c) $\mathbb{P}(B_u) = \mathbb{P}(B_u|A)\mathbb{P}(A) + \mathbb{P}(B_u|A^c)\mathbb{P}(A^c)$

$$= (0.2)(0.05) + (0.3)(0.95) = 0.295$$

$$\boxed{\mathbb{P}(A|B_u) =} \frac{\mathbb{P}(B_u|A)\mathbb{P}(A)}{\mathbb{P}(B_u)} = \frac{(0.2)(0.05)}{0.295} = \boxed{\frac{1}{295}}$$

d) $\mathbb{P}(B_n) = \mathbb{P}(B_n|A)\mathbb{P}(A) + \mathbb{P}(B_n|A^c)\mathbb{P}(A^c)$

$$= (0.1)(0.05) + (0.6)(0.95) = 0.575$$

$$\boxed{\mathbb{P}(A|B_n) =} \frac{\mathbb{P}(B_n|A)\mathbb{P}(A)}{\mathbb{P}(B_n)} = \frac{(0.1)(0.05)}{0.575} = \boxed{\frac{1}{115}}$$

Problem 3.2. We are given $\mathbb{P}(A_h) = \mathbb{P}(A_m) = \mathbb{P}(A_s) = 1/3$.

a) $\boxed{\mathbb{P}(B|A_h) = 0}$, $\boxed{\mathbb{P}(B|A_m) =}(1-p) = \boxed{0.75}$, $\boxed{\mathbb{P}(B|A_s) = 1}$.

b) $\boxed{\mathbb{P}(B) =}\ \mathbb{P}(B|A_h)\mathbb{P}(A_h) + \mathbb{P}(B|A_m)\mathbb{P}(A_m) + \mathbb{P}(B|A_s)\mathbb{P}(A_s)$

$$= 0 + \frac{3}{4}\cdot\frac{1}{3} + 1\cdot\frac{1}{3} = \frac{1}{4} + \frac{1}{3} = \boxed{\frac{7}{12}}.$$

c) $$\boxed{\mathbb{P}(A_h|B) =} \frac{\mathbb{P}(B|A_h)\mathbb{P}(A_h)}{\mathbb{P}(B)} = \boxed{0}$$

$$\boxed{\mathbb{P}(A_m|B) =} \frac{\mathbb{P}(B|A_m)\mathbb{P}(A_m)}{\mathbb{P}(B)} = \frac{(0.75)(1/3)}{(7/12)} = \boxed{\frac{3}{7}}$$

$$\boxed{\mathbb{P}(A_s|B) =} 1 - \mathbb{P}(A_h|B) - \mathbb{P}(A_m|B) = \boxed{\frac{4}{7}}.$$

We could also use the Bayes' rule to compute it:

$$\mathbb{P}(A_s|B) = \frac{\mathbb{P}(B|A_s)\mathbb{P}(A_s)}{\mathbb{P}(B)} = \frac{(1/3)}{(7/12)} = \frac{4}{7} \checkmark$$

d). In this case

$$\mathbb{P}(B|A_h) = \mathbb{P}(B|A_m) = \mathbb{P}(B|A_s) = \frac{1}{2}.$$

Then, $\mathbb{P}(B) = \mathbb{P}(B|A_h)\mathbb{P}(A_h) + \mathbb{P}(B|A_m)\mathbb{P}(A_m) + \mathbb{P}(B|A_s)\mathbb{P}(A_s)$

$$= \frac{1}{2}\cdot\frac{1}{3} + \frac{1}{2}\cdot\frac{1}{3} + \frac{1}{2}\frac{1}{3} = \frac{1}{2}.$$

$$\boxed{\mathbb{P}(A_h|B) =} \frac{\mathbb{P}(B|A_h)\mathbb{P}(A_h)}{\mathbb{P}(B)} = \frac{(1/2)(1/3)}{(1/2)} = \boxed{\frac{1}{3}}$$

$$\boxed{\mathbb{P}(A_s|B) =} \frac{\mathbb{P}(B|A_s)\mathbb{P}(A_s)}{\mathbb{P}(B)} = \boxed{\frac{1}{3}}$$

$$\Rightarrow \boxed{\mathbb{P}(A_m|B) = 1/3}.$$

Posterior = Prior $\Rightarrow$ No learning! and this is expected.

