ORF 418 : OPTIMAL LEARNING

LECTURE 10 : October 6 , 2025

ONE DIMENSIONAL KALMAN FILTER

1) Set-up
2) Formulae
3) Variances

I. ONE-DIMENSIONAL SET-UP

Please see the motivating example in the lecture Notes. Here we start with the one-dimensional problem:

(i) $x_0, x_1, x_2, \dots \in \mathbb{R}^1$ is a stochastic process solving

$$\boxed{\begin{aligned} x_{k+1} &= a x_k + w_{k+1}, \quad & k=0,1,\dots \quad \text{state} \\ z_k &= h x_k + v_k, \quad & k=0,1,\dots \quad \text{observation} \end{aligned}}$$

where $\{w_k\}$ and $\{v_k\}$ are i.i.d sequences with

$$\mathbb{E}[w_k] = \mathbb{E}[v_k] = 0,$$

$$\mathbb{E}[w_k^2] = q, \quad \mathbb{E}[v_k^2] = r.$$

(ii) Goal is to compute

$$\boxed{\hat{x}_k := \mathbb{E}[x_k \mid z_1, \dots, z_k]}$$

for $k = 1, 2, \dots$. By convention, we set $\hat{x}_0 = \mathbb{E}[x_0]$.

(iii) Auxiliary processes

$$\boxed{\begin{aligned} \hat{x}_{k+1|k} &:= \mathbb{E}[x_{k+1} \mid z_1, \dots, z_k] \quad & \text{pre-observation estimate.} \\ i_k &:= z_k - h\, \hat{x}_{k|k-1} \quad & \text{innovation process} \end{aligned}}$$

and we define two more deterministic processes

$$\boxed{\begin{aligned} P_k &:= \mathbb{E}[(x_k-\hat{x}_k)^2] \\ \bar{P}_k &:= \mathbb{E}[(x_k - x_{k|k-1})^2] \end{aligned}}$$

Assumptions:

(a) All constants, a, h, q, r are known. (No learning!)

(b) All random variables $\{w_k\}, \{v_k\}$ and x_0 are independent of each other and Gaussian.

II. FORMULAE.

$$\begin{aligned} \hat{x}_{k+1|k} &= \mathbb{E}[x_{k+1} \mid z_1, \dots z_k] \\ &= \mathbb{E}[a x_k + w_{k+1} \mid z_1 \dots z_k] \\ &= a\,\mathbb{E}[x_k \mid z_1 \dots z_k] + \underbrace{\mathbb{E}[w_{k+1}]}_{=0} \\ &= a\hat{x}_k \end{aligned}$$

So our first equation is

$$\boxed{\hat{x}_{k+1|k} = a\hat{x}_k} \qquad k = 0, 1, 2 \dots .$$

Case $k=1$. Since x_0 are w_1 are Gaussian and independent, x_1 is also Gaussian and independent of v_1. Therefore, $z_1 = hx_1 + v_1$ and x_1 are jointly

Gaussian. So we can use the formula from Chapter 3 to conclude that $x_1|z_1$ is Gaussian with

$$\mu_{x_1|z_1} = \mathbb{E}[x_1] + K_1(z_1 - \mathbb{E}[z_1])$$

where

$$K_1 = \frac{\operatorname{Cov}(x_1, z_1)}{\operatorname{Var}(z_1)}.$$

By definitions,

$$\mathbb{E}[x_1] = a\,\mathbb{E}[x_0] = a\hat{x}_0 = \hat{x}_{1|0}$$

$$\mathbb{E}[z_1] = h\,\mathbb{E}[x_1 + v_1] = h\,\mathbb{E}[x_1] = h\hat{x}_{1|0}.$$

Then,

$$z_1 - \mathbb{E}[z_1] = z_1 - h\hat{x}_{1|0} = i_1.$$

For the variances: $i_1 = h(x_1 - \hat{x}_{1|0}) + v_1$ and $(x_1 - \hat{x}_{1|0})$, v_1 are independent. Therefore,

$$\operatorname{var}(i_1) = \mathbb{E}[i_1^2] = h^2\,\mathbb{E}[(x_1 - \hat{x}_{1|0})^2] + \mathbb{E}[v_1^2]$$

$$= h^2\bar{P}_1 + r.$$

Also, since $\mathbb{E}[x_1] = \hat{x}_{1|0}$

$$\operatorname{cov}(x_1, z_1) = \mathbb{E}[(x_1 - \mathbb{E}[x_1])(z_1 - \mathbb{E}[z_1])]$$

$$= \mathbb{E}[(x_1 - \hat{x}_{1|0})(h(x_1 - \hat{x}_{1|0}) + v_1)],$$

$$= \mathbb{E}[h(x_1 - \hat{x}_{1|0})^2] = h\bar{P}_1.$$

So we have shown that

$$\boxed{\hat{x}_1 = a\hat{x}_0 + K_1 i_1 \,, \quad K_1 = \frac{h\bar{P}_1}{h^2\bar{P}_1 + r}}$$

Note that $i_1 = z_1 - h\hat{x}_{1|0} = z_1 - ha\hat{x}_0$ is computable after observing z_1. Then, we can update $\hat{x}_1$ as long as we know $\bar{P}_1$. (This we address later).

