Synchronization Games

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I would like to thank the organizers and also to the Italian analysis for many wonderful meetings over the decades. Below is one from 1991 in Castiglione della Pescia.

This is joint work with
Rene Carmona (Princeton),
Quentin Cormier (INRIA and Ecole Polytechnique),
Felix Höfer (Princeton).

Outline

Kuramoto Dynamical System

Equation and Order parameter

Kuramoto (1975) considered a population of N coupled phase oscillators θ_t^k having natural frequencies ω^k distributed with a given density, and whose dynamics are governed by

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\theta_t^k = \omega^k + \frac{\kappa}{N}\sum_{j=1}^{N} \sin(\theta_t^j - \theta_t^k), \qquad k = 1, \ldots, N.}$$

The following complex order parameter simplifies the equation :

$$r_t\, e^{i\,\psi_t} := \frac{1}{N}\sum_{j=1}^{N} e^{i\,\theta_t^j} \quad \Rightarrow \quad r_t\, \sin(\psi_t - \theta_t^k) = \frac{1}{N}\sum_{j=1}^{N} \sin(\theta_t^j - \theta_t^k) \quad \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}\theta_t^k = \omega^k + \kappa\, r_t\, \sin(\psi_t - \theta_t^k).$$

Also, when $\omega^k = 0$, the following integral decreases along the solutions.

$$\mathcal{E} := \int\int \sin^2((x-y)/2)\ \mu_t^N(\mathrm{d}x)\mu_t^N(\mathrm{d}y), \quad \text{where} \quad \mu_t^N(\mathrm{d}x) := \frac{1}{N}\sum_{k=1}^{N} \delta_{\theta_t^k}(\mathrm{d}x).$$

There exists a critical threshold $\widehat{\kappa}_c$ (depending on the distribution of ω^is) such that :

- For all $\kappa < \widehat{\kappa}_c$, the oscillators behave as if they are uncoupled. The phases become uniformly distributed and the coherence r_t decays like $1/\sqrt{N}$.
- For all $\kappa > \widehat{\kappa}_c$, the incoherent state becomes unstable and r_t grows to an eventual level $r_\infty < 1$. In the partially synchronized state, most oscillators co-rotate with the average phase ψ_t.
- As $\kappa \uparrow \infty$, synchronization increases and r_∞ gets closer to 1.

A good review of these results can be found in the 2000 paper of S. H. Strogatz and also in,
The Kuramoto model : A simple paradigm for synchronization phenomena
by Acebrón, Bonilla, Pérez, Ritort, Spigler (Review of modern physics, 2005).

This is from the github page of Helge Dietert from Paris.

`https://hdietert.github.io/static/kuramoto-animation/kuramoto.html`

Kuramoto Mean Field Game

1. Start with a deterministic flow of probability measures $\boldsymbol{\mu} = (\mu_t)_{t\geq 0}$ with $\mu_0 = \mu$.
2. Find the optimal control $\boldsymbol{\alpha}^{*,\boldsymbol{\mu}} = (\alpha_t^{*,\boldsymbol{\mu}})_{t\geq 0}$ minimizing,

$$\boldsymbol{\alpha} = (\alpha_t)_{t\geq 0} \quad \mapsto \quad J(\boldsymbol{\alpha}\,;\ \boldsymbol{\mu}) := \mathbb{E}\int_0^\infty e^{-\beta t}\left[\kappa\ L(X_t^{\boldsymbol{\alpha}}, \mu_t) + \tfrac{1}{2}(\alpha_t)^2\right] \mathrm{d}t,$$

where $\mathrm{d}X_t^{\boldsymbol{\alpha}} = \alpha_t \mathrm{d}t + \sigma \mathrm{d}B_t$, $Law(X_0) = \mu_0$, and

$$L(x, \mu) := 2\int_{-\pi}^{\pi} \sin^2((x-y)/2)\ \mu(\mathrm{d}y) = \delta_m \mathcal{E}(\mu)(x).$$

3. Find a fixed point $\mu_t = Law(X_t^{\boldsymbol{\alpha}^{*,\boldsymbol{\mu}}})$.

Synchronization of coupled oscillators is a game, by Yin, Mehta, Meyn, Shanbhag, IEEE (2011).

Synchronization in a Kuramoto Mean Field Game, Carmona, Cormier, Soner, CPDE (2023).

