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VISCOSITY SOLUTIONS FOR MCKEAN–VLASOV CONTROL ON A TORUS*

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Abstract. An optimal control problem in the space of probability measures and the viscosity solutions of the corresponding dynamic programming equations defined using the intrinsic linear derivative are studied. The value function is shown to be Lipschitz continuous with respect to a smooth Fourier–Wasserstein metric. A comparison result between the Lipschitz viscosity sub- and supersolutions of the dynamic programming equation is proved using this metric, characterizing the value function as the unique Lipschitz viscosity solution.

Key words. mean-field games, Wasserstein metric, viscosity solutions, McKean–Vlasov

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1. Introduction.

McKean–Vlasov optimal control is a part of the overarching program of Lasry and Lions [24, 25, 26], as articulated by Lions through his College de France lectures [27] and independently initiated by Huang, Malhamé, and Caines [22]. We refer the reader to the classic book of Carmona and Delarue [8] and to the lecture notes of Cardaliaguet [6] for detailed information and more references.

The main feature of the McKean–Vlasov-type optimization is the dependence of its evolution and cost not only on the position of the state but also on its probability distribution, making the set of probability measures its state space. Thus, the dynamic programming approach results in nonlinear partial differential equations set in the space of probability measures. Without common noise, they are first-order Hamilton–Jacobi–Bellman equations, and the Hamiltonian is defined only when the derivative of the value function is twice differentiable. In fact, this type of unboundedness is almost always the case for optimal control problems set in infinite-dimensional spaces [19] and is the main new technical difficulty.

These dynamic programming equations are analogous to the coupled Hamilton–Jacobi and Fokker–Planck–Kolmogorov systems that characterize the solutions of the mean-field games for which deep regularity results are proved in [7] under some structural conditions. However, in general, the dynamic programming equations for the McKean–Vlasov optimal control problems are not expected to admit classical solutions, as shown in subsection 4.1 below, and a weak formulation is needed.

Because the maximum principle is still the salient feature in these settings as well, the viscosity solutions of Crandall et al. [15, 16, 17, 20] are clearly the appropriate choice. However, due to the unboundedness of the Hamiltonian, the original definition must be modified. In fact, such modifications of viscosity solutions in infinite-dimensional spaces have already been studied extensively, and the book [19]

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provides an exhaustive account of these results. Still, it is believed that more can be achieved in the context of McKean–Vlasov due to the special structure of the set of probability measures. Indeed, an approach developed by Lions lifts the problems from the Wasserstein space to a regular $\mathbb{L}^2$ space and then exploits the Hilbert structure to obtain new comparison results. This procedure also delivers the novel *Lions derivative*, which has many useful properties, and we refer to [8] for its definition and more information. This method is further developed in several papers, including [1, 3, 12, 29, 30]. The choice of the appropriate notion of a derivative is also explored in a recent paper [21], which then utilizes the deep connections to geometry to prove uniqueness results for Hamiltonians that are bounded in the sense discussed above.

Our main goals are to develop a viscosity theory directly on the space of probability measures using the linear derivative, provide a comparison result, and obtain a characterization of the value function as the unique viscosity solution in a certain class of functions. A natural approach toward this goal is to project the problem onto finite-dimensional spaces to leverage the already developed theory on these structures. A second-order problem studied in [14] provides a clear example of this approach because its projections exactly solve the projected finite-dimensional equations. However, in general, these projections are only approximate solutions, and [13] uses the Ekeland variational principle together with Gaussian-smoothed Wasserstein metrics as gauge functions to control the approximation errors. A different technical tool is developed in [4], and [21] studies the pure projection problem. Other approaches include the path-dependent equations used in [34], gradient flows in [11], convergence analysis in [2], and an optimal stopping problem in [32, 33]. A recent paper [10] exploits the semiconvexity and also provides an extensive survey.

We, on the other hand, employ the classical viscosity technique of doubling the variables as done in [5] in lieu of projection. The central difficulty of this approach is to appropriately replace the "distance-square" term $|x-y|^2$ used in the finite-dimensional comparison proofs with the square of a metric on the space of measures. Thus, the crucial ingredient of our method is a novel smooth metric ρ_* defined by a Fourier-based modification of the Wasserstein metrics, which is, in fact, the norm of the dual of a classical Sobolev space. In addition to several properties of ρ_* proved in section 5, our other main results are a comparison between Lipschitz continuous subsolutions and viscosity supersolutions, Theorem 4.1, and the Lipschitz continuity of the value function with respect to a weaker metric, Theorem 4.2. Although the Lipschitz property of the value function is rather elementary for the Wasserstein metrics, it requires detailed analysis for the Fourier-based ones. Indeed, a technical estimate, Proposition 7.1, on the dependence of the solutions of the McKean–Vlasov stochastic differential equation on the initial distribution is needed for this property.

As our approach contains several new steps, we study the simplest problem that allows us to showcase its details and power concisely. In particular, to ease the notation, we omit the dependence of all functions on the time variable, which can be added directly. Additionally, dynamics with jumps can be included as done in [5]. The compact structure of the torus is clearly a simplifying feature as well. We leave the extension of our method to future studies, including [31]. However, our main structural assumption on the regularity of the feedback controls is a strong one, and further studies are needed to remove this restrictive condition. One possible direction is to consider problems with more structure. Indeed, in our recent paper [31], we exploit the separability of the cost function and the dynamics and the uniform ellipticity of the infinitesimal generator of the underlying diffusion process to obtain a comparison result under natural assumptions.

The paper is organized as follow. General structure and notations are given in section 2; in section 3, we define the problem and state the assumptions. The main results are stated in section 4. We construct a family of Fourier–Wasserstein metrics in section 5. The comparison result is proved in section 6 and the Lipschitz property in section 7. The standard results of dynamic programming and the viscosity property are proved in section 8 and, respectively, in section 9.

2. Notations. In this section, we summarize the notations and known results used in the following. We denote the dimension of the ambient space by d and the finite horizon by $T>0$. $\mathbb{Z}^d$ is the set of all d-tuples of integers. $\mathbb{T}^d=\mathbb{R}^d/(2\pi\mathbb{Z})^d$ is the d-dimensional torus with the metric given by $|x-y|_{\mathbb{T}^d}:=\inf_{k\in\mathbb{Z}^d}|x-y-2k\pi|$. We use a filtered probability space $(\Omega,\mathbb{F}=(\mathcal{F}_t)_{0\le t\le T},\mathbb{P})$ that supports Brownian motions. We assume that the initial filtration $\mathcal{F}_0$ is rich enough so that, for any probability measure on $\mathbb{T}^d$, there exists a random variable on Ω whose distribution is equal to this measure.

For a metric space $(E,\hat{d})$, $\mathcal{M}(E)$ is the set of all Radon measures on E, and $\mathcal{P}(E)$ denotes the set of all probability measures on E. Let $\mathbb{L}^0(E)$ be the set of all E-valued random variables. For $X\in\mathbb{L}^0(E)$, $\mathcal{L}(X)\in\mathcal{P}(E)$ is the *law of* X.

We denote the set of all continuous real-valued functions on E by $\mathcal{C}(E)$ and the bounded ones by $\mathcal{C}_b(E)\subset\mathcal{C}(E)$. We write $\mathcal{C}(E,\hat{d})$ when the dependence on the metric is relevant and $\mathcal{C}(E\mapsto Y)$ if the range Y is not the real numbers. For a positive integer n, $\mathcal{C}^n(E)$ is the set of all n-times continuously differentiable, real-valued functions with the usual norm $\|\cdot\|_{\mathcal{C}^n}$ given by the sum of supremum norms of each derivative of order at most n.

We endow $\mathcal{M}(E)$ with the weak* topology $\sigma(\mathcal{P}(E),\mathcal{C}_b(E))$ and write $\mu_n\rightharpoonup\mu$ when $\lim_{n\to\infty}\mu_n(f)=\mu(f)$ for every $f\in\mathcal{C}_b(E)$. Using the standard (linear) derivative on the convex set $\mathcal{P}(E)$, we say that $\phi\in\mathcal{C}(\mathcal{P}(E))$ is *continuously differentiable* if there exists a function $\partial_\mu\phi\in\mathcal{C}(\mathcal{P}(E)\mapsto\mathcal{C}(E)))$ satisfying

$$\phi(\nu)=\phi(\mu)+\int_0^1\int_E\partial_\mu\phi(\mu+\tau(\nu-\mu))(x)\,(\nu-\mu)(\mathrm{d}x)\,\mathrm{d}\tau\qquad\forall\,\mu,\nu\in\mathcal{P}(E).$$

Clearly, $\partial_\mu\varphi(\mu)$ has many representatives. However, when $\partial_\mu\varphi(\mu)$ is twice differentiable, the Hamiltonian $H(\mu,\partial_\mu\varphi(\mu))$ of the dynamic programming equation (3.3) below is independent of this choice.

We set $\mathcal{O}:=(0,T)\times\mathcal{P}(\mathbb{T}^d)$. For $\psi\in\mathcal{C}(\overline{\mathcal{O}})$ and $(t,\mu)\in\mathcal{O}$, $\partial_t\psi(t,\mu)$ denotes the time derivative evaluated at (t,μ), and $\partial_\mu\psi(t,\mu)\in\mathcal{C}(\mathbb{T}^d)$ denotes the derivative in the μ-variable again evaluated at (t,μ). $\mathbb{L}^2(\mathbb{T}^d)$ is the set of measurable functions on $\mathbb{T}^d$ that are square integrable with respect to the Lebesgue measure, with the following orthonormal Fourier basis:

$$e_k(x):=(2\pi)^{-\frac{d}{2}}\ e^{ik\cdot x},\qquad x\in\mathbb{T}^d,\ k\in\mathbb{Z}^d,$$

where $i=\sqrt{-1}$ and z^* is the complex conjugate of z. In particular, for any $\gamma\in\mathbb{L}^2(\mathbb{T}^d)$,

$$\gamma=\sum_{k\in\mathbb{Z}^d}F_k(\gamma)\,e_k,\quad\text{where}\quad F_k(\gamma):=\int_{\mathbb{T}^d}\gamma(x)e_k^*(x)\ \mathrm{d}x,\quad k\in\mathbb{Z}^d.$$

The following metrics on $\mathcal{P}(\mathbb{T}^d)$ are given by their dual representations:

$$\rho_\lambda(\mu,\nu):=\sup\{(\mu-\nu)(\psi)\ :\ \psi\in\mathbb{H}_\lambda(\mathbb{T}^d),\ \|\psi\|_\lambda\le 1\},\qquad\lambda\ge 1,$$
$$\widehat{\rho}_n(\mu,\nu):=\sup\{(\mu-\nu)(\psi)\ :\ \psi\in\mathcal{C}^n(\mathbb{T}^d),\ \|\psi\|_{\mathcal{C}^n}\le 1\},\qquad n=1,2,\dots,$$

where, in view of the Kantorovich duality, $\widehat{\rho}_1$ is the Wasserstein-one distance and, for $\lambda \geq 1$,

$$\mathbb{H}_\lambda(\mathbb{T}^d) := \{f \in \mathbb{L}^2(\mathbb{T}^d) : \|f\|_\lambda < \infty\}, \qquad \|f\|_\lambda := \left(\sum_{k\in\mathbb{Z}^d} (1+|k|^2)^\lambda\, |F_k(f)|^2\right)^{\frac{1}{2}}.$$

A classical Fourier representation of ρ_λ is derived in Corollary 5.2.

