Homework 2

ORF 418 - Fall 2025
Due: October 12 (Sunday), 2025 - midnight
version: October 14, 2025

1 (30 points). Consider the general *Linear-Quadratic problem* in infinite horizon as introduced in Section 2.1.1 of the lecture note, with the following parameters: $d = 2$, $\ell = 1$, $N = 1$, $\rho = 1/2$ and

$$A := \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}, \; B := \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \; M := \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

Let $x_k := (y_k, z_k) \in \mathbb{R}^2$ be the state of the system at time $k \in \mathbb{N}$, with initial condition $x_0 \in \mathbb{R}^2$ known. Let $v(y, z)$ be the value function starting from an initial condition $x_0 = (y, z) \in \mathbb{R}^2$.

1. **[7pt]** *Compute the controllability* matrix and decide whether this system is *controllable* or not.
2. **[10pt]** We know that $v(y, z) = ay^2 + bz^2$ for some constants a, b. Use the dynamic programming equation to *compute* the constants a, b.
3. **[10pt]** Consider the following minimization problem:

$$\widetilde{v}(r_0, r_1) := \inf_u \sum_{k=0}^{\infty} \frac{1}{2^k} \left(r_k^2 + 2r_{k+1}^2 + u_k^2 \right),$$

   where $r_{k+2} + 2r_k = u_k$ for $k = 0, 1, \ldots$, and the initial conditions r_0, r_1 are given. *Formulate* this problem as a Linear Quadratic problem, *i.e.* write down A, B, M, N and ρ.
4. **[3pt]** *Compute* $v(1, 2)$.

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2 (30 points). Suppose that $Y_1, \ldots, Y_n$ are i.i.d. from the *Poisson distribution* with mean $\theta > 0$, i.e., for any $k = 1, \ldots$, conditioned on the value of θ,

$$\mathbb{P}(Y_k = y \,|\, \theta) = \frac{\theta^y}{y!} e^{-\theta}, \qquad y = 0, 1, \ldots$$

The parameter θ is not known and hence, is a random variable. Suppose that the *prior distribution of* θ *is the gamma distribution* with parameters $a, b > 0$, i.e., the p.d.f. is given by,

$$f_\theta(\theta) = \frac{b^a}{\Gamma(a)} \theta^{a-1} e^{-b\theta}, \qquad \theta > 0,$$

where Γ is the *gamma function*:

$$\Gamma(a) := \int_0^\infty z^{a-1} e^{-z} \, dz.$$

In this exercise you do not need the properties of the Γ function.

a. *Show that the posterior distribution of θ* based on the observation that $Y_1 = y_1$ is again gamma-distributed with parameters,

$$a + y_1, \qquad b + 1.$$

b. By iterating part a, *show that the posterior distribution of θ* based on the observations $Y_1 = y_1, Y_2 = y_2, \ldots, Y_n = y_n$ is again gamma-distributed with parameters,

$$a + \sum_{i=1}^{n} y_i, \qquad b + n.$$

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3 (40 points).

This exercise is related to a court case Castaneda vs. Partida *430 U.S. 482 (1977) deciding whether there was bias in juror selection in a particular county. Please read the case in Exercise 3.5 of the Lecture Notes.*

The local population of the given county was 79% Mexican-American. However, over an 11-year period, among the 870 persons summoned to serve as grand jurors, only 339 were Mexican-American. Court assumed that the selection of jurors was random with P being the probability of any selected juror being Mexican-American. If Y denotes the number of Mexican-American jurors, then Y is a Binomial random variable with $n = 870$ and success probability P. Namely, the conditional distribution of Y given $P = p$ is:

$$\mathbb{P}(Y = y \mid P = p) = \binom{n}{y} p^y (1-p)^{n-y}, \qquad \text{for } y = 0, 1, \ldots, n = 870.$$

1. **[4pt]** *Compute* $\mathbb{E}[Y \mid P = 0.79]$. By comparing this value with the observation, comment on potential bias in the jury selection.
2. **[6pt]** Based on many scientific experiments, the Court assumed that the distribution of P prior to the observation is a Beta distribution with parameters a and b. Write this *prior density* function of P and *find* a formula for its expectation in terms of the constants a, b (do not need to prove it).
3. **[10pt]** *Show that the posterior distribution of* P, given an observation $Y = y$ for $y \in \{0, \ldots, n = 870\}$ satisfies the following, for some function C independent of p:

$$f_{P|Y}(p|y) = C(y) p^{a+y-1}(1-p)^{b+n-y-1}, \qquad p \in (0, 1).$$

4. **[10pt]** *Compute* that

$$C(y) = \frac{\Gamma(a+b+n)}{\Gamma(a+y)\Gamma(b+n-y)}.$$

Conclude that the posterior distribution of P, given the observation $Y = 339$, is again a Beta distribution with parameters $a + 339$ and $b + 531$, recall that $n = 870$.

In the followings, as the prior must be consistent with the population, we set $\mathbb{E}[P] = 0.79$, or equivalently $a = 79b/21$, and consider the conditional probability

$$p(b) := \mathbb{P}(P \leq 0.632 \mid Y = 339).$$

As indicated above, this conditional probability is only a function b because a is a function of b given by $a = 79b/21$.

5. **[4pt]** *Numerically compute and plot* $p(b) = \mathbb{P}(P \leq 0.632 \mid Y = 339)$ as a function of $b \in [0.1, 870]$ with step size 0.1.
6. **[4pt]** *Compute numerically*

   $$b^* := \min\{b > 0 \ : \ p(b) = \mathbb{P}(P \leq 0.632 \mid Y = 339) \leq 0.5 \ \}.$$

   is less than 0.5.
7. **[2pt]** For b^* computed above and $a^* = 79b^*/21$, *compute numerically* the unconditional probability

   $$\mathbb{P}(P \leq 0.632).$$

   This corresponds to the prior probability that one places on the jury selection process is unbiased.

Legal context: Mathematically, the bias in juror selection would mean that the random variable P is substantially smaller than 0.79. Say we define the bias to be P being 80% or less than the population percentage of 79%. In other words, the event $P \leq 0.8 \times 0.79 = 0.632$ means bias. Then, given the observation of very low selection rate, to say that there is no bias with probability %50 (i.e., to have a conditional probability of $\mathbb{P}(P \leq 0.632 \mid Y = 339) \leq 0.5$), one needs to have a very very low prior bias probability (i.e., very low prior probability of bias $\mathbb{P}(P \leq 0.632)$).