Let $U(\mathrm{d}x) := \frac{\mathrm{d}x}{2\pi}$ be the uniform measure on the circle. Then,

$$L(x, U) = \int_{-\pi}^{\pi} 2\sin^2(\tfrac{x-y}{2})\ U(\mathrm{d}x) \equiv 1.$$

Then, the control problem corresponding to the stationary flow U is

$$\text{minimize} \ \ \boldsymbol{\alpha} = (\alpha_t)_{t\geq 0} \ \mapsto \ J(\boldsymbol{\alpha}\,;\ U) := \mathbb{E}\int_0^{\infty} e^{-\beta t}\,[\kappa + \tfrac{1}{2}(\alpha_t)^2]\ \mathrm{d}t.$$

Cleary the optimal solution is $\alpha^* \equiv 0$, and the optimal state is $\mathrm{d}X_t^* = 0\,\mathrm{d}t + \sigma\mathrm{d}B_t$. Hence, $X_t^* = X_0^* + \sigma B_t$ and as $Law(X_0^*) = U$, we have $Law(X_t^*) = U$ as well. Hence,

The uniform measure U is a stationary solution of the KMFG for every parameter.

Critical interaction parameter is $\kappa_c := \beta\sigma^2 + \sigma^4/2$.

Theorem (Sub-critical interaction : incoherence)

(Carmona, Cormier, S.)(2023) For $\kappa < \kappa_c$, the uniform measure is locally stable.
Namely, there exist a positive constant $\rho > 0$ depending on β, σ, κ such that for any μ_0 satisfying $d(\mu_0 - U) \leq \rho$, there exists a solution $\boldsymbol{\mu} = (\mu_t)_{t\geq 0}$ of the Kuramoto mean field game with interaction parameter κ with $\mu_0 = \nu$ and μ_t converges in law to the uniform distribution as t tends to infinity.

Theorem (Super-critical interaction : synchronization)

(Carmona, Cormier, S.)(2023) For $\kappa > \kappa_c$, there exists a non-trivial stationary solutions of the KMFG.

- We do not know whether the uniform is the only NE in the subcritical case, or it is globally stable.
- We do not know whether the uniform is unstable in the supercritical case.
- We do not know the minimizers of the associated control problem.
- Only results are by Cesaroni & M. Cirant, CPDE, 2024.

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Synchronization Game

We discretize the Kuramoto dynamics ‘severely’.

- Assume there are two types of particles, $\{0, 1\}$.
- Without control each particle changes type randomly with rate $\sigma > 0$ (thermal noise).
- Particles want to have the same type as the majority of the particles.
- To achieve their goal, particles can increase the rate of change but with a quadratic penalty.

Synchronization Games (2024), by Höfer & Soner.

Exact same model with $\sigma = 0$ is also studied in
Climb on the Bandwagon : Consensus and Periodicity in a Lifetime Utility Model with Strategic Interactions (2019), by Dai Pra, Sartori, & Tolotti,

- We are given a probability flow $p(\cdot)$ representing the proportion of the players in state 1, or equivalently, the probability of being at the state 1 ;
- Note that the probability $p(\cdot)$ is not affected by the type of any individual player ;
- Control is any square integrable function α_t ;
- Given control α,
  i. rate from 0 to 1 is $\sigma^2 + \alpha_t^+$;
  ii. rate from 1 to 0 is $\sigma^2 + \alpha_t^-$;
- Let $X_t^\alpha \in \{0, 1\}$ be the type process of the representative particle corresponding to the control α.
- Running cost is :

$$\frac{1}{2}\alpha_t^2 + \ell(X_t^\alpha, p(t)), \qquad \ell(x, p) := p\chi_{x=0} + (1-p)\chi_{x=1}.$$

- The control problem is

$$\text{minimize } \mathbb{E} \int_0^\infty e^{-\beta s} \left[\frac{1}{2}\alpha_s^2 + \kappa\ell(X_s^\alpha, p(s))\right] \mathrm{d}s.$$

- The parameter κ is as in the original model.

Results - Discounted infinite horizon

For a positive discount factor $\beta > 0$ the critical coupling or misalignment strength is

$$\boxed{\kappa_c = 2\beta\sigma^2 + 4\sigma^4.}$$

Subcritical regime $\kappa < \kappa_c$. The uniform distribution is the unique stationary Nash equilibrium (SNE) and it is stable. Further, for any initial distribution there exists a unique time-inhomogeneous Nash equilibrium (NE), and it converges to the uniform distribution.