It is well known that $\mathbb{H}_\lambda$ is the classical Sobolev space with fractional derivatives. Indeed, for any integer $n \geq 1$, $\mathcal{C}^n(\mathbb{T}^d) \subset \mathbb{H}_n(\mathbb{T}^d) = W^{n,2}(\mathbb{T}^d)$ and $\widehat{\rho}_n \leq c_n \rho_n$ for some constant c_n. Moreover, by the embedding results, $\mathbb{H}_\lambda(\mathbb{T}^d) \subset \mathcal{C}^n(\mathbb{T}^d)$ if $\lambda > n + \frac{d}{2}$. In particular, we set

$$n_*(d) = n_* := 3 + \left\lfloor \frac{d}{2} \right\rfloor, \quad \mathcal{C}_* := \mathcal{C}^{n_*}(\mathbb{T}^d), \quad \rho_* := \rho_{n_*}, \quad \widehat{\rho}_* := \widehat{\rho}_{n_*}, \tag{2.1}$$

where $\lfloor a \rfloor$ is the integer part of a real number a. Then, $\mathbb{H}_{n_*}(\mathbb{T}^d) \subset \mathcal{C}^2(\mathbb{T}^d)$.

3. McKean–Vlasov control.

In this section, we define the *McKean–Vlasov optimal control* problem, and for a general introduction, we refer the reader to Chapter 6 in [8]. Formally, starting from $t \in [0,T]$, the goal is to choose feedback controls $(\alpha_u(\cdot))_{u\in[t,T]}$ so as to minimize

$$\int_t^T \mathbb{E}[\ell(X_u, \mathcal{L}(X_u), \alpha_u(X_u))]\, \mathrm{d}u + \varphi(\mathcal{L}(X_T)),$$

where ℓ is the *running cost*; φ is the *terminal cost*; b, σ are given functions; and, with a Brownian motion B, $\mathrm{d}X_u = b(X_u, \mathcal{L}(X_u), \alpha_u(X_u))\mathrm{d}u + \sigma(X_u, \mathcal{L}(X_u), \alpha_u(X_u))\mathrm{d}B_u$.

We continue by defining this problem properly.

3.1. Controlled processes.

Suppose that A is a closed Euclidean space, let the *control set* $\mathcal{C}_a$ be a subset of $\mathcal{C}(\mathbb{T}^d \to A)$ containing all constant functions, and let the *admissible controls* $\mathcal{A}$ be the set of (deterministic) measurable functions $\boldsymbol{\alpha} : [0,T] \mapsto \mathcal{C}_a$. We denote the value of any $\boldsymbol{\alpha} \in \mathcal{A}$ at time $u \in [0,T]$ by $\alpha_u \in \mathcal{C}_a$. The given functions are the drift vector $b = (b_1, \ldots, b_d) \in \mathbb{R}^d$; the $d \times d'$ volatility matrix $\sigma = (\sigma_{ij})$ with $i = 1, \ldots, d$, $j = 1, \ldots, d'$; and the costs ℓ, φ. We continue by stating our standing regularity assumptions on these functions:

$$b_i, \sigma_{ij}, \ell : \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d) \times A \mapsto \mathbb{R}, \qquad \varphi : \mathcal{P}(\mathbb{T}^d) \mapsto \mathbb{R}.$$

Recall $\mathcal{C}_*, \rho_*, \widehat{\rho}_*$ of (2.1), and, for $\alpha \in \mathcal{C}_a$, $x \in \mathbb{T}$ and $\mu \in \mathcal{P}(\mathbb{T}^d)$, set

$$b^\alpha(x,\mu) := b(x,\mu,\alpha(x)), \quad \sigma^\alpha(x,\mu) := \sigma(x,\mu,\alpha(x)), \quad \ell^\alpha(x,\mu) := \ell(x,\mu,\alpha(x)).$$

Assumption 3.1 (regularity). There exists $c < \infty$ such that, for all $\alpha \in \mathcal{C}_a$ and $\mu \in \mathcal{P}(\mathbb{T}^d)$,

$$\|b^\alpha(\cdot,\mu)\|_{\mathcal{C}_*} + \|\sigma^\alpha(\cdot,\mu)\|_{\mathcal{C}_*} + \|\ell^\alpha(\cdot,\mu)\|_{\mathcal{C}_*} \leq c,$$

and, for $h = b, \sigma, \ell, \varphi$,

$$|h(x,\mu,a) - h(x,\nu,a)| \leq c\ \widehat{\rho}_*(\mu,\nu) \qquad \forall\, x \in \mathbb{T}^d,\ \mu,\nu \in \mathcal{P}(\mathbb{T}^d),\ a \in A.$$

Under this regularity condition, for any $\boldsymbol{\alpha} \in \mathcal{A}$; $t \in [0,T]$; and $\mathcal{F}_t$-measurable, $\mathbb{T}^d$-valued random variable ξ with $\mu = \mathcal{L}(\xi)$, there is a unique $\mathbb{F}$-adapted solution $X_s^{t,\mu,\boldsymbol{\alpha}}$ of the following McKean–Vlasov stochastic differential equation,

$$X_s^{t,\mu,\boldsymbol{\alpha}} = \xi + \int_t^s b^{\alpha_u}(X_u^{t,\mu,\boldsymbol{\alpha}}, \mathcal{L}_u^{t,\mu,\boldsymbol{\alpha}})\,\mathrm{d}u + \int_t^s \sigma^{\alpha_u}(X_u^{t,\mu,\boldsymbol{\alpha}}, \mathcal{L}_u^{t,\mu,\boldsymbol{\alpha}})\,\mathrm{d}B_u, \qquad s \in [t,T], \tag{3.1}$$

where $\mathcal{L}_u^{t,\mu,\boldsymbol{\alpha}} = \mathcal{L}(X_u^{t,\mu,\boldsymbol{\alpha}})$, and B is a d'-dimensional Brownian motion. Note that any function on the torus $\mathbb{T}^d$ can be extended to a 2π-periodic function on $\mathbb{R}^d$. Therefore, we solve the above equation by considering these extensions of b, σ and solving the equation first on $\mathbb{R}^d$ and then mapping it to $\mathbb{T}^d$.

The solution $X_u^{t,\mu,\boldsymbol{\alpha}}$ depends on the choice of the initial condition ξ and the Brownian increments $(B_u - B_t)_{u\in[t,T]}$. However, because these Brownian increments are independent of $\mathcal{F}_t$ and we consider feedback controls, the flow $(\mathcal{L}_u^{t,\mu,\boldsymbol{\alpha}})_{u\in[t,T]}$ depends only on the law $\mu = \mathcal{L}(\xi)$ of the initial condition and not on ξ itself.

Clearly, the existence and uniqueness of solutions of (3.1) can be obtained under weaker assumptions. On the other hand, the stronger condition with n_* derivatives is needed for the comparison and the Lipschitz continuity results.

Remark 3.2. We emphasize that assumption 3.1 puts implicit regularity restrictions of the control set $\mathcal{C}_a$, as further discussed in Remark 3.3 below. It should be seen as a structural assumption rather than a technical regularity restriction. Indeed, one way of verifying it is to impose a regularity condition that the functions $b(x,\mu,\cdot)$, $\sigma(x,\mu,\cdot)$, and $\ell(x,\mu,\cdot)$ are all in $\mathcal{C}_*$ and a structural assumption that $\mathcal{C}_a \subset \{\alpha \in C_* : \|\alpha\|_* \le c_a\}$ for some constant c_a.

3.2. Problem. Starting from $(t,\mu) \in \overline{\mathcal{O}}$, the *pay-off* of a control process $\boldsymbol{\alpha} \in \mathcal{A}$ is given by

$$J(t,\mu,\boldsymbol{\alpha}) := \int_t^T \mathbb{E}[\ell^{\alpha_u}(X_u^{t,\mu,\boldsymbol{\alpha}}, \mathcal{L}_u^{t,\mu,\boldsymbol{\alpha}})]\,\mathrm{d}u + \varphi(\mathcal{L}_T^{t,\mu,\boldsymbol{\alpha}}), \qquad \boldsymbol{\alpha} \in \mathcal{A},\ (t,\mu) \in \overline{\mathcal{O}}.$$

Since $\mathbb{E}[\ell^{\alpha_u}(X_u^{t,\mu,\boldsymbol{\alpha}}, \mathcal{L}_u^{t,\mu,\boldsymbol{\alpha}})] = \mathcal{L}_u^{t,\mu,\boldsymbol{\alpha}}(\ell(\cdot, \mathcal{L}_u^{t,\mu,\boldsymbol{\alpha}}, \alpha_u(\cdot)))$, $J(t,\mu,\boldsymbol{\alpha})$ is a function of $\mu = \mathcal{L}(\xi)$ independent of the choice of the initial random variable ξ. Although this property, called *law-invariance*, holds directly in our setting, in general structures, it is quite subtle. We refer to Proposition 2.4 of [18], and Theorem 3.5 in [12] for its general proof and to section 6.5 and Definition 6.27 of [8] for a discussion.

Then, the McKean–Vlasov optimal control problem is to minimize the pay-off functional J over $\boldsymbol{\alpha} \in \mathcal{A}$, and the value function is given by

$$v(t,\mu) := \inf_{\boldsymbol{\alpha}\in\mathcal{A}} J(t,\mu,\boldsymbol{\alpha}), \qquad (t,\mu) \in \overline{\mathcal{O}}.$$

Remark 3.3. Suppose that $\mathcal{C}_a = \{\alpha \in \mathcal{C}_*(\mathbb{T}^d \to A) :\ \|\alpha\|_{\mathcal{C}_*} \le c_0\}$ for some constant $c_0 \ge 0$. Consider the class of functions of the form $h(x,\mu(f),a)$ for some $f \in \mathcal{C}_*$ and $h : \mathbb{T}^d \times \mathbb{R} \times A \to \mathbb{R}$ satisfying $\|h(\cdot,y,\cdot)\|_{\mathcal{C}_*} + \|h(x,\cdot,a)\|_{1,\infty} \le c_1$ for every $x \in \mathbb{T}^d$, $y \in \mathbb{R}$, and $a \in A$ for some $c_1 \ge 0$. Then, $h^\alpha(x,\mu) = h(x,\mu(f),\alpha(x))$, and $\|h^\alpha(\cdot,\mu)\|_{\mathcal{C}_*}$ is less than a constant c_a depending on c_0, c_1, and n_*. Also, for every $x \in \mathbb{T}^d$,

$$|h(x,\mu(f),\alpha(x)) - h(x,\nu(f),\alpha(x))| \le c_1|(\mu-\nu)(f)| \le c_1\|f\|_{\mathcal{C}_*}\widehat{\rho}_*(\mu,\nu) \le c_1 c_0 \widehat{\rho}_*(\mu,\nu).$$

Hence, this class of functions satisfies the regularity assumption. More generally, under the appropriate assumptions, functions $h(x,\mu(f_1),\ldots,\mu(f_m),a)$ with

$f_1, \dots, f_m \in \mathcal{C}_*(\mathbb{T})$ and $h : \mathbb{T}^d \times \mathbb{R}^m \times A \to \mathbb{R}$ also satisfy the regularity assumption with the above control set $\mathcal{C}_a$. We emphasize that, even when the coefficients depend on μ only through $\mu(f_1), \dots, \mu(f_m)$ of the measure μ, the value function in general is still infinite-dimensional.

The assumptions made above hold in a large class of examples studied in the mean-field games. In particular, for the Kuramoto problem studied in [9], for some constants $\kappa, \sigma > 0$,

$$\ell(x,\mu,a) = \frac{1}{2}a^2 + \kappa[1 - \mu(\cos)\cos(c) - \mu(\sin)\sin(x)], \quad b(x,\mu,a) = a, \quad \sigma(a) = \sigma.$$

3.3. Dynamic programming principle. We next state the dynamic programming principle, which is central to the viscosity approach to optimal control. A general proof in a different setting is given in [18]. However, the deterministic structure allows for a simpler proof that we provide in section 8.