e) $\mathbb{P}(B|A_h) = \mathbb{P}(B|A_m) = \mathbb{P}(B|A_s) = (1-p) = 0.75.$

Then, $\mathbb{P}(B) = 0.75$ and again

$$\boxed{\mathbb{P}(A_h|B) = \mathbb{P}(A_m|B) = \mathbb{P}(A_h|B) = \frac{1}{3}}$$

f) $\mathbb{P}(B|A_h) = 0$, $\mathbb{P}(B|A_m) = \frac{1}{3}$, $\mathbb{P}(B|A_s) = 1$.

$$\mathbb{P}(B) = \mathbb{P}(B|A_h)\mathbb{P}(A_h) + \mathbb{P}(B|A_m)\mathbb{P}(A_m) + \mathbb{P}(B|A_s)\mathbb{P}(A_s)$$

$$= 0 + \frac{1}{3}\cdot\frac{1}{3} + 1\cdot\frac{1}{3} = \frac{4}{9}.$$

$$\boxed{\mathbb{P}(A_h|B) = 0}, \quad \boxed{\mathbb{P}(A_m|B) =} \frac{\frac{1}{3}\cdot\frac{1}{3}}{4/9} = \boxed{\frac{1}{4}}$$

$$\boxed{\mathbb{P}(A_s|B) =} \frac{1\cdot 1/3}{4/9} = \boxed{\frac{3}{4}}$$

Problem 3.3 Let Y be the number of heads. Then, we are given that $\mathbb{P}(Y=y \mid P=p) = \binom{100}{y} p^y (1-p)^{100-y}$ for $y = 0, 1, \ldots, 100$.

a) The prior is given as $f_P(p) = 2p$ for $p \in (0,1)$. Then,

$$f_{P|Y}(p \mid y) = \frac{\mathbb{P}(Y=y \mid P=p)\, f_P(p)}{\mathbb{P}(Y=y)}$$

$$= C\, p^y (1-y)^{100-y} \cdot 2p$$

where $C = \binom{100}{y} / \mathbb{P}(Y=y)$. Then, with $y=52$,

$$\boxed{f_{P|Y}(p \mid 52) = 2C\, p^{53} (1-p)^{48}, \qquad p \in (0,1).}$$

This is beta distribution with parameter $a=54$ and $b=49$. If we want we can calculate C from this observation but is not necessary. Indeed

$$2C = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} = \frac{\Gamma(103)}{\Gamma(54)\Gamma(49)}.$$

b) Bayesian estimate is

$$\mathbb{E}[P \mid Y=52] = \int_0^1 p\, f_{P|Y}(p \mid 52)\, dp$$

$$= \text{mean of } \mathrm{Beta}(a=54, b=49)$$

$$= \frac{a}{a+b} = \boxed{\frac{54}{103} = \hat{p}}$$

Problem 3.4.

We could use induction as in the solutions of Homework 2.

Here we use a direct approach.

Set $Y=(Y_1,\dots,Y_n)$, $y=(y_1,\dots,y_n)$. Since $Y_1,\dots,Y_n$ are independent

$$\mathbb{P}(Y=y \mid P=p) = \prod_{i=1}^{n} \mathbb{P}(Y_i=y_i \mid P=p)$$

$$= \prod_{i=1}^{n} p^{y_i}(1-p)^{1-y_i}$$

$$= p^{\sum_{i=1}^n y_i}(1-p)^{n-\sum_{i=1}^n y_i}, \qquad y_1,\dots,y_n \in \{0,1\}$$