Supercritical regime $\kappa > \kappa_c$. There exist three SNE : the uniform distribution and two symmetric self-organizing SNE given by $\underline{p} < 1/2 < \overline{p}$ where $\underline{p} = 1 - \overline{p}$. We have two sub-regimes :

(A) $\kappa_c < \kappa < \kappa_c + \beta^2/4$. For any non-uniform initial distribution there exists a unique NE and it converges to one of the self-organizing SNE.

(B) $\kappa > \kappa_c + \beta^2/4$. For initial conditions close to the uniform distribution there exist many NE that spiral around the uniform distribution before converging to one of the self-organizing SNE.

Here the critical interaction parameter is

$$\boxed{\kappa_c = 4\sigma^4.}$$

- In the subcritical regime $\kappa < \kappa_c$, the uniform distribution $p^* = 1/2$ is again the unique SNE, and there are no other ergodic NE.
- In the supercritical case $\kappa > \kappa_c$, in addition to the uniform distribution, there are two other symmetric SNE. However, in contrast to the discounted model, there are infinitely many periodic NE rotating around the uniform distribution as well.
- In the subcritical regime, the uniform distribution is the minimizer of the associated ergodic mean-field control (MFC) problem. It fails to be the minimizer in the supercritical case.

Given a probability flow $p(\cdot)$, $t \geq 0$ and $x \in \{0,1\}$, let

$$v(t,x) := \inf_{\alpha} \mathbb{E} \int_t^{\infty} e^{-\beta(s-t)} \left[\frac{1}{2}\alpha_s^2 + \kappa\, \ell(X_s^{\alpha}, p(s))\right] \mathrm{d}s, \qquad X_t^{\alpha} = x.$$

Then,

$$\boxed{\begin{aligned} -v_t(t,0) + \beta v(t,0) &= H(-\, a(t)) + \kappa p(t), \\ -v_t(t,1) + \beta v(t,1) &= H(\, a(t)) + \kappa(1 - p(t)), \\ H(a) := +\inf_{z \geq 0}\{\frac{1}{2}z^2 + (\sigma^2 + z)a\} &= \sigma^2 a - \frac{1}{2}(a^-)^2, \qquad a \in \mathbb{R}. \end{aligned}}$$

where $a(t) := v(t,0) - v(t,1)$ is the optimal control. The above equations imply that

$$\boxed{\dot{a}(t) = (\beta + 2\sigma^2)a(t) + \frac{1}{2}\mathrm{sign}(a(t))a(t)^2 - \kappa(2p(t) - 1).}$$

Let X_t^a be the optimal state process. Then $p(\cdot)$ is a Nash equilibrium if $p(t) = \mathbb{P}(X_t^a = 1)$ for all $t \geq 0$.

Since $p(t) = \mathbb{P}(X_t^a = 1)$, $p(\cdot)$ solves the Fokker-Planck-Kolmogorov equation, which is a one dimensional ODE, coupled with the equation derived earlier for the optimal control $a(\cdot)$:

$$\boxed{\begin{aligned} \dot{a}(t) &= (\beta + 2\sigma^2)a(t) + \frac{1}{2}\mathrm{sign}(a(t))a(t)^2 - \kappa(2p(t) - 1) \\ \dot{p}(t) &= (\sigma^2 + a^+(t))(1 - p(t)) - (\sigma^2 + a^-(t))p(t). \end{aligned}}$$

Theorem

A probability flow $p(\cdot)$ is a Nash equilibrium, if and only if there is a continuous function $a(\cdot)$ such that the pair (a, p) is a bounded solution of the above dynamical system.

Note that $\beta = 0$ corresponds to the ergodic, and also to the finite-horizon problems.

$\Rightarrow$: This follows from the previous calculations. Since the value function $v(t,x)$ is bounded, $a(t) = v(t,0) - v(t,1)$ is also bounded.