THEOREM 3.4 (dynamic programming). *For every $\mu \in \mathcal{P}(\mathbb{T}^d)$ and $0 \le t \le \tau \le T$,*

$$v(t,\mu) = \inf_{\boldsymbol{\alpha} \in \mathcal{A}} \int_t^\tau \mathbb{E}[\ell^{\alpha_u}(X_u^{t,\mu,\boldsymbol{\alpha}}, \mathcal{L}_u^{t,\mu,\boldsymbol{\alpha}})] \, \mathrm{d}u + v(\tau, \mathcal{L}_\tau^{t,\mu,\boldsymbol{\alpha}}). \tag{3.2}$$

It is well known that the dynamic programming can be used directly to show that the value function is a viscosity solution of the dynamic programming equation

$$-\partial_t v(t,\mu) = H(\mu, \partial_\mu v(t,\mu)), \qquad t \in [0,T), \ \mu \in \mathcal{P}(\mathbb{T}^d), \tag{3.3}$$

where, for $\gamma \in \mathcal{C}^2(\mathbb{T}^d)$, $\mu \in \mathcal{P}(\mathbb{T}^d)$, $x \in \mathbb{T}^d$, and $\alpha \in \mathcal{C}_a$,

$$H(\mu,\gamma) := \inf_{\alpha \in \mathcal{C}_a} \left\{ \mu\left(\ell^\alpha(\cdot,\mu) + \mathcal{M}^{\alpha,\mu}[\gamma](\cdot)\right) \right\},$$

$$\mathcal{M}^{\alpha,\mu}[\gamma](x) := b^\alpha(x,\mu) \cdot \partial_x \gamma(x) + \sum_{i,j=1}^{d} \sum_{l=1}^{d'} \sigma_{il}^\alpha(x,\mu)\sigma_{jl}^\alpha(x,\mu)\partial_{x_i x_j}\gamma(x).$$

The value function also trivially satisfies the following terminal condition:

$$v(T,\mu) = \varphi(\mu) \qquad \forall \ \mu \in \mathcal{P}(\mathbb{T}^d). \tag{3.4}$$

Because the value function is not necessarily differentiable, a weak formulation is needed, and we use the notion of viscosity solutions. The definition that we use is exactly the classical one in which the auxiliary test functions are continuously differentiable functions on $\overline{\mathcal{O}} = [0,T] \times \mathcal{P}(\mathbb{T}^d)$, with the linear derivative in $\mathcal{P}(\mathbb{T}^d)$ recalled in section 2. We continue by specifying the auxiliary functions used in the definition of viscosity solutions.

DEFINITION 3.5. *We say that $\psi \in \mathcal{C}(\overline{\mathcal{O}})$ is a* test function *if ψ is continuously differentiable with $\partial_\mu \psi(t,\mu) \in \mathcal{C}^2(\mathbb{T}^d)$ for every $(t,\mu) \in \overline{\mathcal{O}}$ and the map $(t,\mu) \in \overline{\mathcal{O}} \mapsto H(\mu, \partial_\mu \psi(t,\mu))$ is continuous. We denote the set of all test functions by $\mathcal{C}_s(\overline{\mathcal{O}})$.*

DEFINITION 3.6. *A continuous function $u \in \mathcal{C}(\overline{\mathcal{O}})$ is a* viscosity subsolution *of (3.3) if every $\psi \in \mathcal{C}_s(\overline{\mathcal{O}})$, $(t_0,\mu_0) \in [0,T) \times \mathcal{P}(\mathbb{T}^d)$ satisfying $(u-\psi)(t_0,\mu_0) = \max_{\overline{\mathcal{O}}}(u-\psi)$ also satisfies*

$$-\partial_t \psi(t_0,\mu_0) \le H(\mu_0, \partial_\mu \psi(t_0,\mu_0)).$$

A continuous function $w \in \mathcal{C}(\overline{\mathcal{O}})$ *is a* viscosity supersolution of (3.3) *if every* $\psi \in \mathcal{C}_s(\overline{\mathcal{O}})$, $(t_0, \mu_0) \in [0,T) \times \mathcal{P}(\mathbb{T}^d)$ *satisfying* $(w-\psi)(t_0,\mu_0) = \min_{\overline{\mathcal{O}}}(w-\psi)$ *also satisfies*

$$-\partial_t \psi(t_0,\mu_0) \geq H(\mu_0, \partial_\mu \psi(t_0,\mu_0)).$$

Finally, $v \in \mathcal{C}(\overline{\mathcal{O}})$ *is a* viscosity solution of (3.3) *if it is both a sub- and a supersolution.*

4. Main results. Our main result is the characterization of the value function as the unique continuous viscosity solution of the dynamic programming equation (3.3) and the terminal condition (3.4).

Recall the metrics $\rho_*, \widehat{\rho}_*$ of (2.1).

Theorem 4.1 (comparison). *Suppose that the regularity assumption* 3.1 *holds,* $u \in \mathcal{C}(\overline{\mathcal{O}})$ *is a viscosity subsolution of* (3.3) *and* (3.4)*, and* $w \in \mathcal{C}(\overline{\mathcal{O}})$ *is a viscosity supersolution of* (3.3) *and* (3.4)*. If, further,* u *or* w *is Lipschitz continuous in the* μ*-variable with respect to the metric* ρ_**, then* $u \leq v$ *on* $\overline{\mathcal{O}}$*.*

The above comparison result is proved in section 6.

Theorem 4.2 (continuity). *Under the regularity assumption* 3.1*, there exists a constant* $L_v > 0$ *depending only on the horizon* T *and the constant* c_a *of assumption* 3.1 *so that*

$$|v(t,\mu) - v(s,\nu)| \leq L_v \left[\widehat{\rho}_*(\mu,\nu) + |t-s|^{\frac{1}{2}}\right] \qquad \forall\, \mu,\nu \in \mathcal{P}(\mathbb{T}^d),\ t,s \in [0,T]. \tag{4.1}$$

This continuity result, proved in section 7, also implies Lipschitz continuity with respect to ρ_* since $\widehat{\rho}_* \leq c_* \rho_*$ for some constant c_*. The following result follows directly from the standard viscosity theory [20], and its proof is given in section 9.

Theorem 4.3 (viscosity property). *Under the regularity assumption* 3.1*, the value function is a viscosity solution of* (3.3) *in* $\mathcal{O}$ *satisfying the terminal condition* (3.4)*.*

In particular, any continuous viscosity subsolution is less than or equal to the value function v*, and any continuous viscosity supersolution is greater than or equal to* v*.*

Remark 4.4. In the comparison result, we could use any metric ρ_λ with $\lambda > 2 + \frac{d}{2}$. However, our proof for Lipschitz continuity requires us to employ the smaller metric $\widehat{\rho}_m$ and only for integer values of m. This combination of the results dictates the global choice $\lambda = n_*$.

4.1. An example. In this subsection, we provide a simple example to illustrate the notation and also the need for viscosity solutions. We take $T=1$, $d=1$, $A=\mathbb{R}$, $b(x,\mu,a)=a$, $\sigma \equiv 1$, $\varphi \equiv 0$, and

$$\ell(\mu,a) := \frac{1}{2}a^2 + L(m(\mu)), \quad \text{where} \quad m(\mu) := \int_{\mathbb{T}} x\ \mu(\mathrm{d}x)$$

and $L : [-\pi,\pi] \to \mathbb{R}$ is a given Lipschitz function. Next, we show that the value function of the above problem is independent of the control set $\mathcal{C}_a$ and is given by

$$v(t,\mu) = w(t, m(\mu)), \qquad (t,\mu) \in \overline{\mathcal{O}},$$

where

$$w(t,y) := \inf_{\hat{\boldsymbol{\alpha}} \in \widehat{\mathcal{A}}} \hat{J}(t,y,\hat{\boldsymbol{\alpha}}) := \inf_{\hat{\boldsymbol{\alpha}} \in \widehat{\mathcal{A}}} \int_t^1 \left[\frac{1}{2}(\hat{\alpha}_u)^2 + L(Y_u^{t,y,\hat{\boldsymbol{\alpha}}}) \right] \mathrm{d}u, \qquad (t,y) \in [0,1] \times \mathbb{T}$$

and $\widehat{\mathcal{A}}$ is the set of all measurable maps $\hat{\boldsymbol{\alpha}} : [0,1] \mapsto \mathbb{T}$; and $Y_u^{t,y,\hat{\boldsymbol{\alpha}}} = y + \int_t^u \hat{\alpha}_s \mathrm{d}s$. It is well known that w is the unique viscosity solution of the Eikonal equation [20],

$$-\partial_t w(t,y) = -\frac{1}{2}(\partial_y w(t,y))^2 + L(y), \qquad y \in \mathbb{T}, \tag{4.2}$$

and $w(1,\cdot) \equiv 0$. Since w is not always differentiable, we conclude that v is not either, and therefore, a weak theory is needed. On the other hand, when w is differentiable, we have

$$\partial_\mu v(t,\mu)(x) = \partial_y w(t,m(\mu))\, x \quad \Rightarrow \quad \partial_x(\partial_\mu v(t,\mu)(x)) = \partial_y w(t,m(\mu)).$$

Hence, by Jensen's inequality,

$$\begin{aligned} H(\mu,\partial_\mu v(t,\mu)) &= \inf_{\alpha \in \mathcal{C}_a} \int_{\mathbb{T}} \left(\frac{1}{2}\alpha(x)^2 + \alpha(x)\partial_y w(t,m(\mu)) \right) \mu(\mathrm{d}x) + L(m(\mu)) \\ &\geq \inf_{\alpha \in \mathcal{C}_a} \left\{ \frac{1}{2} \left(\int_{\mathbb{T}} \alpha(x)\mu(\mathrm{d}x) \right)^2 + \left(\int_{\mathbb{T}} \alpha(x)\mu(\mathrm{d}x) \right) \partial_y w(t,m(\mu)) \right\} + L(m(\mu)) \\ &= \inf_{a \in \mathbb{R}} \left\{ \frac{1}{2}a^2 + a\partial_y w(t,m(\mu)) \right\} + L(m(\mu)) \\ &= -\frac{1}{2}(\partial_y w(t,m(\mu)))^2 + L(m(\mu)). \end{aligned}$$

Because constant functions $\alpha \equiv a$ are always in $\mathcal{C}_a$, the opposite inequality also holds. Therefore,

$$H(\mu,\partial_\mu v(t,\mu)) = -\frac{1}{2}(\partial_y w(t,m(\mu))^2 + L(m(\mu))$$

for every $\mathcal{C}_a$. Since $\partial_t v(t,\mu) = \partial_t w(t,m(\mu))$, the Eikonal equation (4.2) implies that when w is differentiable, v is a classical solution of the dynamic programming equation (3.3).

5. Fourier–Wasserstein metrics.

In this section, we study the properties of the norms and the metric ρ_λ defined in section 2. Similar metrics are also defined in [28] using a dual representation with Sobolev functions.

Recall that z^* is the complex conjugate of z and the orthonormal basis $\{e_k\}_{k\in\mathbb{Z}^d}$; the Fourier coefficients $F_k(f)$ are defined in section 2. For $\mu \in \mathcal{M}(\mathbb{T}^d)$, $k \in \mathbb{Z}^d$, we also set $F_k(\mu) := \mu(e_k^*)$. Because $\mathbb{T}^d$ is compact, $F_k(\mu)$ is finite for every k and $F_0(\mu) = 1$ for all $\mu \in \mathcal{P}(\mathbb{T}^d)$.