(Note that $p^{y_i}(1-p)^{y_i}$ is equal to p if $y_i=1$ and is equal to $(1-p)$ if $y_i=0$). By Bayes' formula,

$$f_{P|Y}(p|y) = \frac{\mathbb{P}(Y=y \mid P=p)\, f_P(p)}{\mathbb{P}(Y=y)}$$

$$= C\left(p^{\sum_{i=1}^n y_i}(1-p)^{n-\sum_{i=1}^n}\right)\left(p^{a-1}(1-p)^{b-1}\right)$$

$$= C\, p^{\hat{a}-1}(1-p)^{\hat{b}-1},$$

where $\hat{a}=a+\sum_{i=1}^n y_i$, and $\hat{b}=b+n-\sum_{i=1}^n y_i$. This shows that $P|Y$ is Beta with parameters $\hat{a},\hat{b}$.

Problem 3.5.

Let Y be the number of defective items among eight chosen ones. $\mathbb{P}(Y=k \mid \theta) = \binom{8}{k}\theta^k(1-\theta)^{8-k}$

for $k=0,1,\ldots,8$. Here $Y=3$.

a) $$f_{\theta|Y}(\theta|Y=3) = \frac{\mathbb{P}(Y=3|\theta)\, f_\theta(\theta)}{\mathbb{P}(Y=3)}$$

$$= c_1 (\theta^3 (1-\theta)^5)(1)$$

Therefore, $\theta|Y=3$ has Beta distribution with $a=4$ and $b=6$. Then,

$$\boxed{\hat{\theta} =} \mathbb{E}[\theta|Y=3] = \text{mean of Beta with } a=4, b=6$$

$$= \frac{a}{a+b} = \boxed{\frac{4}{10}}$$

b) $$f_{\theta|Y}(\theta|Y=3) = c_1 (\theta^3 (1-\theta)^5)(2(1-\theta))$$

$$= 2c\ \theta^3 (1-\theta)^6.$$

So, $\theta|Y=3$ is Beta with $a=4$, $b=7$. Therefore, its mean is $4/11$. That is $\boxed{\hat{\theta} = 4/11.}$

c) $\mathbb{P}(Y=3|\theta) = c_1\ \theta^3 (1-\theta)^5$. (We know $c = \binom{8}{3}$ but it is a not needed in this calculation). The maximum likelihood estimator $\hat{\theta}_{MLE}$ maximizes the function $\theta \mapsto \mathbb{P}(Y=3|\theta)$. But that is equivalent to maximize

$$\theta \mapsto h(\theta) = \ln(\mathbb{P}(Y=3|\theta)) = \ln c_1 + 3\ln\theta + 5\ln(1-\theta).$$

Therefore, $$0 = \frac{3}{\hat{\theta}_{MLE}} - \frac{5}{(1-\hat{\theta}_{MLE})}$$

$\Rightarrow 3(1-\hat{\theta}_{MLE}) = 5\hat{\theta}_{MLE} \Rightarrow 3 = 8\hat{\theta}_{MLE}$

$$\Rightarrow \boxed{\hat{\theta}_{MLE} = \frac{3}{8}}$$

Problem 3.6 Let Y be the number of voters among the chosen sample of 1,000, who are in favor of the proposition.

We observed that $Y = 710$ and

$$\mathbb{P}(Y=710 \mid \theta) = c\, \theta^{710} (1-\theta)^{290}.$$

The Bayes' formula implies that for any prior $f_\theta(\cdot)$,

$$f_{\theta|Y}(\theta \mid Y=710) = \frac{\mathbb{P}(Y=710 \mid \theta) f_\theta(\theta)}{\mathbb{P}(Y=710)}$$

$$= C.\ \theta^{710} (1-\theta)^{290} \cdot f_\theta(\theta).$$

a) $f_\theta(\theta) = 2\theta$. Then,

$$f_{\theta|Y}(\theta \mid Y=710) = 2C\ \theta^{711} (1-\theta)^{290}.$$

which is beta with $a=712, b=291$. As $\hat{\theta}$ is equal to its mean, $\boxed{\hat{\theta} = \frac{a}{a+b} = \frac{712}{1002}}$

