^ϵ takes values in the bounded set $M = \{0,1,2,\dots,m\}$.

The value function for the control problem is given by

$$v^\epsilon(x,i) = \inf J^\epsilon(x,i,u.) \tag{1.4}$$

with the infimum taken over all $\mathscr{F}_t^\epsilon$-adapted policies u subject to $u_t \in K(\alpha_t^\epsilon)$.

We shall assume that

$$\epsilon = o\left(\frac{1}{\rho}\right), \tag{1.5}$$

i.e., the time constant ϵ of the machine availability process is much smaller than the time constant $1/\rho$ of the discounting process (see Remark 3 in §3). It should be emphasized that the demand is assumed to be constant for convenience in exposition. In fact, demand could be assumed to be a stochastic process as long as its fluctuations are much slower than the capacity fluctuations due to machine breakdown and repair.

2. Properties of the value function. The Bellman equation associated with this problem is given by

(2.1)

$$\rho v^\epsilon(x,i) + \sup_{u\in K(i)}\{-(u-d)\cdot\nabla v^\epsilon(x,i) - G(x,u)\} - \frac{1}{\epsilon}\mathbf{Q}v^\epsilon(x,i) = 0,$$

$$x \in \mathbf{R}^n,\ i = 0,1,\dots,m$$

where

$$\mathbf{Q}v^\epsilon(x,i) = \sum_{j=0}^{m} q_{ij}v^\epsilon(x,j) = \sum_{j\neq i} q_{ij}[v^\epsilon(x,j) - v^\epsilon(x,i)]. \tag{2.2}$$

The value function $v^\epsilon(\cdot,i)$ is in general not differentiable, and hence is not a solution of (2.1) in the classical sense. However, in the following lemma, we make precise the notion of a nondifferentiable solution to (2.1). This involves the concept of a subdifferential used in convex analysis. A vector $p \in \mathbf{R}^n$ is called a subdifferential of a convex function $f(\cdot)$ at a point x if $f(y) - f(x) \geqslant p\cdot(y-x)$ for all $y \in \mathbf{R}^n$. The set of all subdifferentials of $f(\cdot)$ at x is denoted by $\partial f(x)$.

LEMMA 2.1. *$v^\epsilon(\cdot,i)$ is Lipschitz continuous and is convex for each i. There exists a constant C such that*

(2.3)

$$|v^\epsilon(x,i)| + \frac{1}{\epsilon}|v^\epsilon(x,i) - v^\epsilon(x,j)| + \frac{1}{|x-y|}|v^\epsilon(x,i) - v^\epsilon(y,i)| \leqslant C(1+|x|^a)$$

for every i, j and $x \neq y$ satisfying $|x-y| < b$, where a and b are the same as in (A1) *and* (A2). *Moreover,* (2.1) *holds whenever $v^\epsilon(\cdot,i)$ is differentiable, while at the points of nondifferentiability*

$$\rho v^\epsilon(x,i) + \sup_{u\in K(i)}\{-(u-d)\cdot p - G(x,u)\} - \frac{1}{\epsilon}\mathbf{Q}v^\epsilon(x,i) \geqslant 0 \tag{2.4}$$

for all $p \in \partial v^\epsilon(x,i)$.

PROOF. The convexity of $v^\epsilon(x,i)$ in x follows from the assumption of convexity of G and the linear dynamics of (1.2).

Since both $G \geqslant 0$ and $v \geqslant 0$, by choosing $u_t \equiv 0$ and using (A1) we obtain

$$0 \leqslant v^\epsilon(x,i) \leqslant C_1(1+|x|^a). \tag{2.5}$$

Fix $i \in M$, $x \neq y \in \mathbf{R}^n$ with $|x-y| < b$, and $\xi > 0$. Pick $u.$ so that

$$J^\epsilon(x,i,u.) \leqslant v^\epsilon(x,i) + \xi. \tag{2.6}$$

Since $u.$ is still feasible for the initial condition (y,i), by using (2.6) we obtain

$$\begin{aligned} v^\epsilon(y,i) - v^\epsilon(x,i) &\leqslant J^\epsilon(y,i,u.) - J^\epsilon(x,i,u.) + \xi \\ &= \xi + E\int_0^\infty e^{-\rho t}[G(y_t,u_t) - G(x_t,u_t)]\,dt \\ &\leqslant \xi + E\int_0^\infty e^{-\rho t}k(1+G(x_t,u_t))|y_t - x_t|\,dt \quad (\text{by (A.2)}). \end{aligned}$$