For $\lambda \geq 1$, we define a norm on $\mathcal{M}(\mathbb{T}^d)$ dual to $\|\cdot\|_\lambda$ by

$$|\eta|_\lambda := \sup\{\eta(\psi) \;:\; \psi \in \mathbb{H}_\lambda(\mathbb{T}^d),\ \|\psi\|_\lambda \leq 1\}, \qquad \eta \in \mathcal{M}(\mathbb{T}^d)$$

so that $\rho_\lambda(\mu,\nu) = |\mu - \nu|_\lambda$.

LEMMA 5.1. *For $\lambda > \frac{d}{2}$, $\eta \in \mathcal{M}(\mathbb{T}^d)$, $|\eta|_\lambda < \infty$, and the following dual representation holds:*

$$|\eta|_\lambda = \left(\sum_{k\in\mathbb{Z}^d} (1+|k|^2)^{-\lambda} |F_k(\eta)|^2 \right)^{\frac{1}{2}}. \tag{5.1}$$

Proof. We first note as $2\lambda > d$, $c_\lambda := \sum_{k\in\mathbb{Z}^d}(1+|k|^2)^{-\lambda} < \infty$. Let $d(\eta)$ be the expression in the right-hand side of (5.1) and $TV(\eta)$ be the total variation of the measure η. Then, $|F_k(\eta)| \le TV(\eta)$, and therefore, $d(\eta) \le \sum_{k\in\mathbb{Z}^d}(1+|k|^2)^{-\lambda}\, TV(\eta) = c_\lambda TV(\eta)$.

For $\psi \in \mathcal{C}(\mathbb{T}^d)$, the Fourier representation $\psi = \sum_{k\in\mathbb{Z}^d} F_k(\psi) e_k$ implies that

(5.2)

$$\begin{aligned}\eta(\psi) &= \sum_{k\in\mathbb{Z}^d} F_k(\psi)\,\eta(e_k) = \sum_{k\in\mathbb{Z}^d} F_k(\psi)\, F_k^*(\eta)\\ &= \sum_{k\in\mathbb{Z}^d} [(1+|k|^2)^{\frac{\lambda}{2}} F_k(\psi)]\,[(1+|k|^2)^{-\frac{\lambda}{2}} F_k^*(\eta)]\\ &\le \left(\sum_{k\in\mathbb{Z}^d}(1+|k|^2)^{\lambda}|F_k(\psi)|^2\right)^{\frac{1}{2}} \left(\sum_{k\in\mathbb{Z}^d}(1+|k|^2)^{-\lambda}\,|F_k^*(\eta)|^2\right)^{\frac{1}{2}} = \|\psi\|_\lambda\, d(\eta).\end{aligned}$$

In view of the definition of $|\cdot|_\lambda$, $|\eta|_\lambda \le d(\eta)$ for any $\eta \in \mathcal{M}(\mathbb{T}^d)$.

To prove the opposite inequality, fix $\eta \in \mathcal{M}(\mathbb{T}^d)$, and define a function $\tilde\psi$ by

$$\tilde\psi(x) := \sum_{k\in\mathbb{Z}^d}(1+|k|^2)^{-\lambda}\, F_k(\eta)\, e_k(x), \quad \Rightarrow \quad F_k(\tilde\psi) = (1+|k|^2)^{-\lambda}\, F_k(\eta), \quad k\in\mathbb{Z}^d.$$

Since $c_\lambda < \infty$, $\tilde\psi$ is well defined. Moreover,

$$\|\tilde\psi\|_\lambda^2 = \sum_{k\in\mathbb{Z}^d}(1+|k|^2)^{\lambda}|F_k(\tilde\psi)|^2 = \sum_{k\in\mathbb{Z}^d}(1+|k|^2)^{-\lambda}\,|F_k(\eta)|^2 = d^2(\eta) < \infty.$$

Hence, $\tilde\psi \in \mathbb{H}_\lambda(\mathbb{T}^d)$, and, by (5.2),

$$\eta(\tilde\psi) = \sum_{k\in\mathbb{Z}^d} F_k(\tilde\psi)\, F_k^*(\eta) = \sum_{k\in\mathbb{Z}^d}(1+|k|^2)^{-\lambda}\,|F_k(\eta)|^2 = d^2(\eta) = \|\tilde\psi\|_\lambda\; d(\eta).$$

Because $\eta(\tilde\psi) \le |\eta|_\lambda \|\tilde\psi\|_\lambda$ by the definition of $|\cdot|_\lambda$, we have $d(\eta)\|\tilde\psi\|_\lambda = \eta(\tilde\psi) \le |\eta|_\lambda \|\tilde\psi\|_\lambda$. ∎

An immediate corollary is the following.

COROLLARY 5.2. *For any $\lambda > \frac{d}{2}$, ρ_λ is a metric on $\mathcal{P}(\mathbb{T}^d)$ with a dual representation*

$$\rho_\lambda(\mu,\nu) = \max\{(\mu-\nu)(\psi)\ :\ \|\psi\|_\lambda \le 1\} = \left(\sum_{k\in\mathbb{Z}^d}(1+|k|^2)^{-\lambda}\,|F_k(\mu-\nu)|^2\right)^{\frac{1}{2}}.$$

Proof. The dual representation follows directly from the previous lemma. Suppose that $\rho_\lambda(\mu,\nu) = 0$; then $F_k(\mu) = F_k(\nu)$ for every $k\in\mathbb{Z}^d$. Because μ,ν have the same Fourier series, we conclude that $\mu = \nu$. The fact that ρ_λ is a metric now follows from the dual representation. ∎

The following provides a connection between the two metrics that we consider. Also, with $m = 1$, it implies that the classical Wasserstein-one metric $\widehat\rho_1$ is dominated by ρ_1.

LEMMA 5.3. *For any integer $m \ge 1$, there exists $c_{m,d} > 0$ such that $\widehat\rho_m(\mu,\nu) \le c_{m,d}\,\rho_m(\mu,\nu)$ for every $\mu,\nu \in \mathcal{P}(\mathbb{T}^d)$.*

Proof. Fix the $m \geq 1$, and let $D^m\psi$ be the mth order derivatives of $\psi \in \mathcal{C}^m(\mathbb{T}^d)$. Then, since $|k|^{2m}|F_k(\psi)|^2 = |F_k(D^m\psi)|^2$,

$$\sum_{k\in\mathbb{Z}^d} |k|^{2m}|F_k(\psi)|^2 = \sum_{k\in\mathbb{Z}^d} |F_k(D^m\psi)|^2 = \| \mathbb{D}^m\psi\|^2_{\mathbb{L}^2(\mathbb{T}^d)} \leq d^m \ (2\pi)^d \|\psi\|^2_{\mathcal{C}^m(\mathbb{T}^d)}.$$

Because $(1+|k|^2)^m \leq 2^m(1+|k|^{2m})$, for any $k \in \mathbb{Z}^d$, $\psi \in \mathcal{C}^m(\mathbb{T}^d)$,

$$\|\psi\|^2_m \leq 2^m \sum_{k\in\mathbb{Z}^d} |F_k(\psi)|^2 + 2^m \sum_{k\in\mathbb{Z}^d} |k|^{2m}|F_k(\psi)|^2 \leq c^2_{m,d}\|\psi\|^2_{\mathcal{C}^m(\mathbb{T}^d)},$$

where $c^2_{m,d} = 2^m[1 + d^m(2\pi)^d]$. Hence,

$$\begin{aligned}\widehat{\rho}_m(\mu,\nu) &= \sup\{(\mu-\nu)(\psi) \ : \ \|\psi\|_{\mathcal{C}^m(\mathbb{T}^d)} \leq 1\} \\ &\leq \sup\{(\mu-\nu)(\psi) \ : \ \psi \in \mathcal{C}^m(\mathbb{T}^d), \ \|\psi\|_m \leq c_{m,d}\} \\ &\leq \sup\{(\mu-\nu)(\psi) \ : \ \psi \in \mathbb{H}_m(\mathbb{T}^d), \ \|\psi\|_m \leq c_{m,d}\} = c_{m,d}\rho_m(\mu,\nu). \qquad \square\end{aligned}$$

Our next result is on the differentiability of ρ_λ. Recall the test functions $\mathcal{C}_s(\overline{\mathcal{O}})$ of Definition 3.5, $n_*(d)$ of (2.1), and the basis e_k of section 2.

LEMMA 5.4. *Fix $\lambda > \frac{d}{2}$ and $\nu \in \mathcal{P}(\mathbb{T}^d)$, and set $h(\mu) := \frac{1}{2}\rho^2_\lambda(\mu,\nu)$. Then,*

$$\partial_\mu h(\mu)(x) = \sum_{k\in\mathbb{Z}^d} (1+|k|^2)^{-\lambda} F_k(\mu-\nu)\, e^*_k(x), \qquad x \in \mathbb{T}^d,$$

and $\|\partial_\mu h(\mu)\|_\lambda = \rho_\lambda(\mu,\nu)$. Moreover, if $\lambda = n_*(d)$, then $\partial_\mu h(\mu) \in \mathcal{C}^2(\mathbb{T}^d)$.

Proof. Fix $\nu \in \mathcal{P}(\mathbb{T}^d)$. For each $k \in \mathbb{Z}^d$, set $a_k(\mu) := \frac{1}{2}|F_k(\mu-\nu)|^2$. Then, we directly calculate that $\partial_\mu a_k(\mu)(\cdot) = F_k(\mu-\nu)\, e^*_k(\cdot)$. Then, for any $x \in \mathbb{T}^d$,

$$\partial_\mu h(\mu)(x) = \sum_{k\in\mathbb{Z}^d} (1+|k|^2)^{-\lambda} \ \partial_\mu a_k(\mu)(x) = \sum_{k\in\mathbb{Z}^d} (1+|k|^2)^{-\lambda} F_k(\mu-\nu)\, e^*_k(x).$$

The above formula implies that $F_k(\partial_\mu h(\mu)) = (1+|k|^2)^{-\lambda} F^*_k(\mu-\nu)$ for every $k \in \mathbb{Z}^d$. Hence,

$$\|\partial_\mu h(\mu)\|^2_\lambda = \sum_{k\in\mathbb{Z}^d} (1+|k|^2)^{\lambda}\, |F_k(\partial_\mu h(\mu))|^2 = \sum_{k\in\mathbb{Z}^d} (1+|k|^2)^{-\lambda}\, |F_k(\mu-\nu)|^2 = \rho^2_\lambda(\mu,\nu).$$

In view of the Sobolev embedding of $\mathbb{H}_{n_*}(\mathbb{T}^d)$ into $\mathcal{C}^2(\mathbb{T}^d)$, $\partial_\mu h(\mu) \in \mathcal{C}^2(\mathbb{T}^d)$. $\square$

6. Comparison. In this section, we prove Theorem 4.1 in several steps. Recall the test functions $\mathcal{C}_s(\overline{\mathcal{O}})$ of Definition 3.5 and n_*, ρ_* of (2.1). Then, $2(n_*-2) \geq d+1$, and consequently,

$$c(d) := \sum_{k\in\mathbb{Z}^d} (1+|k|^2)^{2-n_*} < \infty. \tag{6.1}$$

Step 1 (set-up). Let u, w be as in the statement of the theorem. Toward a contraposition, suppose that $\sup_{\overline{\mathcal{O}}}(u-w) > 0$. We fix a sufficiently small $\delta > 0$ satisfying

$$l := \max_{(t,\mu)\in\overline{\mathcal{O}}} \{(u-w)(t,\mu) - \delta(T-t)\} > 0.$$

Set $\bar u(t,\mu) := u(t,\mu) - \delta(T-t)$. Then, $\bar u$ is a continuous viscosity subsolution of

$$-\partial_t \bar u(t,\mu) = H(\mu, \partial_\mu \bar u(t,\mu)) - \delta. \tag{6.2}$$

Step 2 (doubling the variables). For $\epsilon > 0$, set

$$\Phi_\epsilon(t,\mu,s,\nu) := \bar u(t,\mu) - w(s,\nu) - \frac{1}{2\epsilon}\ \left(\rho_*^2(\mu,\nu) + (t-s)^2\right).$$