$\Leftarrow$: Suppose (a,p) is a bounded solution. We set

$$V(t,0) := \int_t^\infty e^{\beta(t-u)}[H(-a(s)) + \kappa p(s)]\,\mathrm{d}s,$$
$$V(t,1) := \int_t^\infty e^{\beta(t-u)}[H(a(s)) + \kappa(1-p(s))]\,\mathrm{d}s.$$

Then, $A(t) := V(t,0) - V(t,1)$ solves

$$\frac{\mathrm{d}}{\mathrm{d}t}(e^{-\beta t}A(t)) = 2\sigma^2 a(t) + \frac{1}{2}\mathrm{sign}(a(t))a(t)^2 - \kappa(2p(t)-1) = \frac{\mathrm{d}}{\mathrm{d}t}(e^{-\beta t}a(t)).$$

Using the boundedness of a we conclude that $A \equiv a$. Hence, $V(t,x)$ is a solution of the dynamic programming equation. Therefore, $a = A$ is the optimal control.

Phase transitions and Qualitative behavior

In the next slights we use the above characterization of the Nash equilibria to prove the phase transition results. Main tool is the analysis of the two dimensional dynamical system.

To achieve symmetry, we set

$$\boxed{q(t) := 2p(t) - 1.}$$

Then, the ODE for the pair (a, q) can be rewritten as

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}(a, q) = f(a, q) := (\beta + 2\sigma^2)a + \operatorname{sign}(a)a^2/2 - \kappa q\ ,\ a - (2\sigma^2 + |a|)q).}$$

We note that $div(f) \equiv \beta$, and the stationary solutions are the zeroes of f.

Recall that

$$f(a,q) := (\beta + 2\sigma^2)a + \mathrm{sign}(a)a^2/2 - \kappa q \ , \ a - (2\sigma^2 + |a|)q).$$

A direct calculation reveals that origin is the only solution for $\kappa \le \kappa_c$, where

$$\boxed{\kappa_c := 2\beta\sigma^2 + 4\sigma^4.}$$

However, if $\kappa > \kappa_c$, there are three stationary points $\{(-\bar{a}, -\bar{q}), (0,0), (\bar{a}, \bar{q})\}$, where

$$\bar{q} = \frac{\bar{a}}{\bar{a} + 2\sigma^2}, \quad \bar{a} = -(\beta + 3\sigma^2) + \sqrt{(\beta + 3\sigma^2)^2 + 2(\kappa - \kappa_c)} > 0.$$

Phase transition at $\kappa = \kappa_c$?

Phase Diagrams

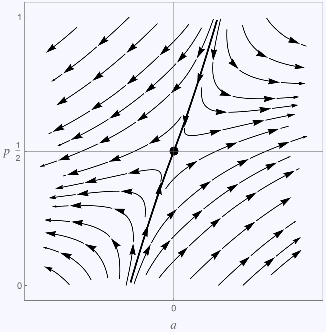


Figure 1: $\kappa < \kappa_c$

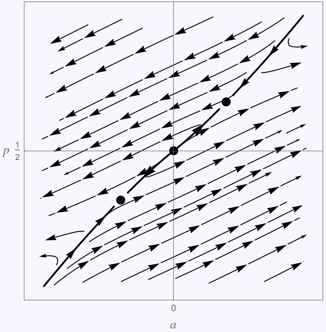


Figure 2: $\kappa_c < \kappa < \kappa_c + \beta^2/4$

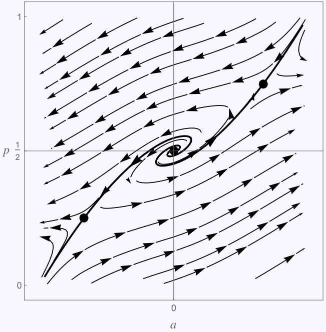


Figure 3: $\kappa > \kappa_c + \beta^2/4$.