Observe that $y_t - x_t = y - x$. Hence,

$$\begin{aligned} v^\epsilon(y,i) - v^\epsilon(x,i) &\leqslant \xi + k|y-x|\left\{E\int_0^\infty e^{-\rho t}G(x_t,u_t)\,dt + \frac{1}{\rho}\right\} \\ &= \xi + k|y-x|\left\{J^\epsilon(x,i,u.) + \frac{1}{\rho}\right\} \\ &\leqslant \xi + k|y-x|\left[v^\epsilon(x,i) + \frac{1}{\rho} + \xi\right]. \end{aligned}$$

The above estimate together with (2.5) yields

$$\frac{1}{|y-x|}|v^\epsilon(y,i) - v^\epsilon(x,i)| \leqslant C_2(1+|x|^a) \tag{2.7}$$

for suitable C_2. This proves that $v^\epsilon(\cdot,i)$ is (locally) Lipschitz continuous. We need

(2.4) to complete the proof of (2.3). By using the Lipschitz continuity of $v^\epsilon(\cdot, i)$, (2.4) follows easily from the dynamic programming principle (see [12], [10], [6] for similar results).

Now let x_0 be a point of differentiability of $v^\epsilon(\cdot, i)$ for each i. Then equation (2.1) holds, and we have

$$\mathbf{Q}v^\epsilon(x_0, i) = \epsilon l(\epsilon, i), \qquad i = 0, 1, \dots, m, \tag{2.8}$$

where

$$l(\epsilon, i) = \rho v^\epsilon(x_0, i) + \sup_{u \in K(i)} \{-(u-d) \cdot \nabla v^\epsilon(x_0, i) - G(x_0, u)\}. \tag{2.9}$$

First observe that due to (2.5) and (2.7),

$$|l(\epsilon, i)| \leqslant C_3(1 + |x_0|^a) \quad \text{for each } i, \epsilon > 0 \text{ and suitable } C_3. \tag{2.10}$$

Also, the irreducibility of $\mathbf{Q}$ implies that the kernel of $\mathbf{Q}$ is the one-dimensional subspace spanned by the vector $\mathbf{1} = (1, 1, \dots, 1) \in \mathbf{R}^{m+1}$. Hence, for any i, j we have

$$\frac{1}{\epsilon}|v^\epsilon(x_0, i) - v^\epsilon(x_0, j)| \leqslant \sup_{j'} |l(\epsilon, j')| \leqslant C_3(1 + |x_0|^a). \tag{2.11}$$

Recall that we have assumed that x_0 is a point of differentiability of $v^\epsilon(\cdot, i)$ for each i. But these points are dense because of the Lipschitz continuity of the value function. Hence, (2.11) holds for every x_0. Combining (2.11) with (2.5) and (2.7) yields (2.3) with a suitable choice of C.

In what follows, we need the concept of viscosity solutions of the dynamic programming equation (2.1). The definition below is a straightforward generalization of the original definition given by M. G. Crandall and P.-L. Lions [4] (see also [3], [10], and [12] for more information).

Let v be a continuous function on $R^n \times M$. For each (x, i) we define convex subsets $D_x^\pm v(x, i)$ or $\mathbf{R}^n$, as follows:

$$D_x^+ v(x, i) = \left\{ r \in \mathbf{R}^n \colon \limsup_{h \to 0} (v(x+h, i) - v(x, i) - r \cdot h)|h|^{-1} \leqq 0 \right\},$$

$$D_x^- v(x, i) = \left\{ r \in \mathbf{R}^n \colon \liminf_{h \to 0} (v(x+h, i) - v(x, i) - r \cdot h)|h|^{-1} \geqq 0 \right\}.$$

We say that any continuous function v is a *viscosity solution* of (2.1), if for each x, i:

(i) $\rho v(x, i) + \sup_{u \in K(i)} \{-(u-d) \cdot r - G(x, u) - (1/\epsilon)\mathbf{Q}v(x, i)\} \leqslant 0 \quad \forall \; r \in D_x^+ v(x, i)$,

(ii) $\rho v(x, i) + \sup_{u \in K(i)} \{-(u-d) \cdot r - G(x, u) - (1/\epsilon)\mathbf{Q}v(x, i)\} \geqslant 0 \quad \forall \; r \in D_x^- v(x, i)$.