Because $\overline{\mathcal{O}}$ is compact and $\bar u, w$ are continuous, there exists $(t_\epsilon, s_\epsilon, \mu_\epsilon, \nu_\epsilon) \in \overline{\mathcal{O}} \times \overline{\mathcal{O}}$ satisfying

$$\Phi_\epsilon(t_\epsilon,\mu_\epsilon,s_\epsilon,\nu_\epsilon) = \max_{\overline{\mathcal{O}}\times\overline{\mathcal{O}}} \Phi_\epsilon \ge l > 0.$$

Set $M := \max \bar u$, $m := \min w$, and $\zeta_\epsilon := \rho_*^2(\mu_\epsilon,\nu_\epsilon) + (t_\epsilon - s_\epsilon)^2$ so that

$$0 \le \zeta_\epsilon \le 2\epsilon\ (M + m - l). \tag{6.3}$$

Step 3 (letting ϵ to zero). Since $\overline{\mathcal{O}}$ is compact, there is a subsequence $\{(t_\epsilon,\mu_\epsilon,s_\epsilon,\nu_\epsilon)\} \subset \overline{\mathcal{O}} \times \overline{\mathcal{O}}$, denoted by ϵ again, and $(t^*,\mu^*,s^*,\nu^*) \in \overline{\mathcal{O}} \times \overline{\mathcal{O}}$ such that

$$\mu_\epsilon \rightharpoonup \mu^*, \quad \nu_\epsilon \rightharpoonup \nu^*, \quad t_\epsilon \to t^*, \quad s_\epsilon \to s^* \qquad \text{as} \ \ \epsilon \downarrow 0.$$

By (6.3), it is clear that $t^* = s^*$ and $\rho_*(\mu^*,\nu^*) = 0$. Then, by Lemma 5.3, $\mu^* = \nu^*$.

If t^* were to be equal to T, by the terminal condition (3.4), we would have

$$0 < l \le \liminf_{\epsilon\downarrow 0} \Phi_\epsilon(t_\epsilon,\mu_\epsilon,s_\epsilon,\nu_\epsilon) \le \lim_{\epsilon\downarrow 0} [\bar u(t_\epsilon,\mu_\epsilon) - w(s_\epsilon,\nu_\epsilon)] = \bar u(T,\mu^*) - w(T,\mu^*) \le 0.$$

Hence, $t^* < T$ and $t_\epsilon, s_\epsilon < T$ for all sufficiently small $\epsilon > 0$.

Step 4 (distance estimate). Without loss of generality, suppose that w is Lipschitz. Indeed, if instead, u were to be Lipschitz, the argument below with obvious changes would also yield the estimate (6.4) below, and this estimate is the only place where the Lipschitz assumption is used. Then,

$$|w(t,\mu) - w(t,\nu)| \le \frac{1}{2} L_w\, \rho_*(\mu,\nu), \qquad \mu,\nu \in \mathcal{P}(\mathbb{T}^d),\ t \in [0,T],$$

and, for each $\epsilon > 0$,

$$\begin{aligned}\bar u(t_\epsilon,\mu_\epsilon) - w(s_\epsilon,\nu_\epsilon) - \frac{1}{2\epsilon}\,\zeta_\epsilon &= \Phi_\epsilon(t_\epsilon,\mu_\epsilon,s_\epsilon,\nu_\epsilon) \ge \Phi_\epsilon(t_\epsilon,\mu_\epsilon,s_\epsilon,\mu_\epsilon) \\ &= \bar u(t_\epsilon,\mu_\epsilon) - w(s_\epsilon,\mu_\epsilon) - \frac{1}{2\epsilon}\,(t_\epsilon - s_\epsilon)^2.\end{aligned}$$

Therefore, $\rho_*^2(\mu_\epsilon,\nu_\epsilon) = \zeta_\epsilon - (t_\epsilon - s_\epsilon)^2 \le 2\epsilon\,[w(s_\epsilon,\mu_\epsilon) - w(s_\epsilon,\nu_\epsilon)] \le 2\epsilon\, L_w\, \rho_*(\mu_\epsilon,\nu_\epsilon)$. Hence,

$$\rho_*(\mu_\epsilon,\nu_\epsilon) \le 2\epsilon\, L_w \qquad \forall\, \epsilon > 0. \tag{6.4}$$

Step 5 (viscosity property). Set

$$\psi_\epsilon(t,\mu) := \frac{1}{2\epsilon}[\rho_*^2(\mu,\nu_\epsilon) + (t - s_\epsilon)^2], \qquad \phi_\epsilon(s,\nu) := -\frac{1}{2\epsilon}[\rho_*^2(\mu_\epsilon,\nu) + (t_\epsilon - s)^2].$$

By Lemma 5.4, both $\partial_\mu \psi_\epsilon(t,\mu), \partial_\mu \phi_\epsilon(s,\nu) \in \mathcal{C}^2(\mathbb{T}^d)$. Moreover, by the regularity assumption 3.1, maps $(t,\mu) \mapsto H(\mu, \partial_\mu \psi_\epsilon(t,\mu))$ and $(t,\nu) \mapsto H(\nu, \partial_\mu \phi_\epsilon(t,\nu))$ are continuous. Hence, ψ_ϵ and ϕ_ϵ are smooth test functions. Set

$$\kappa_\epsilon(x) := \partial_\mu \psi_\epsilon(t_\epsilon,\mu_\epsilon)(x) = \partial_\mu \phi_\epsilon(s_\epsilon,\nu_\epsilon)(x) = \frac{1}{\epsilon}\sum_{k\in\mathbb{Z}^d} \frac{F_k(\mu_\epsilon-\nu_\epsilon)}{(1+|k|^2)^{n_*}}\, e_k^*(x), \qquad x \in \mathbb{T}^d.$$

Clearly, $\bar{u}(t,\mu) - \psi_\epsilon(t,\mu)$ is maximized at t_ϵ, μ_ϵ. Since $t_\epsilon < T$, $\psi_\epsilon \in \mathcal{C}_s(\overline{\mathcal{O}})$ and $\bar{u}$ is a viscosity subsolution of (6.2), then

$$-\frac{t_\epsilon - s_\epsilon}{\epsilon} \le H(\mu_\epsilon, \kappa_\epsilon) - \delta.$$

By the viscosity property of w, a similar argument implies that

$$-\frac{t_\epsilon - s_\epsilon}{\epsilon} \ge H(\nu_\epsilon, \kappa_\epsilon).$$

We subtract the above inequalities to arrive at

$$0 < \delta \le H(\mu_\epsilon, \kappa_\epsilon) - H(\nu_\epsilon, \kappa_\epsilon). \tag{6.5}$$

Step 6 (estimation). Since $H(\mu,\kappa_\epsilon) = \inf_{\alpha\in\mathcal{C}_a} \{\mu(\ell^\alpha(\cdot,\mu) + \mathcal{M}^{\alpha,\mu}[\kappa_\epsilon](\cdot))\}$,

$$|H(\mu_\epsilon,\kappa_\epsilon) - H(\nu_\epsilon,\kappa_\epsilon)| \le \sup_{\alpha\in\mathcal{C}_a} \mathcal{T}_\epsilon^\alpha + \sup_{\alpha\in\mathcal{C}_a} \mathcal{I}_\epsilon^\alpha + \sup_{\alpha\in\mathcal{C}_a} \mathcal{J}_\epsilon^\alpha,$$

where

$$\begin{aligned}
\mathcal{T}_\epsilon^\alpha &:= |\mu_\epsilon(\ell^\alpha(\cdot,\mu_\epsilon)) - \nu_\epsilon(\ell^\alpha(\cdot,\nu_\epsilon))|,\\
\mathcal{I}_\epsilon^\alpha &:= |(\mu_\epsilon-\nu_\epsilon)(\mathcal{M}^{\alpha,\mu_\epsilon}[\kappa_\epsilon](\cdot))|,\\
\mathcal{J}_\epsilon^\alpha &:= |\nu_\epsilon(\mathcal{M}^{\alpha,\mu_\epsilon}[\kappa_\epsilon](\cdot) - \mathcal{M}^{\alpha,\nu_\epsilon}[\kappa_\epsilon](\cdot))|.
\end{aligned}$$

Step 7 (estimating $\mathcal{T}_\epsilon^\alpha$). By the regularity assumption 3.1 and the estimate (6.4),

$$\begin{aligned}
|\mu_\epsilon(\ell^\alpha(\cdot,\mu_\epsilon)) - \nu_\epsilon(\ell^\alpha(\cdot,\nu_\epsilon))| &\le |(\mu_\epsilon-\nu_\epsilon)(\ell^\alpha(\cdot,\mu_\epsilon))| + |\nu_\epsilon(\ell^\alpha(\cdot,\mu_\epsilon) - \ell^\alpha(\cdot,\nu_\epsilon))|\\
&\le \rho_*(\mu_\epsilon,\nu_\epsilon)\, \|\ell^\alpha(\cdot,\mu_\epsilon)\|_{\mathcal{C}_*} + \sup_{x\in\mathbb{T}^d} |\ell^\alpha(x,\mu_\epsilon) - \ell^\alpha(x,\nu_\epsilon)|\\
&\le 2c_a\, \rho_*(\mu_\epsilon,\nu_\epsilon) \le 2c_a L_w\, \epsilon.
\end{aligned}$$

Hence, we have $\lim_{\epsilon\downarrow 0}\ \sup_{\alpha\in\mathcal{C}_a} \mathcal{T}_\epsilon^\alpha = 0$.

Step 8 (estimating $\mathcal{I}_\epsilon^\alpha$). For $x\in\mathbb{T}^d$, $\mu\in\mathcal{P}(\mathbb{T}^d)$, $\alpha\in\mathcal{C}_a$, and $k\in\mathbb{Z}^d$, set

$$\beta_k^\alpha(x,\mu) := \mathcal{M}^{\alpha,\mu}[e_k^*](x) = [ik\cdot b^\alpha(x,\mu) - a_k^\alpha(x,\mu)]e_k^*(x),$$

where, for $x\in\mathbb{T}^d, \mu\in\mathcal{P}(\mathbb{T}^d), \alpha\in\mathcal{C}_a, k\in\mathbb{Z}^d$,

$$a_k^\alpha(x,\mu) := \frac{1}{2}\sum_{i,j=1}^{d}\sum_{l=1}^{d'} \sigma_{il}(x,\mu,\alpha(x))\sigma_{jl}(x,\mu,\alpha(x))k_i k_j. \tag{6.6}$$

Then,

$$\mathcal{M}^{\alpha,\mu_\epsilon}[\kappa_\epsilon](x) = \frac{1}{\epsilon}\sum_{k\in\mathbb{Z}^d} \frac{1}{(1+|k|^2)^{n_*}}\, F_k(\mu_\epsilon-\nu_\epsilon)\, \beta_k^\alpha(x,\mu_\epsilon).$$