Case $k=2$. In the lecture notes, it is shown that for any random variable ξ

$$\mathbb{E}[\xi \mid z_1, z_2] = \mathbb{E}[\xi \mid i_1, i_2].$$

Hence,

$$\hat{x}_2 = \mathbb{E}[x_2 - \hat{x}_{2|1} \mid i_2, i_1] + \mathbb{E}[\hat{x}_{2|1} \mid i_2, i_1].$$

However, $\hat{x}_{2|1} = \mathbb{E}[x_2 \mid z_1] = \mathbb{E}[x_2 \mid i_1]$. Hence $\mathbb{E}[\hat{x}_{2|1} \mid i_1, i_2] = \hat{x}_{2|1}$. Also,

$$\mathbb{E}[(x_2 - \hat{x}_{2|1}) \mid i_1] = \mathbb{E}[ax_1 + w_2 - a\hat{x}_1 \mid i_1]$$

$$= a\,\mathbb{E}[x_1 - \hat{x}_1 \mid i_1] + \underbrace{\mathbb{E}[w_2 \mid i_1]}_{=0}$$

$$= a\,\mathbb{E}[x_1 \mid i_1] - a\hat{x}_1 = 0.$$

Since $(x_2-\hat{x}_{2|1})$, i_1 are jointly Gaussian, this implies that they are independent, and

$$\mathbb{E}[(x_2-\hat{x}_{2|1})\,|\,i_1,i_2] = \mathbb{E}[(x_2-\hat{x}_{2|1})\,|\,i_2]$$

$$= \underbrace{\mathbb{E}[x_2-\hat{x}_{2|1}]}_{=0} + \underbrace{\frac{\mathrm{Cov}((x_2-\hat{x}_{2|1}, i_2)}{\mathrm{var}(i_2)}}_{=:K_2}[i_2-\underbrace{\mathbb{E}[i_2]}_{=0})$$

$$= K_2 i_2$$

In the lecture notes we show that

$$K_2 = \frac{h\bar{P}_2}{h^2\bar{P}_2+r}.$$

Hence,

$$\hat{x}_2 = \hat{x}_{2|1} + \mathbb{E}[(x_2-\hat{x}_{2|1})\,|\,i_1,i_2]$$

$$= a\hat{x}_1 + K_2 i_2.$$

Summarizing

$$\boxed{\hat{x}_2 = a\hat{x}_1 + K_2 i_2, \quad K_2 = \frac{h\bar{P}_2}{h^2\bar{P}_2+r}.}$$

This formula holds for every $k=1,2,\ldots$!

III. VARIANCES.

$$\bar{P}_{k+1} = \mathbb{E}\left[(x_{k+1} - \hat{x}_{k+1|k})^2 \right]$$

$$= \mathbb{E}\left[(a x_k + w_{k+1} - a \hat{x}_k)^2 \right]$$

$$= a^2 \mathbb{E}\left[(x_k - \hat{x}_k)^2 \right] + \mathbb{E}\left[w_{k+1}^2 \right]$$

$$= a^2 P_k + q.$$

A tedious calculation shows that

$$P_k = \frac{r \bar{P}_k}{h^2 \bar{P}_k + r}.$$

Summarizing: We have

$$P_0 = \mathbb{E}\left[(x_0 - \hat{x}_0)^2 \right] = \mathrm{var}(x_0).$$

This gives

$$\bar{P}_1 = a^2 P_0 + q \Rightarrow P_1 = \frac{r \bar{P}_1}{h^2 \bar{P}_1 + r}$$

and iterate to compute all $P_k, \bar{P}_k$. We can then compute all gains $K_k = h \bar{P}_k / (h^2 \bar{P}_k + r)$.

Then starting with $\hat{x}_0 = \mathbb{E}[x_0]$,

$$\hat{x}_k = a \hat{x}_{k-1} + K_k i_k, \quad i_k = z_k - h \hat{x}_{k|k-1}, \quad \hat{x}_{k|k-1} = a \hat{x}_{k-1}.$$