Note that v is differentiable in the x-direction at (x, i) if and only if $D_x^+ v(x, i)$ and $D_x^- v(x, i)$ are both singletons. In this case, the singleton is the gradient $\nabla v(x, i)$. Moreover, if v is convex in x, then $D_x^+ v(x, i)$ is empty unless v is differentiable there and $D_x^- v(x, i)$ coincides with the set of subdifferentials $\partial v(x, i)$ defined earlier.

From Lemma 2.1, it follows that $v^\epsilon(\cdot, \cdot)$ is a viscosity solution of (2.1).

3. Limiting control problem. In this section, we prove the convergence of $v^\epsilon(x, i)$ to the value function of a limiting deterministic control problem $v(x)$, for each i. First, we define the limiting control problem.

Let $\nu = (\nu_0, \nu_1, \nu_2, \dots, \nu_m)$ be the unique stationary measure (or, the equilibrium distribution) of the process $\{\alpha_t^\epsilon, t \geqslant 0\}$. Observe that this measure is independent of

ϵ, and it is the unique solution of

$$\sum_{i=0}^{m} q_{ij}\nu_i = 0, \qquad j = 0, 1, \ldots, m, \tag{3.1}$$

and

$$\sum_{j=0}^{m} \nu_j = 1. \tag{3.2}$$

Define

$$\bar{\nu} = \sum_{j=0}^{m} \nu_j j \tag{3.3}$$

and

$$\overline{G}(x,u) = \begin{cases} \inf\left\{ \sum_{j=0}^{m} \nu_j G(x, w_j) : w_j \in K(j) \text{ and } \sum_{j=0}^{m} \nu_j w_j = u \right\}, \\ +\infty \quad \text{if empty.} \end{cases}$$

Finally define $v(x)$ by

$$v(x) = \inf\left\{ \int_0^\infty e^{-\rho t}\overline{G}(x_t, u_t)\, dt : u_t \in K(\bar{\nu}),\ \frac{dx^j}{dt} = u_t^j - d^j, \right. \\ \left. j = 1, \ldots, n \text{ and } x_0 = x \right\}, \tag{3.4}$$

where $K(\bar{\nu})$ is defined by formula (1.1) with $\bar{\nu}$ replacing i. Note that $\bar{\nu}$ is not necessarily an integer.

It is interesting to note that the limiting control problem does not depend on the explicit form of $\mathbf{Q}$, only on $\bar{\nu}$, which is the mean machine availability.

THEOREM 3.1. *As ϵ tends to zero, $v^\epsilon(x, i)$ converges to $v(x)$ (defined by (3.4)), uniformly for every i and bounded x. Moreover, $v(\cdot)$ is the unique convex function satisfying for $|x - y| < b$,*

$$|v(x)| + \frac{1}{|x-y|}|v(x) - v(y)| \leqslant C(1 + |x|^a), \tag{3.5}$$

$$\rho v(x) + \sup_{u \in K(\bar{\nu})} \left\{ -(u - d) \cdot \nabla v(x) - \overline{G}(x,u) \right\} = 0 \tag{3.6}$$

whenever v is differentiable, and

$$\rho v(x) + \sup_{u \in K(\bar{\nu})} \left\{ -(u - d) \cdot p - \overline{G}(x,u) \right\} \geqslant 0 \tag{3.7}$$

for every x and $p \in \partial v(x)$.