This, in turn, implies that

$$
\begin{aligned}
\mathcal{I}_\epsilon^\alpha &\le \frac{1}{\epsilon}\sum_{k\in\mathbb{Z}^d}\frac{1}{(1+|k|^2)^{n_*}}\ |F_k(\mu_\epsilon-\nu_\epsilon)|\ |(\mu_\epsilon-\nu_\epsilon)(\beta_k^\alpha(\cdot,\mu_\epsilon))| \\
&\le \frac{1}{\epsilon}\left(\sum_{k\in\mathbb{Z}^d}\frac{|F_k(\mu_\epsilon-\nu_\epsilon)|^2}{(1+|k|^2)^{n_*}}\right)^{\frac12}\left(\sum_{k\in\mathbb{Z}^d}\frac{((\mu_\epsilon-\nu_\epsilon)(\beta_k^\alpha(\cdot,\mu_\epsilon)))^2}{(1+|k|^2)^{n_*}}\right)^{\frac12} \\
&\le \frac{\rho_*(\mu_\epsilon,\nu_\epsilon)}{\epsilon}\left(\sum_{k\in\mathbb{Z}^d}(1+|k|^2)^{2-n_*}\ \beta_{k,\epsilon}^2\right)^{\frac12},
\end{aligned}
$$

where

$$
\beta_{k,\epsilon} := (1+|k|^2)^{-1}\ \sup_{\alpha\in\mathcal{C}_a}|(\mu_\epsilon-\nu_\epsilon)(\beta_k^\alpha(\cdot,\mu_\epsilon))| \qquad k\in\mathbb{Z}.
$$

Again, by assumption 3.1, $|\beta_{k,\epsilon}|\le c_a+c_a^2$ and $\beta_{k,\epsilon}^\alpha$ is Lipschitz continuous with a Lipschitz constant c_k uniformly in α. Hence, by the Kantorovich duality, $\beta_{k,\epsilon}\le c_k\widehat{\rho}_1(\mu_\epsilon,\nu_\epsilon)$. As $\mu_\epsilon-\nu_\epsilon$ converges weakly to 0, we conclude that $\beta_{k,\epsilon}$ also converges to 0 for every $k\in\mathbb{Z}$. Also, $c(d)=\sum_{k=1}^\infty(1+|k|^2)^{2-n_*}$ is finite by (6.1), and we have argued that $|\beta_{k,\epsilon}|$ is uniformly bounded. Hence, we may use dominated convergence to conclude that the sequence $\sum_{k=1}^\infty(1+|k|^2)^{2-n_*}\beta_{k,\epsilon}^2$ converges to 0 as $\epsilon\downarrow 0$. Then, by (6.4),

$$
\lim_{\epsilon\downarrow 0}\ \sup_{\alpha\in\mathcal{C}_a}\mathcal{I}_\epsilon^\alpha \le \lim_{\epsilon\downarrow 0}\ L_w\left(\sum_{k=1}^\infty(1+|k|^2)^{2-n_*}\ \beta_{k,\epsilon}^2\right)^{\frac12}=0.
$$

Step 9 (estimating $\mathcal{J}_\epsilon^\alpha$). The definition of $\mathcal{J}_\epsilon^\alpha$ implies that

$$
\mathcal{J}_\epsilon^\alpha \le \sup_{x\in\mathbb{T}^d}\{|\mathcal{M}^{\alpha,\mu_\epsilon}[\kappa_\epsilon](x)-\mathcal{M}^{\alpha,\nu_\epsilon}[\kappa_\epsilon](x)|\}.
$$

Let a_k^α be as in (6.6), and, for $\alpha\in\mathcal{C}_a$, $x\in\mathbb{T}^d$, $k\in\mathbb{Z}^d$, set

$$
\begin{aligned}
\gamma_{k,\epsilon}^\alpha(x) &:= \mathcal{M}^{\alpha,\mu_\epsilon}[e_k^*](x)-\mathcal{M}^{\alpha,\nu_\epsilon}[e_k^*](x) \\
&= ik\cdot[b^\alpha(x,\mu_\epsilon)-b^\alpha(x,\nu_\epsilon)]e_k^*(x)+[a_k^\alpha(x,\nu_\epsilon)-a_k^\alpha(x,\mu_\epsilon)]e_k^*(x).
\end{aligned}
$$

By the regularity assumption 3.1, there exists c_2 such that

$$
\sup_{x\in\mathbb{T}^d}|\gamma_{k,\epsilon}^\alpha(x)|\le c_2(1+|k|^2)\widehat{\rho}_*(\mu_\epsilon,\nu_\epsilon) \qquad \forall\alpha\in\mathcal{C}_a,\ k\in\mathbb{Z}^d.
$$

Hence, for every $\boldsymbol{\alpha}\in\mathcal{A}$,

$$
\begin{aligned}
\mathcal{J}_\epsilon^\alpha &\le \frac{1}{\epsilon}\sum_{k\in\mathbb{Z}^d}\frac{|F_k(\mu_\epsilon-\nu_\epsilon)|}{(1+|k|^2)^{n_*}}\ \sup_{x\in\mathbb{T}^d}|\gamma_{k,\epsilon}^\alpha(x)| \\
&\le \frac{c_2}{\epsilon}\left(\sum_{k\in\mathbb{Z}^d}\frac{|F_k(\mu_\epsilon-\nu_\epsilon)|^2}{(1+|k|^2)^{n_*}}\right)^{\frac12}\left(\sum_{k\in\mathbb{Z}^d}(1+|k|^2)^{2-n_*}\right)^{\frac12}\widehat{\rho}_*(\mu_\epsilon,\nu_\epsilon) \\
&\le c_2L_w\ c(d)\,\widehat{\rho}_*(\mu_\epsilon,\nu_\epsilon) =: \hat{c}\ \widehat{\rho}_*(\mu_\epsilon,\nu_\epsilon),
\end{aligned}
$$

where $c(d)$ is as in (6.1). Therefore, $\lim_{\epsilon\downarrow 0}\ \sup_{\alpha\in\mathcal{C}_a}\mathcal{J}_\epsilon^\alpha\le\hat{c}\lim_{\epsilon\downarrow 0}\ \widehat{\rho}_*(\mu_\epsilon,\nu_\epsilon)=0$.

Step 10 (conclusion). By (6.5) and the above steps, $0<\delta\le\lim_{\epsilon\downarrow 0}\ [H(\mu_\epsilon,\kappa_\epsilon)-H(\nu_\epsilon,\kappa_\epsilon)]\le 0$. This clear contradiction implies that $\max_{\overline{\mathcal{O}}}\ (u-w)\le 0$.

7. Lipschitz continuity. In this section, we prove Theorem 4.2.

7.1. Regularity in space. We first prove the continuous dependence of the solutions of the McKean–Vlasov stochastic differential equation (3.1) on its initial data.

PROPOSITION 7.1. *Suppose that the regularity assumption* 3.1 *holds. Then, there exists* $\hat{c} > 0$ *depending on* T *and the constant* c_a *of assumption* 3.1 *such that*

$$\widehat{\rho}_*(\mathcal{L}_u^{t,\mu,\boldsymbol{\alpha}}, \mathcal{L}_u^{t,\nu,\boldsymbol{\alpha}}) \le \hat{c}\, \widehat{\rho}_*(\mu,\nu) \quad \forall\ 0 \le t \le u \le T,\ \mu,\nu \in \mathcal{P}(\mathbb{T}^d),\ \boldsymbol{\alpha} \in \mathcal{A}.$$

Proof. We complete the proof in several steps.

Step 1 (setting). We fix $t \in [0,T]$, $\mu,\nu \in \mathcal{P}(\mathbb{T}^d)$, and $\boldsymbol{\alpha} \in \mathcal{A}$ and set

$$Y_u := X_u^{t,\mu,\boldsymbol{\alpha}},\ \ \mu_u := \mathcal{L}_u^{t,\mu,\boldsymbol{\alpha}}, \qquad Z_u := X_u^{t,\nu,\boldsymbol{\alpha}},\ \ \nu_u := \mathcal{L}_u^{t,\nu,\boldsymbol{\alpha}}, \qquad u \in [t,T].$$

By the definition of $\widehat{\rho}_*$, we need to prove the following estimate for every $u \in [t,T]$:

$$(\mu_u - \nu_u)(\psi) \le \hat{c}\, \widehat{\rho}_*(\mu,\nu)\, \|\psi\|_{\mathcal{C}_*} \qquad \forall\, \psi \in \mathcal{C}_*.$$

Step 2 (stochastic differential equations). For $x \in \mathbb{T}^d$, let Y^x, Z^x be the solutions of the stochastic differential equations

$$Y_u^x = x + \int_t^u \left[b^{\alpha_s}(Y_s^x,\mu_s)\mathrm{d}s + \sigma^{\alpha_s}(Y_s^x,\mu_s)\mathrm{d}B_s\right],$$
$$Z_u^x = x + \int_t^u \left[b^{\alpha_s}(Z_s^x,\nu_s)\mathrm{d}s + \sigma^{\alpha_s}(Z_s^x,\nu_s)\mathrm{d}B_s\right].$$

Set $L_u^\mu(x) := \mathbb{E}[\psi(Y_u^x)]$ and $L_u^\nu(x) := \mathbb{E}[\psi(Z_u^x)]$. Then, by conditioning, we have

$$\mu_u(\psi) = \mathbb{E}[\psi(Y_u)] = \mu(L_u^\mu), \qquad \nu_u(\psi) = \mathbb{E}[\psi(Z_u)] = \nu(L_u^\nu).$$

Therefore,

$$(\mu_u - \nu_u)(\psi) = (\mu-\nu)(L_u^\mu) + \nu(L_u^\mu - L_u^\nu) =: \mathcal{I}_u(\psi) + \mathcal{J}_u(\psi).$$

Step 3 ($\mathcal{I}_u$ estimate). By the regularity assumption 3.1, there exists a constant $\hat{c}_1$ satisfying

$$\|b^{\alpha_u}(\cdot,\mu_u)\|_{\mathcal{C}_*} + \|\sigma^{\alpha_u}(\cdot,\mu_u)\|_{\mathcal{C}_*} \le \hat{c}_1 \qquad \forall\ u \in [t,T].$$

Hence, the map $x \in \mathbb{T}^d \to Y_u^x$ is n_*-times differentiable [23]. Therefore, $L_u^\mu \in \mathcal{C}_*$. From the chain rule, there exists a constant $\hat{c}_2 > 0$ depending only on c_a of assumption 3.1, satisfying

$$\|L_u^\mu\|_{\mathcal{C}_*} \le \hat{c}_2\, \|\psi\|_{\mathcal{C}_*} \qquad \forall\ u \in [t,T], \mu \in \mathcal{P}(\mathbb{T}^d).$$

This implies that

$$\mathcal{I}_u(\psi) = (\mu-\nu)(L_u^\mu) \le \hat{c}_2\, \widehat{\rho}_*(\mu,\nu)\, \|\psi\|_{\mathcal{C}_*}.$$

Step 4 ($\mathcal{J}_u$ estimate). By definitions, $\mathcal{J}_{\le} \sup_x |L_u^\mu(x) - L_u^\nu(x)|$ and

$$|L_u^\mu - L_u^\nu| \le \mathbb{E}[|\psi(Y_u^x) - \psi(Z_u^x)|] \le \mathbb{E}[|Y_u^x - Z_u^x|]\, \|\psi\|_1 \le (\mathbb{E}[(Y_s^x - Z_s^x)^2])^{\frac{1}{2}}\, \|\psi\|_*.$$

For $x \in \mathbb{T}^d$ and setting $m_s^2(x) := \mathbb{E}[(Y_s^x - Z_s^x)^2]$, we directly estimate that

$$m_u^2(x) \leq 2T \int_t^u \mathbb{E}[(b^{\alpha_s}(Y_s^x,\mu_s) - b^{\alpha_s}(Z_s^x,\nu_s))^2]\mathrm{d}s + 2\int_t^u \mathbb{E}[|\sigma^{\alpha_s}(Y_s^x,\mu_s) - \sigma^{\alpha_s}(Z_s^x,\nu_s)|^2]\mathrm{d}s.$$

By the regularity assumption 3.1,

$$|b^{\alpha_s}(Y_s^x,\mu_s) - b^{\alpha_s}(Z_s^x,\nu_s)| \leq c_a \left[|Y_s^x - Z_s^x| + \widehat{\rho}_*(\mu_s,\nu_s)\right].$$