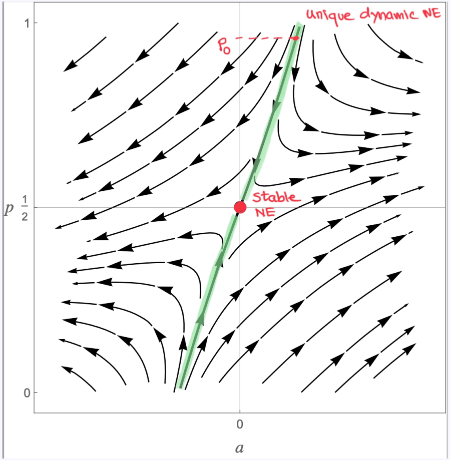


Figure 4: $\kappa < \kappa_c$

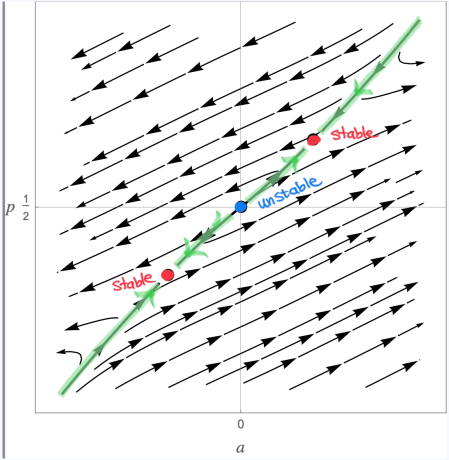


Figure 5: $\kappa_c < \kappa < \kappa_c + \beta^2/4$

Phase Diagram - large κ

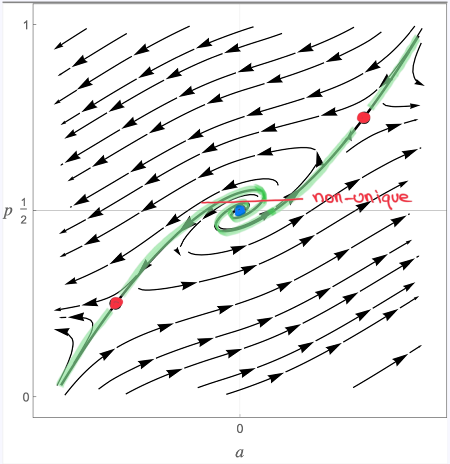


Figure 6: $\kappa > \kappa_c + \beta^2/4$

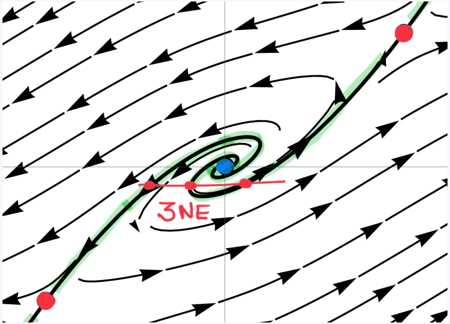


Figure 7: $\kappa > \kappa_c + \beta^2/4$

Phase Diagram - large κ

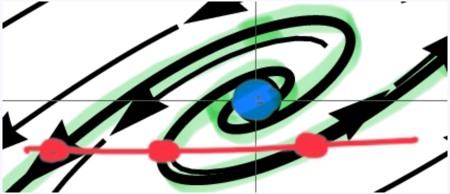


Figure 8: $\kappa > \kappa_c + \beta^2/4$

The local behavior of the dynamical system around the origin is described by the spectral properties of the linearized system,

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} a(t) \\ q(t) \end{pmatrix} = \begin{pmatrix} \beta + 2\sigma^2 & -\kappa \\ 1 & -2\sigma^2 \end{pmatrix} \begin{pmatrix} a(t) \\ q(t) \end{pmatrix}.$$

An analysis of the eigenvalues of this system shows that :

- *Subcritical regime* $\kappa < \kappa_c$: The origin is a saddle point (one positive, one negative eigenvalue).
- *Supercritical regime (A)* $\kappa_c < \kappa < \kappa_c + \beta^2/4$: The origin is unstable (two positive eigenvalues). The other stationary points are saddles.
- *Supercritical regime (B)* $\kappa > \kappa_c + \beta^2/4$: The origin is a spiral source (two complex eigenvalues with positive real part). The other two stationary points remain saddles.

We have the following corollary to the previous calculations of the stationary Nash equilibria (SNE) :

Lemma

Both in the ergodic and discounted models, the uniform distribution $q = 0$ is a SNE for any $\kappa > 0$. Moreover,

Subcritical case : For $\kappa < \kappa_c$, the uniform distribution is the only SNE, and it is the global attractor.

Supercritical case : For $\kappa > \kappa_c$, there are three SNE given by $q = -\bar{q}, 0, \bar{q}$. The origin is unstable and the other two are local attractors.

Full synchronization : As $\kappa \uparrow \infty$, $\bar{q} \uparrow 1$.

The uniform distribution is interpreted as incoherence. Hence, this abrupt transition at κ_c is analogous to the phase transitions for the Kuramoto dynamical system and MFG.