b) Now $f_\theta(\theta) = 4\theta^3$:

$$f_{\theta|Y}(\theta \mid Y=710) = 4C\ \theta^{713} (1-\theta)^{290} = \text{Beta}(a=714, b=291)$$

$$\boxed{\hat{\theta} = \frac{714}{1005}}$$

c) As in the previous problem, $\hat{\theta}_{MLE}$ maximizes

$$L(\theta) = \ln(\mathbb{P}(Y=712|\theta)) = \ln(C) + 712\theta + 290(1-\theta)$$

and the maximizer is $\boxed{\hat{\theta}_{MLE} = \frac{712}{1000}}$

Problem 3.7 As $Y_1, .., Y_n$ are independent, with $y=(y_1,..,y_n)$,

$$f_{Y|\theta}(y|\theta) = \prod_{k=1}^{n} f_{Y_k|\theta}(y_k|\theta)$$
$$= \theta^n \left(\prod_{k=1}^{n} y_k\right)^{\theta-1} \qquad y_k \in (0,1).$$

The maximum likelihood estimator $\hat{\theta}_{MLE}$ maximizes the function $\theta \mapsto f_{Y|\theta}(y|\theta)$ or equivalently

$$L(\theta) = \ln(f_{Y|\theta}(y|\theta)) = n\ln\theta + \sum_{k=1}^{n}(\theta-1)\ln y_k.$$

We directly compute that

$$L'(\theta) = \frac{n}{\theta} + \sum_{k=1}^{n} \ln(y_k).$$

As $L'(\hat{\theta}_{MLE}) = 0$, we have

$$\boxed{\hat{\theta}_{MLE} = -\left(\frac{1}{n}\sum_{k=1}^{n} \ln(y_k)\right)^{-1}}$$

Problem 4.1. This is the solution with $h=1$!

In one-dimesion, the covariance equations are:

$$\bar{P}_{k+1} = a^2 P_k + q_k = P_k + 1 \qquad k=0,1\ldots$$

$$K_k = \frac{\bar{P}_k}{\bar{P}_k + r} = \frac{\bar{P}_k}{\bar{P}_k+1}, \qquad P_k = (I - K_k h)\bar{P}_k = \frac{\bar{P}_k}{\bar{P}_k+1}$$

a) As $P_0 = 1$, $\bar{P}_1 = 2 \Rightarrow P_1 = \frac{2}{3} \Rightarrow \bar{P}_2 = \frac{5}{3} \Rightarrow P_2 = \frac{5}{8}$

b) Passing to the limit in the above equations, we obtain the following limiting equations for $\bar{p}^*$, K^* and p^*:

$$\bar{p}^* = p^* + 1, \quad K^* = p^* = \frac{\bar{p}^*}{\bar{p}^*+1}.$$

Set $x := \bar{p}^*$. Then, x is the positive solution of

$$x = \frac{x}{x+1} + 1 \Rightarrow x^2 + x = 2x + 1 \Rightarrow x^2 - x - 1 = 0 \Rightarrow x = \frac{1+\sqrt{5}}{2}$$

$$\boxed{\bar{p}^* = \frac{1+\sqrt{5}}{2}} \Rightarrow \boxed{K^* = p^*} = \frac{\bar{p}^*}{\bar{p}^*+1} = \boxed{\frac{1+\sqrt{5}}{3+\sqrt{5}}}$$

c) If $P_0 = p^*$, then

$$\bar{P}_1 = P_0 + 1 = p^* + 1 = \frac{1+\sqrt{5}}{3+\sqrt{5}} + 1 = \frac{4+2\sqrt{5}}{3+\sqrt{5}} \, \frac{3-\sqrt{5}}{3-\sqrt{5}} = \frac{2(1+\sqrt{5})}{9-5} = \bar{p}^*,$$

and $$K_1 = P_1 = \frac{\bar{P}_1}{\bar{P}_1 + 1} = \frac{1+\sqrt{5}}{3+\sqrt{5}} = K^* = p^*.$$

Hence, $\bar{P}_k = p^*$ and $K_k = P_k = p^*$ for every $k = 1,2\ldots$

d) We know that i_k are independent and have mean zero.