The same estimate also holds for $|\sigma^{\alpha_s}(Y_s^x,\mu_s) - \sigma^{\alpha_s}(Z_s^x,\nu_s)|$. Hence, there exists a constant $\hat{c}_3 > 0$, independent of x, satisfying $m_u^2 \leq \hat{c}_3 \int_t^u [m_s^2 + \widehat{\rho}_*(\mu_s,\nu_s)^2]\mathrm{d}s$ for every $u \in [t,T]$. By Grönwall's inequality, there exists $\hat{c}_4 > 0$ satisfying $m_u^2 \leq \hat{c}_4^2 \int_t^u \widehat{\rho}_*(\mu_s,\nu_s)^2 \mathrm{d}s$. Hence,

$$\mathcal{J}_u \leq (\mathbb{E}[(Y_s^x - Z_s^x)^2])^{\frac{1}{2}} \|\psi\|_* \leq \hat{c}_4 \left(\int_t^u \widehat{\rho}_*(\mu_s,\nu_s)^2 \ \mathrm{d}s\right)^{\frac{1}{2}} \|\psi\|_{\mathcal{C}_*} \qquad \forall u \in [t,T].$$

Step 5 (conclusion). By the previous steps,

$$(\mu_u - \nu_u)(\psi) \leq \left(\hat{c}_2\, \widehat{\rho}_*(\mu,\nu) + \hat{c}_4 \left(\int_t^u \widehat{\rho}_*(\mu_s,\nu_s)^2 \ \mathrm{d}s\right)^{\frac{1}{2}}\right) \|\psi\|_{\mathcal{C}_*} \qquad \forall \psi \in \mathcal{C}_*.$$

Since the above holds for every $\psi \in \mathcal{C}_*$, the definition of $\widehat{\rho}_*$ implies that

$$\widehat{\rho}_*(\mu_u,\nu_u) \leq \hat{c}_2\, \widehat{\rho}_*(\mu,\nu) + \hat{c}_4 \left(\int_t^u \widehat{\rho}_*(\mu_s,\nu_s)^2 \ \mathrm{d}s\right)^{\frac{1}{2}} \qquad \forall u \in [t,T].$$

Hence,

$$\widehat{\rho}_*(\mu_u,\nu_u)^2 \leq 2\hat{c}_2^2\, \widehat{\rho}_*(\mu,\nu)^2 + 2\hat{c}_4^2 \int_t^u \widehat{\rho}_*(\mu_s,\nu_s)^2 \mathrm{d}s \qquad \forall u \in [t,T].$$

Again, by Grönwall, $\widehat{\rho}_*(\mu_u,\nu_u)^2 \leq \hat{c}^2\, \widehat{\rho}_*(\mu,\nu)^2$ for some $\hat{c} > 0$ for all $u \in [t,T]$. ◻

The following is an immediate consequence of the above estimate.

LEMMA 7.2. *Under the regularity assumption* 3.1*, there exists* $L_1 > 0$ *such that*

$$|J(t,\mu,\boldsymbol{\alpha}) - J(t,\nu,\boldsymbol{\alpha})| \leq L_1\, \widehat{\rho}_*(\mu,\nu) \qquad \forall \boldsymbol{\alpha} \in \mathcal{A},\ \mu,\nu \in \mathcal{P}(\mathbb{T}^d),\ t \in [0,T].$$

Consequently,

$$|v(t,\mu) - v(t,\nu)| \leq L_1\, \widehat{\rho}_*(\mu,\nu) \qquad \forall \mu,\nu \in \mathcal{P}(\mathbb{T}^d),\ t \in [0,T].$$

Proof. We fix $\boldsymbol{\alpha} \in \mathcal{A}$, $\mu,\nu \in \mathcal{P}(\mathbb{T}^d)$, and $t \in [0,T]$ and use the same notation as in Proposition 7.1. For $u \in [t,T]$, the regularity assumption 3.1 implies that

$$\begin{aligned}
&|\mathbb{E}[\ell^{\alpha_u}(Y_u,\mu_u) - \ell^{\alpha_u}(Z_u,\nu_u)]| \\
&\qquad \leq |\mathbb{E}[\ell^{\alpha_u}(Y_u,\mu_u) - \ell^{\alpha_u}(Z_u,\mu_u)]| + |\mathbb{E}[\ell^{\alpha_u}(Z_u,\mu_u) - \ell^{\alpha_u}(Z_u,\nu_u)]| \\
&\qquad \leq |(\mu_u - \nu_u)(\ell^{\alpha_u}(\cdot,\mu_u))| + c_a\, \widehat{\rho}_*(\mu_u,\nu_u) \\
&\qquad \leq \widehat{\rho}_*(\mu_u,\nu_u) \|\ell^{\alpha_u}(\cdot,\mu_u)\|_{\mathcal{C}_*} + c_a\, \widehat{\rho}_*(\mu_u,\nu_u) \\
&\qquad \leq 2c_a\, \widehat{\rho}_*(\mu_u,\nu_u) \leq 2c_a\, \hat{c}\, \widehat{\rho}_*(\mu,\nu).
\end{aligned}$$

We now directly estimate using the above to obtain the following inequalities:

$$\begin{aligned}|J(t,\mu,\boldsymbol{\alpha})-J(t,\nu,\boldsymbol{\alpha})| &\le \int_t^T |\mathbb{E}[\ell^{\alpha_u}(Y_u,\mu_u)-\ell^{\alpha_u}(Z_u,\nu_u)]|\,\mathrm{d}u \;+\; |\mathbb{E}[\varphi(\mu_T)-\varphi(\nu_T)]|\\ &\le 2c_a\,\hat{c}\,(T-t)\,\widehat{\rho}_*(\mu,\nu)+c_a\,\widehat{\rho}_*(\mu_T,\nu_T)\\ &\le c_a\,\hat{c}\,(2(T-t)+1)\widehat{\rho}_*(\mu,\nu).\end{aligned}$$

Because $|v(t,\mu)-v(t,\nu)|\le \sup_{\boldsymbol{\alpha}\in\mathcal{A}}|J(t,\mu,\boldsymbol{\alpha})-J(t,\nu,\boldsymbol{\alpha})|$, the proof of the lemma is complete. ∎

7.2. Time regularity.

PROPOSITION 7.3. *Suppose that the regularity assumption* 3.1 *holds. Then, there exists* $L_2>0$ *depending on* T *and the constant* c_a *in assumption* 3.1 *such that*

$$|v(t,\mu)-v(\tau,\mu)|\le L_2\,|t-\tau|^{\frac{1}{2}} \quad \forall\ t,\tau\in[0,T],\ \mu\in\mathcal{P}(\mathbb{T}^d).$$

Proof. Fix $0\le t\le\tau\le T$, $\mu\in\mathcal{P}(\mathbb{T}^d)$, and $\boldsymbol{\alpha}\in\mathcal{A}$, and set $h:=\tau-t$. With an arbitrary constant $a_*\in A$, we define

$$\tilde{\alpha}_u(\cdot):=\begin{cases}\alpha_{u+h}(\cdot) & \text{if } u\in[t,T-h],\\ a_* & \text{if } u\in[T-h,T].\end{cases}$$

It is clear that $\tilde{\boldsymbol{\alpha}}\in\mathcal{A}$. Set

$$\tilde{\mu}_u:=\mathcal{L}_u^{t,\mu,\tilde{\boldsymbol{\alpha}}},\quad u\in[t,T],\quad \text{and}\quad \mu_u:=\mathcal{L}_u^{\tau,\mu,\boldsymbol{\alpha}},\quad u\in[\tau,T].$$

Then, $\tilde{\mu}_u=\mu_{u+h}$ for every $u\in[t,T-h]$. In particular,

$$\mathbb{E}[\ell^{\tilde{\alpha}_u}(X_u^{t,\mu,\tilde{\boldsymbol{\alpha}}})]=\mathbb{E}[\ell^{\alpha_u}(X_{u+h}^{\tau,\mu,\boldsymbol{\alpha}})]\qquad \forall\, u\in[t,T-h].$$

Since $\mu_T=\tilde{\mu}_{T-h}=\mathcal{L}(X_{T-h}^{t,\mu,\tilde{\boldsymbol{\alpha}}})$ and $\tilde{\mu}_T=\mathcal{L}(X_T^{t,\mu,\tilde{\boldsymbol{\alpha}}})$,

$$\widehat{\rho}_1(\tilde{\mu}_T,\mu_T)\le\mathbb{E}[|X_T^{t,\mu,\tilde{\boldsymbol{\alpha}}}-X_{T-h}^{t,\mu,\tilde{\boldsymbol{\alpha}}}|]\le\left(\mathbb{E}[(X_T^{t,\mu,\tilde{\boldsymbol{\alpha}}}-X_{T-h}^{t,\mu,\tilde{\boldsymbol{\alpha}}})^2]\right)^{\frac{1}{2}}.$$

Because b,σ are bounded by c_a, there is $\tilde{c}_1>0$ satisfying $\widehat{\rho}_1(\tilde{\mu}_T,\mu_T)\le\tilde{c}_1\sqrt{h}$. Therefore,

$$|\varphi(\tilde{\mu}_T)-\varphi(\mu_T)|\le c_a\widehat{\rho}_*(\tilde{\mu}_T,\mu_T)\le c_a\widehat{\rho}_1(\tilde{\mu}_T,\mu_T)\le\tilde{c}_1\,c_a\sqrt{h}.$$

The above estimates imply that, for any $\boldsymbol{\alpha}\in\mathcal{A}$,

$$\begin{aligned}v(t,\mu)-J(\tau,\mu,\boldsymbol{\alpha})&\le J(t,\mu,\tilde{\boldsymbol{\alpha}})-J(\tau,\mu,\boldsymbol{\alpha})\\ &=\int_{T-h}^T\mathbb{E}[\ell^{\tilde{\alpha}_u}(X_u^{t,\mu,\tilde{\boldsymbol{\alpha}}})]\,\mathrm{d}u+\varphi(\tilde{\mu}_T)-\varphi(\mu_T)\le c_a h+\tilde{c}_1\,c_a\sqrt{h}.\end{aligned}$$

Hence,

$$v(t,\mu)-v(\tau,\mu)=\sup_{\boldsymbol{\alpha}\in\mathcal{A}}\left(v(t,\mu)-J(t,\mu,\boldsymbol{\alpha})\right)\le c_a h+\tilde{c}_1\,c_a\sqrt{h}.$$

We prove the opposite inequality by using the control

$$\hat{\alpha}_u(\cdot):=\begin{cases}\alpha_{u-h}(\cdot) & \text{if } u\in[h,T],\\ a_* & \text{if } u\in[0,h].\end{cases}$$

Again, $\hat{\boldsymbol{\alpha}} \in \mathcal{A}$, and we set

$$\hat{\mu}_u := \mathcal{L}_u^{\tau,\mu,\hat{\boldsymbol{\alpha}}}, \quad u \in [\tau, T], \quad \text{and} \quad \mu_u := \mathcal{L}_u^{t,\mu,\boldsymbol{\alpha}}, \quad u \in [t, T].$$

Then, $\hat{\mu}_u = \mu_{u-h}$ for every $u \in [\tau, T]$ and $\hat{\mu}_T = \mu_{T-h}$. Following the above steps *mutatis mutandis*, we obtain the following inequality for any $\boldsymbol{\alpha} \in \mathcal{A}$:

$$\begin{aligned} v(\tau,\mu) - J(t,\mu,\alpha) &\le J(\tau,\mu,\hat{\boldsymbol{\alpha}}) - J(t,\mu,\alpha) \\ &= -\int_t^\tau \mathbb{E}[\ell^{\tilde{\alpha}_u}(X_u^{t,\mu,\boldsymbol{\alpha}})]\,\mathrm{d}u + \varphi(\hat{\mu}_t) - \varphi(\mu_T) \le c_a h + \tilde{c}_1\, c_a \sqrt{h}. \end{aligned}$$

Hence,

$$v(\tau,\mu) - v(t,\mu) = \sup_{\boldsymbol{\alpha}\in\mathcal{A}} \left(v(\tau,\mu) - J(t,\mu,\boldsymbol{\alpha})\right) \le c_a h + \tilde{c}_1\, c_a \sqrt{h}. \qquad \square$$

8. Dynamic programming. In this section, we prove Theorem 3.4. For a general result but in a different setting, we refer the reader to [18].