Ergodic Cost

$$\begin{aligned}\dot{a}(t) &= 2\sigma^2 a(t) + \frac{1}{2}\mathrm{sign}(a(t))a(t)^2 - \kappa q(t)\\ \dot{q}(t) &= a(t) - (2\sigma^2 + |a(t)|)q(t).\end{aligned}$$

This system is conservative.

It admits the following first integral which is constant along any solution ;

$$E(a,q) = \kappa\frac{q^2}{2} + \frac{a^2}{2} - 2\sigma^2 aq - \frac{a^2}{2}\mathrm{sign}(a)q.$$

- In the subcritical regime, the uniform distribution is the unique SNE, and there are no other ergodic NE.
- In the supercritical case, in addition to the uniform distribution, there are two other symmetric SNE.
- In the supercritical case, there are also infinitely many periodic NE rotating around the uniform distribution. In dynamical systems terminology, the origin is a center of the differential equations.

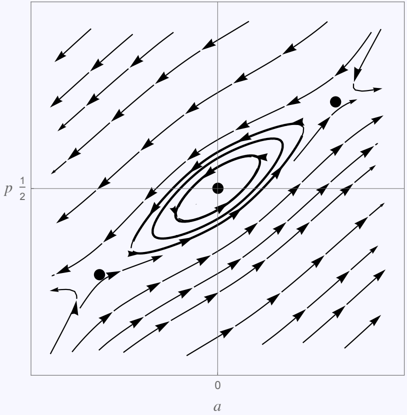
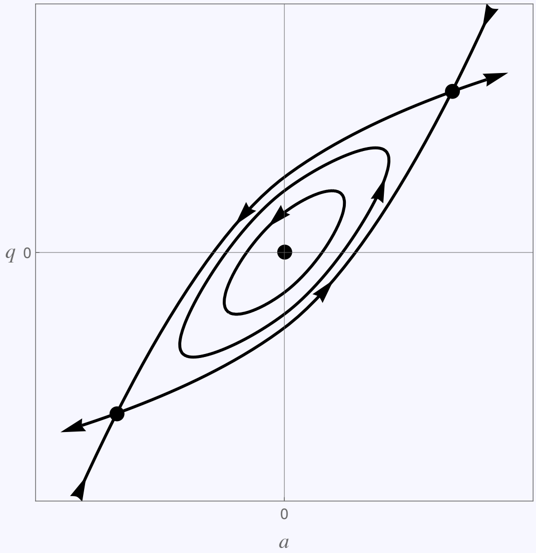


Figure 9: Phase diagram of the supercritical ergodic.


q
0
0
a

Potential Structure

- Two state synchronization games as well the Kuramoto mean field games are potential.
- In the Kuramoto, the running cost is given by

$$\ell(x,\mu) = 1 - \mu(\cos)\cos(x) - \mu(\sin)\sin(x) = \delta_\mu L(\mu),$$

where

$$L(\mu) = \int\int \sin^2\left(\tfrac{x-y}{2}\right)\,\mu(\mathrm{d}x)\,\mu(\mathrm{d}y).$$

- Similarly in the two state synchronization,

$$\ell(x,p) = p\chi_{\{x=0\}} + (1-p)\chi_{\{x=1\}}.$$

- Given $p \in [0,1]$, let $\mu(\{1\}) = p$. Then,

$$\ell(x,p) = \delta_\mu L(\mu)(x),$$

where

$$L(\mu) := \frac{1}{2}\sum_x \sum_y \ |x-y|\ \mu(\mathrm{d}x)\ \mu(\mathrm{d}y) = p(1-p).$$

Recall that any $\alpha(t)$ is a feedback control :

$$\alpha(t,0)=\alpha(t)^{+}, \quad \alpha(t,1)=\alpha(t)^{-}.$$

Hence, the control problem is,

$$\boxed{\text{minimize } \mathbb{E}\int_0^{\infty} e^{-\beta t}[\frac{1}{2}\alpha(t,X_t^{\alpha})^2+\frac{\kappa}{2}L(\mu_t^{\alpha})]\,\mathrm{d}t,}$$

where μ_t^{α} is the law of X_t^{α}. This is a deterministic optimal control problem :

$$\boxed{\begin{aligned}&\text{minimize } \int_0^{\infty} e^{-\beta t}[(p(t)(\alpha^{-}(t))^2+(1-p(t))(\alpha^{+}(t))^2)+\frac{\kappa}{2}p(t)(1-p(t))]\,\mathrm{d}t,\\ &\text{where} \quad \dot{p}(t)=-p(t)(\sigma^2+\alpha^{-}(t))+(1-p(t))(\sigma^2+\alpha^{+}(t)).\end{aligned}}$$

Uniform is the minimizer when...