PROOF. Using the estimate (2.3), there is a subsequence, denoted by ϵ again, and a Lipschitz continuous function $\bar{v}(\cdot)$ such that

$$\lim_{\epsilon \downarrow 0} v^{\epsilon}(x, i) = \bar{v}(x)$$

uniformly for all i and bounded x. In view of (2.3), this means that $\bar{v}(\cdot)$ satisfies (3.5). Let $x_0 \in \mathbf{R}^n$ and $p_0 \in \partial \bar{v}(x_0)$. Set

$$\varphi(x) = \bar{v}(x_0) + p_0 \cdot (x - x_0) - \tfrac{1}{2}|x - x_0|^2.$$

Then

$$\bar{v}(x_0) - \varphi(x_0) = \min_{x \in \mathbf{R}^n} \{\bar{v}(x) - \varphi(x)\} = 0. \tag{3.8}$$

In fact, the above minimum is strict due to the quadratic term in φ.

For each $i \in M$ and $\epsilon > 0$ choose x_i^{ϵ} for which

$$v^{\epsilon}(x_i^{\epsilon}, i) - \varphi(x_i^{\epsilon}) = \min_{x \in \mathbf{R}^n} \{v^{\epsilon}(x, i) - \varphi(x)\}. \tag{3.9}$$

Since $v^{\epsilon}(\cdot, i)$ converges to $\bar{v}(\cdot)$ uniformly for all i and bounded x, and x_0 is the strict minimum in (3.8), we have

$$\lim_{\epsilon \downarrow 0} x_i^{\epsilon} = x_0 \quad \forall\, i \in M. \tag{3.10}$$

Note that x_i^{ϵ} is easily seen to be bounded in ϵ.

Also, (3.9) implies

$$\nabla \varphi(x_i^{\epsilon}) = p_0 - (x_i^{\epsilon} - x_0) \in \partial v^{\epsilon}(x_i^{\epsilon}, i). \tag{3.11}$$

We now use (2.4) to conclude that

(3.12)

$$\rho v^{\epsilon}(x_i^{\epsilon}, i) + \sup_{u \in K(i)} \left\{-(u - d) \cdot \left[p_0(x_i^{\epsilon} - x_0)\right] - G(x_i^{\epsilon}, u)\right\} - \frac{1}{\epsilon}\mathbf{Q} v^{\epsilon}(x_i^{\epsilon}, i) \geqslant 0.$$

Multiply (3.12) by ν_i and then sum over i, to obtain

(3.13)

$$\begin{aligned} \rho \sum_{i=0}^{m} \nu_i v^{\epsilon}(x_i^{\epsilon}, i) &+ \sum_{i=0}^{m} \nu_i \sup_{u \in K(i)} \left\{-(u - d) \cdot \left[p_0 - (x_i^{\epsilon} - x_0)\right] - G(x_i^{\epsilon}, u)\right\} \\ &\geqslant \frac{1}{\epsilon} \sum_{i=0}^{m} \nu_i B^{\epsilon}(x_i^{\epsilon}, i), \end{aligned}$$

where

$$B^{\epsilon}(x, i) = \mathbf{Q} v^{\epsilon}(x, i) = \sum_{j \neq i} q_{ij}[v^{\epsilon}(x, j) - v^{\epsilon}(x, i)].$$

Now use (3.9) to derive

$$v^\epsilon(x_i^\epsilon, j) \geqslant v^\epsilon(x_j^\epsilon, j) - \varphi(x_j^\epsilon) + \varphi(x_i^\epsilon).$$

Substituting the above inequality into $B^\epsilon(x_i^\epsilon, i)$, we obtain

$$\begin{aligned} B^\epsilon(x_i^\epsilon, i) &\geqslant \sum_{j \neq i} q_{ij}\big[\big[v^\epsilon(x_j^\epsilon, j) - v^\epsilon(x_i^\epsilon, i)\big] - \big[\varphi(x_j^\epsilon) - \varphi(x_i^\epsilon)\big]\big] \\ &= \sum_{j=0}^{m} q_{ij}\big[v^\epsilon(x_j^\epsilon, j) - \varphi(x_j^\epsilon)\big]. \end{aligned}$$

Now multiply the above inequality by ν_i and sum over i. Then use (3.1) to arrive at

$$\begin{aligned} \sum_{i=0}^{m} \nu_i B^\epsilon(x_i^\epsilon, i) &\geqslant \sum_{i=0}^{m} \nu_i \sum_{j=0}^{m} q_{ij}\big[v^\epsilon(x_j^\epsilon, j) - \varphi(x_j^\epsilon)\big] \\ &= \sum_{j=0}^{m} \big[v^\epsilon(x_j^\epsilon, j) - \varphi(x_j^\epsilon)\big]\left(\sum_{i=0}^{m} q_{ij}\nu_i\right) = 0. \end{aligned}$$