Also $\operatorname{cov}(i_k) = \bar{P}_k + r = \bar{p}^* + 1 = (3+\sqrt{5})/2$. Hence, identical.

Problem 4.2. Covariance equations are

(1) $\bar{P}_{k+1} = A_k P_k A_k^T + Q_k = P_k + I$

(2) $K_k = \bar{P}_k H^T (H\bar{P}_k H^T + R_k)^{-1} = \bar{P}_k \begin{bmatrix}1\\1\end{bmatrix}\left([1,1]\bar{P}_k \begin{bmatrix}1\\1\end{bmatrix}+1\right)^{-1}$

(3) $P_k = (I - K_k H_k)\bar{P}_k = (I - K_k[1,1])\bar{P}_k$.

a) As $P_0 = I$, we have $\boxed{\bar{P}_1 =} P_0 + I = \boxed{2I}$. Then,

$$[1,1]\bar{P}_1\begin{bmatrix}1\\1\end{bmatrix} = 4 \Rightarrow \boxed{K_1 =} \bar{P}_1\begin{bmatrix}1\\1\end{bmatrix}\frac{1}{5} = \boxed{\frac{2}{5}\begin{bmatrix}1\\1\end{bmatrix}}$$

Then, $\boxed{P_1 =} \left(I - \frac{2}{5}\begin{bmatrix}1\\1\end{bmatrix}[1\ 1]\right)2I = \begin{bmatrix}3/5 & -2/5\\ -2/5 & 3/5\end{bmatrix}2I = \boxed{\begin{bmatrix}6/5 & -4/5\\ -4/5 & 6/5\end{bmatrix}}$

$$\boxed{\bar{P}_2 =} P_1 + I = \boxed{\begin{bmatrix}11/5 & -4/5\\ -4/5 & 11/5\end{bmatrix}}$$

$$[1,1]\bar{P}_2\begin{bmatrix}1\\1\end{bmatrix}+1 = \frac{19}{5} \Rightarrow \boxed{K_2 =} \bar{P}_2\begin{bmatrix}1\\1\end{bmatrix}\frac{5}{19} = \boxed{\frac{11}{19}\begin{bmatrix}1\\1\end{bmatrix}}$$

b) $\boxed{\hat{x}_{1|0} =} A x_0 = x_0 = \boxed{\begin{bmatrix}0\\0\end{bmatrix}}$.

$$\boxed{\hat{x}_1 =} \hat{x}_{1|0} + K_1 i_1 = \frac{2}{5}\begin{bmatrix}1\\1\end{bmatrix}(z_1 - [1,1]\hat{x}_{1|0}) = \boxed{\frac{2z_1}{5}\begin{bmatrix}1\\1\end{bmatrix}}$$

$$\boxed{\hat{x}_{2|1} =} A\hat{x}_1 = \boxed{\hat{x}_1}$$

$$\boxed{\hat{x}_2 =} \hat{x}_{2|1} + K_2 i_2 = \hat{x}_1 + \frac{11}{19}\begin{bmatrix}1\\1\end{bmatrix}(z_2 - [1,1]\hat{x}_1)$$

$$= \left(1 - \frac{22}{19}\right)\hat{x}_1 + \frac{11z_2}{19}\begin{bmatrix}1\\1\end{bmatrix} = \boxed{\left(-\frac{6}{5\cdot 19}z_1 + \frac{11}{19}z_2\right)\begin{bmatrix}1\\1\end{bmatrix}}$$

c) Let $\mathcal{R} = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$ and set

$$\bar{U}_k = \mathcal{R}\bar{P}_k\mathcal{R}, \quad \tilde{K}_k = \mathcal{R}K_k, \quad U_k = \mathcal{R}P_k\mathcal{R}.$$

