ORF 418 : OPTIMAL LEARNING

LECTURE 4 : September 15, 2025

RICATTI EQUATIONS

1) Dynamic Programming
2) Linear Ansatz
3) Solution

I. DYNAMIC PROGRAMMING

We introduce dynamic programming in the LQ setting as a general solution technique. It will be revisited later in the context of MDPs.

We use the same notation as before and consider the infinite horizon problem.

$$x_{k+1} = Ax_k + Bu_k ,$$

$$J(x_0, u) = \sum_{k=0}^{\infty} \rho^k \left(x_k^T M x_k + u_k^T N u_k \right)$$

and define

$$v(x_0) := \inf_u J(x_0, u) \quad (= v_\infty(x_0) \text{ but } \infty \text{ is dropped})$$

THEOREM (2.2.1) Suppose $v(x_0) < \infty$. Then it is the unique solution to

$$\boxed{v(x_0) = x_0^T M x_0 + \inf_{u \in \mathbb{R}^\ell} \left\{ u^T N u + \rho v(Ax_0 + Bu) \right\}}$$

Moreover, the optimal control is given by

$$u_0^*(x_0) \in \operatorname*{argmin}_{u \in \mathbb{R}^\ell} \left\{ u^T N u + \rho v(Ax_0 + Bu) \right\}.$$

Basic idea of the proof is this:

$$J(x_0,u) = x_0^T M x_0 + u_0^T M u_0$$

$$+\rho \underbrace{\sum_{k=1}^{\infty} \rho^{k-1} \left(x_k^T M x_k + u_k^T N u_k \right)}_{= J(x_1, \bar{u})}$$

where $\bar{u} = (u_1, u_2, \ldots)$, i.e., u shifted by one.

The formal proof is given in the notes.

II. LINEAR ANSATZ (In this section, we take $\rho = 1$)

We postulate that

$$v(x) = x^T V x, \qquad x \in \mathbb{R}^d$$

for some $(d \times d)$, symmetric, positive-definite matrix V, where x^T is the transpose. We substitute this form into the dynamic programming equation:

$$x^T V x = x^T M x + \inf_u \left\{ u^T N u + (Ax + Bu)^T V (Ax + Bu) \right\}.$$

The minimizer u^* satisfies

$$N u^* + B^T V A x + B^T V B u^* = 0$$

Hence,

$$u^* = -F.x \quad , \quad F = (N+B^T V B)^{-1} B^T V A$$

Substitute this back into the equation to obtain

$$x^T V x = x^T M x + x^T F^T N F x + x^T (A-BF)^T V (A-BF) x .$$

Since above holds for every x, we conclude that

$$\begin{aligned} V &= M + F^T N F + (A^T - F^T B^T) V (A - BF) \\ &= M + F^T (N + B^T V B) F + A^T V A \\ &\quad - F^T B^T V A - A^T V B F . \end{aligned}$$

We substitute the formula for F into the above equation. First observe that

$$\begin{aligned} F^T (N+B^T V) F &= A^T V B (N+B^T V)^{-1} B^T V A, \\ F^T B^T V A &= A^T V B (N+B^T V B)^{-1} B^T V A, \\ A^T V B F &= A^T V B (N+B^T V B)^{-1} B^T V A . \end{aligned}$$

Hence, V satisfies

$$\boxed{V = M + A^T V A - A^T V B (N+B^T V B)^{-1} B^T V A}$$

This equation is known as the homogenous Ricatti equation.

III. SOLUTION.

We summarize the above calculations as follows:

(i) Suppose that the homogenous Ricatti equation has a solution V.

(ii) Then $v(x) = x^T V x$ solves the dynamic programming equation. Since this equation has a unique solution which is equal to the value function, V is unique.

(iii) Optimal feedback control is given by

$$u^*(x) = -Fx \,, \quad F = \rho\,(N + \rho B^T V B)^{-1} B^T V A.$$

a) One-dimensional example.

There $d = \ell = 1$, $A = 4$, $B = M = N = 1$, $\rho = \frac{1}{2}$. Then the dynamic programming equation is

$$v(x) = x^2 + \inf_u \left\{ u^2 + \frac{1}{2} v(4x+u) \right\}.$$

We postulate that $v(x) = v x^2$. Then,

$$v x^2 = x^2 + \inf_u \left\{ u^2 + \frac{1}{2} v\,(4x+u)^2 \right\}$$

By differentiating at the minimizer u^*:

$$2u^* + v\,(4x + u^*) = 0 \quad \Rightarrow \quad \boxed{u^* = -\frac{4v}{2+v}\, x =: -f x}$$

Hence $f = 4v(2+v)^{-1}$ which agrees with the general formula. Also

$$vx^2 = x^2 + f^2x^2 + \frac{1}{2}v(4-f)^2x^2$$

$$= x^2\left[1 + f^2(1+\tfrac{1}{2}v) + 8v - 4fv\right]$$

$$\Rightarrow \quad \boxed{v = 1 + 8v - \frac{8v^2}{2+v}}$$

This equation also agrees with the general formula. Moreover, it reduces to the following quadratic equation:

$$8v^2 = (2+v)(7v+1) = 7v^2 + 15v + 2$$

$$\Rightarrow \quad v^2 - 15v - 2 = 0$$

Its positive solution $v \approx 15.533$ agrees with the previous calculation from the previous lecture.