Finally, substitute the above inequality into (3.13), then pass to the limit as ϵ tends to zero. Since x_i^ϵ converges to x_0 for every i, this gives

$$\rho\bar{v}(x_0) + \sum_{i=0}^{m} \nu_i \sup_{u \in K(i)} \{-(u - d)\cdot p_0 - G(x_0, u)\} \geqslant 0. \tag{3.14}$$

Then (3.7) follows from (3.14) using the equality

$$\sup_{u \in K(\bar{\nu})} \{-(u-d)\cdot p - \bar{G}(x,u)\} = \sum_{i=0}^{m} \nu_i \sup_{u \in K(i)} \{-(u-d)\cdot p - G(x,u)\}$$

for every $x, p \in \mathbf{R}^n$.

To prove (3.6) at the points of differentiability, first observe that $\partial\bar{v}(x_0) = \{\nabla\bar{v}(x_0)\}$ if $\bar{v}$ is differentiable at x_0 and consequently (3.7) holds. For the reverse inequality, pick a continuously differentiable function φ such that

$$\bar{v}(x_0) - \varphi(x_0) = \max_{x \in \mathbf{R}^n} [\bar{v}(x) - \varphi(x)]$$

and the maximum is strict. Existence of such functions is proven in [3]. Moreover $\nabla\varphi(x_0) = \nabla\bar{v}(x_0)$, and an argument very similar to the one above yields

$$\rho\bar{v}(x_0) + \sup_{u \in K(\bar{\nu})} \{-(u-d)\cdot\nabla\varphi(x_0) - \bar{G}(x_0, u)\} \leqslant 0.$$

Hence, together with (3.7) this yields (3.6).

We have proved that any limit point $\bar{v}(\cdot)$ of the sequence $\{v^\epsilon(\cdot, i)\colon \epsilon > 0\}$ for $i \in M$ satisfies (3.5), (3.6), and (3.7). Since any convex function satisfying (3.6) and (3.7) is a viscosity solution of the Bellman equation (3.6), and the value function $v(\cdot)$, defined by (3.4), is the only viscosity solution of (3.6) satisfying (3.5), we conclude that any limit point $\bar{v}$ is indeed equal to the value function v. Note that the uniqueness of viscosity solutions was first proved in [4], and the problem considered here is covered in [8].

Certain remarks are in order at this point.

REMARK 1. Let $u^*(x)$ be a maximizing argument in (3.6). Suppose (1.2) has a solution with $u_t = u^*(x_t)$. Then $u^*(x)$ is the optimal feedback policy for the limiting problem.

REMARK 2. Consider

$$u(x,i) = iu^*(x)/\bar{\nu} \tag{3.15}$$

to be a feedback production rate for the original problem (1.1)–(1.3). Now let x_t^ϵ be the solution of (1.2) with $u_t^\epsilon = u(x_t^\epsilon, \alpha_t^\epsilon)$. Then u_t^ϵ is an admissible control for the original problem (1.1)–(1.3). This allows us to construct a feedback control for the stochastic problem from the feedback control of the limiting deterministic problem. Moreover, we believe that the performance of the constructed feedback control approximates the optimal performance.

REMARK 3. To bring out the role discounting plays in the model, we allow ρ to depend on ϵ in a way so that $\rho(\epsilon)\epsilon \to 0$ as $\epsilon \to 0$. In this case, the convergence result in Theorem 3.1 is modified to

$$\lim_{\epsilon \to 0} \rho(\epsilon)|v^\epsilon(x,i) - v(x)| = 0. \tag{3.16}$$

The proof of (3.16) requires a change of variable from time t to time $s = \rho(\epsilon)t$, which reduces the problem with variable $\rho(\epsilon)$ to one with a constant discount rate of 1. The limit in (3.16) suggests that if $\rho(\epsilon) = o(1/\epsilon)$, then we can use the approximate control constructed in (3.15) in place of the optimal control for the problem (1.1)–(1.3). We should also observe that $1/\rho$, the order of the *half life* of the exponential decay $e^{-\rho t}$, measures the *time constant* associated with the discounting process and ϵ, the order of the mean time between successive breakdown/repair events, measures the time constant associated with the stochastic machine availability process α_t^ϵ. In other words, our suggestion implies that the time constant of the discounting process should be large in comparison to the time constant of the machine availability process for our approach to be reasonable. This justifies our assumption (1.5).