Then, as $\mathcal{R}\mathcal{R} = I$,

(I) $\bar{U}_{k+1} = \mathcal{R}\bar{P}_{k+1}\mathcal{R} = \mathcal{R}[P_k + I]\mathcal{R} = U_k + I$,

$$\tilde{K}_k = \mathcal{R}K_k = \mathcal{R}\left\{\bar{P}_k \begin{bmatrix}1\\1\end{bmatrix}\left([1,1]\bar{P}_k\begin{bmatrix}1\\1\end{bmatrix}+1\right)^{-1}\right\}$$

$$= \bar{U}_k\begin{bmatrix}1\\1\end{bmatrix}\left([1,1]\bar{P}_k\begin{bmatrix}1\\1\end{bmatrix}+1\right)^{-1}.$$

Also, $[1,1]\bar{P}_k\begin{bmatrix}1\\1\end{bmatrix} = [1,1]\bar{U}_k\begin{bmatrix}1\\1\end{bmatrix}$. Hence,

(II) $\tilde{K}_k = \bar{U}_k\begin{bmatrix}1\\1\end{bmatrix}\left([1,1]\bar{U}_k\begin{bmatrix}1\\1\end{bmatrix}+1\right)^{-1}$.

Finally,

$$U_k = \mathcal{R}P_k\mathcal{R} = \mathcal{R}[(I - K_k[1,1])\bar{P}_k]\mathcal{R}$$

$$= \mathcal{R}[\bar{P}_k - K_k[1,1]\bar{P}_k]\mathcal{R}$$

$$= \bar{U}_k - \tilde{K}_k[1,1]\bar{P}_k\mathcal{R}$$

Since $\mathcal{R}\mathcal{R} = I$

$$[1,1]\bar{P}_k\mathcal{R} = [1,1]\mathcal{R}\mathcal{R}\bar{P}_k\mathcal{R} = [1,1]\mathcal{R}\bar{U}_k = [1,1]\begin{bmatrix}0&1\\1&0\end{bmatrix}\bar{U}_k$$

$$= [1,1]\bar{U}_k.$$

Hence,

(III) $U_k = \bar{U}_k - \tilde{K}_k[1,1]\bar{U}_k = (I - \tilde{K}_k[1,1])\bar{U}_k$.

Equations (I), (II), (III) are exactly equal to (1), (2), (3).

Moreover, the initial condition satisfies:

$$U_0 = \mathcal{R}P_0\mathcal{R} = \begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}0&1\\1&0\end{bmatrix} = I = P_0.$$

Therefore, $\bar{U}_k = P_k$, $\tilde{K}_k = K_k$ and $U_k = P_k$.

Set

$$P_k = \begin{bmatrix} a_k & c_k \\ c_k & b_k \end{bmatrix} \Rightarrow \quad U_k = R P_k R = \begin{bmatrix} b_k & c_k \\ c_k & a_k \end{bmatrix} = P_k$$

This proves that $a_k = b_k$ and $P_k = \begin{bmatrix} b_k & c_k \\ c_k & b_k \end{bmatrix}$.

Similarly one proves that $\bar{P}_k = R\bar{P}_k R = \begin{bmatrix} \bar{b}_k & \bar{c}_k \\ \bar{c}_k & \bar{b}_k \end{bmatrix}$.

Finally, if $K_k = \begin{bmatrix} m_k \\ n_k \end{bmatrix}$ then

$$\begin{bmatrix} m_k \\ n_k \end{bmatrix} = K_k = \tilde{K}_k = R K_k = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} m_k \\ n_k \end{bmatrix} = \begin{bmatrix} n_k \\ m_k \end{bmatrix}.$$

So $m_k = n_k$ and $K_k = \begin{bmatrix} m_k \\ m_k \end{bmatrix}$.

