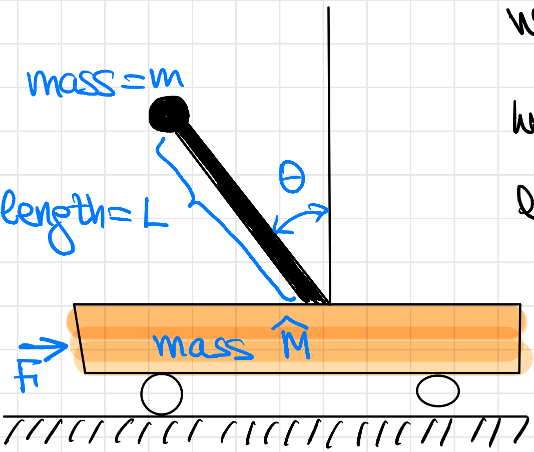
ORF 418 : OPTIMAL LEARNING

LECTURE 5 : September 17, 2025

Inverted Pendelum

1. Formulation
2. Contrability
3. Connection to Ricatti Equation and LQR.

I. INVERTED PENDELUM



We want to keep an inverted pendelum on a cart close to the vertical line (i.e., $\theta=0$) by moving the cart horizontally. Clearly this is an unstable configuration.

a) Problem formulation. State variables are

$\theta(t)$ = angle, $\theta'(t)$ = derivative of the angle

$p(t)$ = position of the cart, $p'(t)$ = cart velocity.

So we set

$$x(t) = (\theta(t), \theta'(t), p(t), p'(t))^T.$$

Control variable is the force $F(t)$ applied to the cart.

Dynamics come from classical mechanics:

$$\boxed{\begin{aligned} F(t) &= (\hat{M}+m)\,p''(t) - mL\,\theta''(t)\cos\theta(t) + mL\,|\theta'(t)|^2 \sin\theta(t) \\ L\theta''(t) &= g\sin\theta(t) + p''(t)\cos\theta(t), \end{aligned}}$$

where g is the gravitational constant $\simeq 9.81 \approx 10\ m/sec^2$

Note that $x(t) \equiv (0,0,0,0)^T$ is an unstable solution.

Difficulties are several:

(i) dynamics are not linear.

(ii) it is naturally set in continuous time.

(iii) objective function is not specified (yet).

(b) Linearization Since the pendelum is going to operate $\theta(t) \approx 0$, we linearize the above functions around the $\theta = 0$ by using Taylor's formula:

$$\cos\theta \approx 1 - \theta^2 \approx 1, \quad \sin\theta \approx \theta. \quad |\theta'(t)|^2 \approx 0.$$

This yields the simplified equations

$$F(t) = (\hat{M} + m)\,p''(t) - m\ell\theta''(t),$$

$$\ell\theta(t)'' = g\theta(t) + p''(t).$$

Let $x(t)$ be as above and set $\boxed{u(t) := \frac{F(t)}{L\hat{M}}}$

Then,

$$\frac{d}{dt}x(t) = \frac{d}{dt}\begin{bmatrix} p(t) \\ p'(t) \\ \theta(t) \\ \theta'(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & b & 0 \end{bmatrix}}_{=A_c} x(t) + \underbrace{\begin{bmatrix} 0 \\ L \\ 0 \\ 1 \end{bmatrix}}_{=B_c} u(t)$$

where $a = mg/\hat{M}$, $b = (m+\hat{M})g/\ell\hat{M}$.

Not finished yet! We still need to discretize time.

Choose a small time step $h = \Delta t$. Then,

$$\frac{1}{h}\left[x(t+h) - x(t)\right] \approx \frac{d}{dt} x(t) = A_c x(t) + B_c u(t)$$

By algebra we obtain

$$x(t+h) \approx x(t) + h A_c x(t) + h B_c u(t)$$

$$= \underbrace{[I + hA_c]}_{A} x(t) + \underbrace{hB_c}_{B} u(t).$$

Hence

$$A = I + A_c = \begin{bmatrix} 1 & h & 0 & 0 \\ 0 & 1 & ah & 0 \\ 0 & 0 & 1 & h \\ 0 & 0 & bh & 1 \end{bmatrix}, \qquad B = hB = \begin{bmatrix} 0 \\ hL \\ 0 \\ h \end{bmatrix}$$

This establishes a linear equation for the state process in discrete time.