- Fact : Minimizers are Nash equilibria.
- So whenever there is a unique Nash equilibrium, it must be the minimizer.
- Ergodic case : uniform is the minimizer only when $\kappa < \kappa_c$.
- Discounted case : uniform is the minimizer if $\kappa < \kappa_c + \frac{1}{4}\beta^2$.
- Note that for $\kappa_c < \kappa < \kappa_c + \frac{1}{4}\beta^2$, the uniform is *unstable* but it still *is the minimizer*.
- Notion of a stable Nash equilibrium is an interesting question.

- Mean Field formalism have exactly the same solution structure as the dynamical system approach.
- As the uniform solutions are the desynchronized states, our results indicate a bifurcation from incoherence to self-organization at κ_c, and then convergence to full synchronization for very large interaction parameters.
- We describe all equilibria and minimizers for two state synchronization game.

THANK YOU FOR YOUR ATTENTION.

Synchronization in a Kuramoto Mean Field Game

with Rene Carmona and Quentin Cormier,
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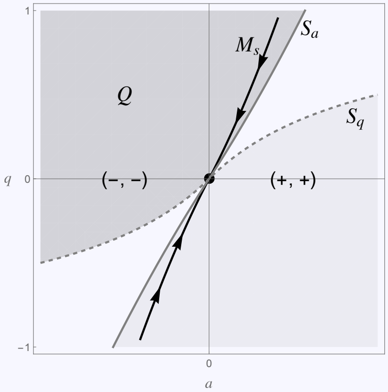
Proposition

Suppose that $\kappa < \kappa_c$. Then, there exist a strictly increasing function $\mathfrak{n} : [-1, 1] \to \mathbb{R}$ with $\mathfrak{n}(0) = 0$ such that its graph $\{(\mathfrak{n}(q), q) : q \in [-1, 1]\}$ is the stable manifold of the origin of the dynamical system. All Nash equilibria are included in the graph of $\mathfrak{n}$. In particular, for any $q \in [-1, 1]$ there exists exactly one discounted Nash equilibrium starting from q, and all Nash equilibria converge to the origin.



There exist three SNE : the uniform distribution and two symmetric self-organizing SNE given by $\underline{p} < 1/2 < \overline{p}$ where $\underline{p} = 1 - \overline{p}$. The local behavior around the uniform distribution depends on two sub-regimes :

(A) $\kappa_c < \kappa < \kappa_c + \beta^2/4$. For any non-uniform initial distribution there exists a unique NE and it converges to one of the self-organizing SNE.

(B) $\kappa > \kappa_c + \beta^2/4$. For initial conditions close to the uniform distribution there exist many NE that spiral around the uniform distribution before converging to one of the self-organizing SNE.

Super-critical Phase Diagrams

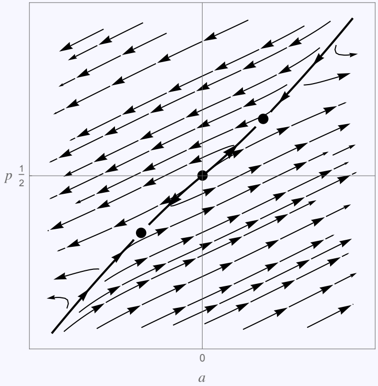


Figure 10: Supercritical case (A) : $\kappa_c < \kappa < \kappa_c + \beta^2/4$. The thick line corresponds to the monotone curve $\mathcal{C}$, and the dots show the stationary equilibria.

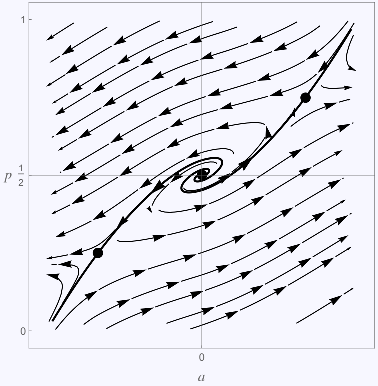


Figure 11: Supercritical case (B) : $\kappa > \kappa_c + \beta^2/4$. The thick line again corresponds to the curve $\mathcal{C}$, which spirals around $(0, 1/2)$.