For the state estimates, we prove by induction.

For $k=0$, $\hat{x}_0 = (\hat{v}_0, \hat{s}_0) = (0,0)$. Hence, $\hat{v}_0 = \hat{s}_0$. Now,

suppose that $\hat{v}_k = \hat{s}_k$. Then,

$$\begin{bmatrix} \hat{v}_{k+1} \\ \hat{s}_{k+1} \end{bmatrix} = \hat{x}_{k+1} = \hat{x}_k + K_{k+1} i_{k+1} = \begin{bmatrix} \hat{v}_k \\ \hat{v}_k \end{bmatrix} + \begin{bmatrix} m_{k+1} \\ m_{k+1} \end{bmatrix} \left(z_{k+1} - [1, 1] \begin{bmatrix} \hat{v}_k \\ \hat{s}_k \end{bmatrix} \right)$$

$$= \begin{bmatrix} \hat{v}_k \\ \hat{v}_k \end{bmatrix} + \begin{bmatrix} m_{k+1} \\ m_{k+1} \end{bmatrix} (z_{k+1} - 2\hat{v}_k).$$

Therefore $\hat{v}_{k+1} = \hat{s}_{k+1} = \hat{v}_k + m_{k+1}(z_{k+1} - 2\hat{v}_k)$.

d) As $x_{k+1} = \begin{bmatrix} v_{k+1} \\ s_{k+1} \end{bmatrix} = \begin{bmatrix} v_k \\ s_k \end{bmatrix} + \begin{bmatrix} w^{(1)}_{k+1} \\ w^{(2)}_{k+1} \end{bmatrix}$

The new state $y_k = v_k + s_k$ solves

$$y_{k+1} = y_k + \underbrace{(w^{(1)}_{k+1} + w^{(2)}_{k+1})}_{=: \bar{w}_{k+1}} \Rightarrow \bar{w}_{k+1} \sim \mathcal{N}(0, 2)$$

Observation is

$$z_k = [1,1]\, x_k + v_k = (v_k + s_k) + v_k = y_k + v_k.$$

Therefore, this is a one-dimensional problem with $a=1, h=1$, $q=2$, $r=1$ and $y_0 = (v_0 + s_0) \sim \mathcal{N}(0,2)$. Covariance equations are $P_0 = 2$ and

$$\bar{P}_{k+1} = P_k + 2, \quad K_k = \frac{\bar{P}_k}{\bar{P}_k + 1}, \quad P_k = \frac{\bar{P}_k}{\bar{P}_k + 1}.$$

e)

$$P_k = \mathbb{E}[(y_k - \hat{y}_k)^2] = \mathbb{E}[(v_k + s_k - \hat{v}_k - \hat{s}_k)^2]$$

$$= \mathbb{E}[(v_k - \hat{v}_k)^2 + (s_k - \hat{s}_k)^2 + 2(v_k - \hat{v}_k)(s_k - \hat{s}_k)]$$

Also,

$$P_k = \begin{bmatrix} b_k & c_k \\ c_k & b_k \end{bmatrix} = \mathbb{E}[(x_k - \hat{x}_k)(x_k - \hat{x}_k)^T]$$

$$= \mathbb{E}\left[\begin{bmatrix} v_k - \hat{v}_k \\ s_k - \hat{s}_k \end{bmatrix} [v_k - \hat{v}_k, s_k - \hat{s}_k] \right]$$

Hence,

$$b_k = \mathbb{E}(v_k - \hat{v}_k)^2 = \mathbb{E}[(s_k - \hat{s}_k)^2]$$

$$c_k = \mathbb{E}[(v_k - \hat{v}_k)(s_k - \hat{s}_k)].$$

These imply that $P_k = 2(b_k + c_k)$.