For LQ control, we need a quadratic cost function or equivalently matrices M and N. But the problem is a question of controllability:

is there a control that bring the state to zero?

II. CONTROLLABILITY

We say that the system in $\mathbb{R}^d$:

$$x_{k+1} = Ax_k + Bu_k \qquad (*)$$

with given matrices A, B is controllable if for every initial point x_{in} and a terminal point x_{out} there are $n \geq 1$ and $u = (u_0, \dots u_n) \in \mathcal{U}$ such that the solution to $(*)$ starting at $x_0 = x_{in}$ satisfies $x_n = x_{out}$. In words, by choosing appropriate controls we can move the state from any point x_{in} to any other point x_{out} in finitely many steps.

THEOREM 2.3.1 The dynamical system $(*)$ is controllable if and only if the $d \times \ell d$ controllability matrix

$$\boxed{\mathcal{C} = [B \,\vdots\, AB \,\vdots\, \dots \,\vdots\, A^{d-1}B]}$$

has (full) rank d. Moreover, when the system is controllable then the Ricatti equation with any M and N has a unique solution.

See the notes for a proof.

Pendelum Example. We have $d=4$ $\ell=1$. Then,

$$A = \begin{bmatrix} 1 & h & 0 & 0 \\ 0 & 1 & ah & 0 \\ 0 & 0 & 1 & h \\ 0 & 0 & bh & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ hL \\ 0 \\ h \end{bmatrix}$$

We compute that

$$AB = \begin{bmatrix} h^2 L \\ hL \\ h^2 \\ h \end{bmatrix} \quad A^2 B = \begin{bmatrix} 2h^2 L \\ hL + ah^3 \\ 2h^3 \\ bh^3 + h \end{bmatrix} \quad A^3 B = \begin{bmatrix} 3h^2 L + ah^4 \\ hL + 3ah^3 \\ 3h^2 + bh^4 \\ 3bh^3 + h \end{bmatrix}$$

$$\mathcal{C} = h \begin{bmatrix} 0 & Lh & 2Lh & 3Lh + ah^3 \\ L & L & L + ah^2 & L + 3ah^2 \\ 0 & h & 2h^2 & 3h + bh^3 \\ 1 & 1 & bh^2 + 1 & 3bh^2 + 1 \end{bmatrix}$$

has full rank (has to be checked).

III. Connection to LQR

THEOREM 2.3.2. Suppose that the dynamical system with matrices A and B is controllable.

Then, for any positive semi-definite, symmetric matrices M and N, the Ricatti equations is solvable.

Proof. For $n \geq 1$ consider

$$v_n(x_0) := \inf_u J_n(x_0, u)$$
$$= \inf_u \sum_{k=0}^{n-1} \rho^k [x_k^T M x_k + u_k^T N u_k]$$

And

$$v_\infty(x_0) = \inf J_\infty(x_0, u)$$
$$= \inf_u \sum_{k=0}^{\infty} \rho^k [x_k^T M x_k + u_k^T N u_k]$$

Since the system is controllable, there is $u \in \mathcal{U}$ such that $x_u = 0$. Hence,

$v_\infty(x_0) < \infty$ and $\lim_{n \to \infty} v_n(x_0) = v_\infty(x_0)$.

By induction we can show that

$$v_n(x_0) = x_0^T V_n x_0$$

for some positive definite symmetric matrix V_n. By convergence of $v_n(x_0)$, we conclude that matrices V_n's also converge. Hence,

$-V_\infty(x_0) = x_0^T V_\infty x_0$ and we then show that V_∞ solves the Ricatti equation.

$\square$

Inverted Pendelum Example.

Suppose $m=1$, $\hat{M}=100$, $L=1$ and take $g \doteq 10$. Then, $a=0.1$ and $b=10.1$. We arbitrarly choose $h=\Delta t = 0.1$, $\rho = 0.36$, $M=I$ and $N=1$. Let V be the solution of the Ricatti equation which we know to exist. Then the gain matrix F is given by

$$F = (N + B^T V B)^{-1} B^T V A$$

Using the package linalg from spicy in python and the command

solve_discrete_are(.....)

we compute V. Here "are" stands for Algebraic Ricatti Equations. We can then compute F. Results are in lecture notes.