4. An example. In this section, we present an example and compute explicitly the value function for the limiting control problem. Take

$$n = 2, \qquad d = (d,1),$$

$$G(x,y,u_1,u_2) = \alpha|x| + |y|, \qquad \rho = 1,$$

$$\gamma_1 = \beta\bar{\nu}/K, \qquad \gamma_2 = \bar{\nu}/K.$$

Recall that the specific form of the **Q**-matrix is not important, only $\bar{\nu}$ is, and the limiting deterministic control problem is given by

$$v(x,y) = \inf\left[\int_0^\infty e^{-t}(\alpha|x_t| + |y_t|)\,dt : \frac{d}{dt}x_t = u_{1t} - d,\ \frac{d}{dt}y_t = u_{2t} - 1,\right.$$
$$\left. u_{1t}, u_{2t} \geqslant 0,\ \beta u_{1t} + u_{2t} \leqslant K\right].$$

The parameters β, d, and K must satisfy $\beta d + 1 < K$ or else the demand cannot be met in the long run.

The value function is relatively simple to compute in this problem by tracing the trajectories of the inventories. When an inventory reaches 0, it is held at 0 by setting $u_1 = d$ or $u_2 = 1$. When an inventory is strictly positive, the control is set to 0. Full production is devoted to x (resp. y) if it is negative when corresponding $\beta < \alpha$ (resp. $\beta > \alpha$).

The value function and the optimal control is given by the following set of expressions.

Case 1: $\beta < \alpha$.

$$v(x,y) = \begin{cases} \alpha x + y + \alpha d e^{-x/d} + e^{-y} - (\alpha d + 1), & x \geqslant 0,\ y \geqslant 0, \\ \alpha x - y + (K-1)e^{y/(K-1)} + \alpha d e^{-x/d} - (K - 1 + \alpha d), & \\ \qquad & x \geqslant 0,\ -\dfrac{K-1}{d}x \leqslant y \leqslant 0, \\ \alpha x - y + (\alpha + \beta) d e^{-x/d} + \theta - (K - 1 + \alpha d), & \\ \qquad & x \geqslant 0,\ y \leqslant -\dfrac{K-1}{d}x, \\ -\alpha x - y + \dfrac{\alpha - \beta}{\beta}(K - \beta d)e^{\beta x/(K-\beta d)} + \theta - \dfrac{K\alpha - \alpha\beta d - \beta}{\beta}, & \\ \qquad & x \leqslant 0,\ y \leqslant 0, \\ -\alpha x + y + 2e^{-y} + \dfrac{(\alpha - \beta)}{\beta}(K - \beta d)e^{Bx/(K-\beta d)} & \\ +\theta - \dfrac{K\alpha - \alpha\beta d - \beta}{\beta}, & y \geqslant 0,\ x \leqslant -\dfrac{K - \beta d}{\beta}y, \\ -\alpha x + y + e^{-y} + \dfrac{\alpha}{\beta}(K - \beta d)e^{\beta x/(K-\beta d)} - \dfrac{K\alpha - \alpha\beta d + \beta}{\beta}, & \\ \qquad & y \geqslant 0,\ -\dfrac{K-\beta d}{\beta}y \leqslant x \leqslant 0, \end{cases}$$

where

$$\theta = (K - 1 - \beta d)e^{(\beta x + y)/(K - 1 - \beta d)}.$$

The optimal control for the limiting problem in this case is

$$u^*(x,y) = \begin{cases} (0,0), & x > 0,\ y > 0, \\ (0,1), & x > 0,\ y = 0, \\ (0,K), & x > 0,\ y < 0, \\ (d, K - \beta d), & x = 0,\ y < 0, \\ (K/\beta, 0), & x < 0,\ -\infty < y < \infty, \\ (d,0), & x = 0,\ y > 0, \\ (d,1), & x = 0,\ y = 0. \end{cases}$$

Case 2: $\beta > \alpha$.

$$
v(x,y) = \begin{cases}
\alpha x + y + \alpha d e^{-x/d} + e^{-y} - (\alpha d + 1), & x \geqslant 0,\ y \geqslant 0, \\
\alpha x - y + (K-1)e^{y/(K-1)} + \alpha d e^{-x/d} - (K - 1 + \alpha d), & \\
\qquad x \geqslant 0,\ -\dfrac{K-1}{d}x \leqslant y \leqslant 0, & \\
\alpha x - y + \dfrac{K-1}{\beta}(\beta - \alpha)e^{y/(K-1)} + 2\alpha d e^{-x/d} + \dfrac{\alpha}{\beta}\theta - (K - 1 + \alpha d), & \\
\qquad x \geqslant 0,\ y \leqslant -\dfrac{K-1}{d}x, & \\
-\alpha x - y + \dfrac{(K-1)}{\beta}(\beta - \alpha)e^{y/(K-1)} + \dfrac{\alpha}{\beta}\theta - (K - 1 - \alpha d), & \\
\qquad x \leqslant 0,\ y \leqslant 0, & \\
-\alpha x + y + \dfrac{\alpha+\beta}{\beta}e^{-y} + \dfrac{\alpha}{\beta}\theta - \left(\dfrac{K\alpha - \alpha\beta d + \beta}{\beta}\right), & \\
\qquad y \geqslant 0,\ x \leqslant -\dfrac{K - \beta d}{\beta}y, & \\
-\alpha x + y + e^{-y} + \dfrac{\alpha}{\beta}(K - \beta d)e^{\beta x/(K-\beta d)} - \left(\dfrac{K\alpha - \alpha\beta d + \beta}{\beta}\right), & \\
\qquad y \geqslant 0,\ -\dfrac{K-\beta d}{\beta}y \leqslant x \leqslant 0. &
\end{cases}
$$

In this case, the optimal control for the limiting problem is given by

$$
u^*(x,y) = \begin{cases}
(0,0), & x > 0,\ y > 0, \\
(0,1), & x \leqslant 0,\ y = 0, \\
(0,K), & y < 0, \\
((K-1)/\beta, 1), & x < 0,\ y = 0, \\
(K/\beta, 0), & x < 0,\ y > 0, \\
(d,0), & x = 0,\ y > 0, \\
(d,1), & x = 0,\ y = 0.
\end{cases}
$$

The value function is continuously differentiable in either case, and one can verify the explicit formulae by showing that the Bellman equation holds at every point. Here the Bellman equation is

$$
v(x,y) + \sup_{u_1,u_2}\{(-u_1 - d, u_2 - 1)\cdot \nabla v(x,y)\} = \alpha|x| + |y|,
$$

where the supremum is taken over (u_1, u_2) satisfying

$$
\beta u_1 + u_2 \leqslant K, \qquad u_1, u_2 \geqslant 0.
$$

Also, the optimal control is unique except in the case $\alpha = \beta$, and when $\alpha = \beta$ any control of the form $(u_1, K - \beta u_1)$ in the 3rd quadrant is optimal. Consequently, when

$x < 0$ and $y < 0$, u^* is specified by

$$u^* = \begin{cases} (0, K), & \alpha < \beta, \\ (K/\beta, 0) & \alpha > \beta, \\ (u_1, K - \beta u_1), & \alpha = \beta. \end{cases}$$

The above policy and value function completely characterize the optimal solution of the limiting control problem. This limiting problem will always be much simpler to solve than the stochastic problem, because the randomness has been averaged out. The optimal policy u is, of course, not always feasible for the stochastic problem, for example when $x < 0$, $y < 0$ and all machines are in repair. One can simply modify this policy by reducing u_1 and u_2 proportionately to satisfy the capacity constraint. Thus, $u^*(x, y, i) = iu^*(x, y)/\bar{\nu}$ provides the feedback stochastic control for which, as is easily seen, $\gamma_1 u_1^*(x, y, i) + \gamma_2 u_2^*(x, y, i) \leqslant i$. It is important to determine the rate of convergence in ϵ and explicit error bounds for $|v^\epsilon(x, i) - v(x)|$. These are open research problems.

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