Proof of Theorem 3.4. We fix $(t,\mu) \in \overline{\mathcal{O}}$, $\tau \in [t,T]$ and set

$$Q(\boldsymbol{\alpha}) := \int_t^\tau \mathbb{E}[\ell^{\alpha_s}(X_s^{t,\mu,\boldsymbol{\alpha}}, \mathcal{L}_s^{t,\mu,\boldsymbol{\alpha}})]\mathrm{d}s + v(\tau, \mathcal{L}_\tau^{t,\mu,\boldsymbol{\alpha}}), \qquad \boldsymbol{\alpha} \in \mathcal{A}.$$

Then, the dynamic programming principle can be stated as $v(t,\mu) = \inf_{\boldsymbol{\alpha}\in\mathcal{A}} Q(\boldsymbol{\alpha})$. Recall that $v(t,\mu) = \inf_{\boldsymbol{\alpha}\in\mathcal{A}} J(t,\mu,\boldsymbol{\alpha})$. For any $\boldsymbol{\alpha} \in \mathcal{A}$ and $s \in [\tau, T]$, the Markov property implies that $X_s^{t,\mu,\boldsymbol{\alpha}} = X_s^{\tau,\mathcal{L}_\tau^{t,\mu,\boldsymbol{\alpha}},\boldsymbol{\alpha}}$, and consequently, $\mathcal{L}_s^{t,\mu,\boldsymbol{\alpha}} = \mathcal{L}_s^{\tau,\mathcal{L}_\tau^{t,\mu,\boldsymbol{\alpha}},\boldsymbol{\alpha}}$ [8]. Hence,

$$\begin{aligned} &\int_\tau^T \mathbb{E}[\ell^{\alpha_s}(X_s^{t,\mu,\boldsymbol{\alpha}}, \mathcal{L}_s^{t,\mu,\boldsymbol{\alpha}})]\ \mathrm{d}s + \varphi(\mathcal{L}_T^{t,\mu,\boldsymbol{\alpha}}) \\ &\qquad = \int_\tau^T \mathbb{E}[\ell^{\alpha_s}(X_s^{\tau,\mathcal{L}_\tau^{t,\mu,\boldsymbol{\alpha}},\boldsymbol{\alpha}}, \mathcal{L}_s^{\tau,\mathcal{L}_\tau^{t,\mu,\boldsymbol{\alpha}},\boldsymbol{\alpha}})]\ \mathrm{d}s + \varphi(\mathcal{L}_T^{\tau,\mathcal{L}_\tau^{t,\mu,\boldsymbol{\alpha}},\boldsymbol{\alpha}}) \\ &\qquad = J(\tau, \mathcal{L}_\tau^{\tau,\mathcal{L}_\tau^{t,\mu,\boldsymbol{\alpha}},\boldsymbol{\alpha}}, \boldsymbol{\alpha}) \ge v(\tau, \mathcal{L}_\tau^{\tau,\mathcal{L}_\tau^{t,\mu,\boldsymbol{\alpha}},\boldsymbol{\alpha}}). \end{aligned}$$

This implies that

$$\begin{aligned} J(t,\mu,\boldsymbol{\alpha}) &= \int_t^\tau \mathbb{E}[\ell^{\alpha_s}(X_s^{t,\mu,\boldsymbol{\alpha}}, \mathcal{L}_s^{t,\mu,\boldsymbol{\alpha}})]\ \mathrm{d}s + \left(\int_\tau^T \mathbb{E}[\ell^{\alpha_s}(X_s^{t,\mu,\boldsymbol{\alpha}}, \mathcal{L}_s^{t,\mu,\boldsymbol{\alpha}}))]\ \mathrm{d}s + \varphi(\mathcal{L}_T^{t,\mu,\boldsymbol{\alpha}})\right) \\ &\ge \int_t^\tau \mathbb{E}[\ell^{\alpha_s}(X_s^{t,\mu,\boldsymbol{\alpha}}, \mathcal{L}_s^{t,\mu,\boldsymbol{\alpha}}))]\ \mathrm{d}s + v(\tau, \mathcal{L}_\tau^{\tau,\mathcal{L}_\tau^{t,\mu,\boldsymbol{\alpha}},\boldsymbol{\alpha}}) = Q(\boldsymbol{\alpha}). \end{aligned}$$

Therefore, $v(t,\mu) = \inf_{\boldsymbol{\alpha}\in\mathcal{A}} J(t,\mu,\boldsymbol{\alpha}) \ge \inf_{\boldsymbol{\alpha}\in\mathcal{A}} Q(\boldsymbol{\alpha})$.

To prove the opposite inequality, we fix $\epsilon > 0$ and choose controls $\tilde{\boldsymbol{\alpha}} \in \mathcal{A}$ satisfying $Q(\tilde{\boldsymbol{\alpha}}) \le \inf_{\boldsymbol{\alpha}\in\mathcal{A}} Q(\boldsymbol{\alpha}) + \frac{\epsilon}{2}$ and $\widehat{\boldsymbol{\alpha}} \in \mathcal{A}$ satisfying

$$J(\tau, \mathcal{L}_\tau^{t,\mu,\tilde{\boldsymbol{\alpha}}}, \widehat{\boldsymbol{\alpha}}) \le v(\tau, \mathcal{L}_\tau^{t,\mu,\tilde{\boldsymbol{\alpha}}}) + \frac{\epsilon}{2}.$$

Set $\boldsymbol{\alpha}^*_u := \tilde{\boldsymbol{\alpha}}_u \ \chi_{[t,\tau]}(u) + \widehat{\boldsymbol{\alpha}}_u \ \chi_{[\tau,T]}(u)$. Then,

$$
\begin{aligned}
v(t,\mu) \le J(t,\mu,\boldsymbol{\alpha}^*) &= \int_t^T \ell^{\tilde{\boldsymbol{\alpha}}_s}(X_s^{t,\mu,\tilde{\boldsymbol{\alpha}}}, \mathcal{L}_s^{t,\mu,\tilde{\boldsymbol{\alpha}}}) \ \mathrm{d}s + J(\tau, \mathcal{L}_\tau^{t,\mu,\tilde{\boldsymbol{\alpha}}}, \widehat{\boldsymbol{\alpha}}) \\
&\le \int_t^T \ell^{\tilde{\boldsymbol{\alpha}}_s}(X_s^{t,\mu,\tilde{\boldsymbol{\alpha}}}, \mathcal{L}_s^{t,\mu,\tilde{\boldsymbol{\alpha}}}) \ \mathrm{d}s + v(\tau, \mathcal{L}_\tau^{t,\mu,\tilde{\boldsymbol{\alpha}}}) + \frac{\epsilon}{2} = Q(\tilde{\boldsymbol{\alpha}}) + \frac{\epsilon}{2} \\
&\le \inf_{\boldsymbol{\alpha}\in\mathcal{A}} Q(\boldsymbol{\alpha}) + \epsilon.
\end{aligned}
$$

9. Viscosity property. In this section, we prove the viscosity property of the value function. Although the below proof follows the standard one very closely, we provide it for completeness.

The following version of Itô's formula along flows of measures follows from Proposition 5.102 of [8]. Recall that $X^{t,\mu,\boldsymbol{\alpha}}$ is the solution of (3.1), $\mathcal{L}_u^{t,\mu,\boldsymbol{\alpha}} = \mathcal{L}(X_u^{t,\mu,\boldsymbol{\alpha}})$, and the operator $\mathcal{M}^{a,\mu}$ is defined in subsection 3.3.

LEMMA 9.1. *For every* $\psi \in \mathcal{C}_s(\mathbb{T}^d)$, $(t,\mu) \in \overline{\mathcal{O}}$, $u \in [t,T]$, *and* $\boldsymbol{\alpha} \in \mathcal{A}$,

$$
\psi(u, \mathcal{L}_u^{t,\mu,\boldsymbol{\alpha}}) = \psi(t,\mu) + \int_t^u \Big(\partial_t \psi(s, \mathcal{L}_s^{t,\mu,\boldsymbol{\alpha}}) + \mathbb{E}[\mathcal{M}^{\alpha_s, \mathcal{L}_s^{t,\mu,\boldsymbol{\alpha}}}[\partial_\mu \psi(s, \mathcal{L}_s^{t,\mu,\boldsymbol{\alpha}})](X_s^{t,\mu,\boldsymbol{\alpha}})]\Big)\,\mathrm{d}s.
$$

9.1. Subsolution. Suppose that, for $(t_0,\mu_0) \in [0,T) \times \mathcal{P}(\mathbb{T}^d)$ and test function $\psi \in \mathcal{C}_s(\overline{\mathcal{O}})$,

$$
0 = (v - \psi)(t_0,\mu_0) = \max_{\overline{\mathcal{O}}}(v-\psi).
$$

For $\alpha \in \mathcal{C}_a$, set

$$
k^\alpha(t,x,\mu) := \ell(x,\mu,\alpha(x)) + \mathcal{M}^{\alpha,\mu}[\partial_\mu \psi(t,\mu)]])(x), \qquad t \in [0,T],\ x \in \mathbb{T}^d,\ \mu \in \mathcal{P}(\mathbb{T}^d).
$$

Because $H(\mu_0, \partial_\mu \psi(t_0,\mu_0)) = \inf_{\alpha\in\mathcal{C}_a} \mu_0(k^\alpha(t_0,\cdot,\mu_0))$, for any $\epsilon > 0$, there is $\alpha^* \in \mathcal{C}_a$ satisfying

$$
\mu_0(k^{\alpha^*}(t_0,\cdot,\mu_0)) \le H(\mu_0, \partial_\mu \psi(t_0,\mu_0)) + \epsilon.
$$

Set $\boldsymbol{\alpha}^* \equiv \alpha_*$, and let $X_u^* := X_u^{t_0,\mu_0,\boldsymbol{\alpha}^*}$ and $\mu_u^* := \mathcal{L}_u^{t_0,\mu_0,\boldsymbol{\alpha}^*}$ for $u \in [t_0,T]$. Since $v \le \psi$, the dynamic programming principle Theorem 3.4 with $\tau = t_0 + h \le T$ implies that

$$
v(t_0,\mu_0) \le \int_{t_0}^{t_0+h} \mathbb{E}[\ell(X_s^*, \mu_s^*, \alpha^*(X_s^*)]\mathrm{d}s + \psi(t_0+h, \mu_{t_0+h}^*).
$$

By Lemma 9.1,

$$
\psi(t_0+h, \mu_{t_0+h}^*) = \psi(t_0,\mu_0) + \int_{t_0}^{t_0+h} \Big(\partial_t \psi(s,\mu_s^*) + \mathbb{E}[\mathcal{M}^{\alpha^*,\mu_s^*}[\partial_\mu \psi(s,\mu_s^*)](X_s^*)]\Big)\,\mathrm{d}s.
$$

Since $\psi(t_0,\mu_0) = v(t_0,\mu_0)$, the above inequalities imply that

$$
0 \le \frac{1}{h} \int_{t_0}^{t_0+h} \Big(\partial_t \psi(s,\mu_s^*) + \mathbb{E}[k^{\alpha^*}(s, X_s^*, \mu_s^*)]\Big)\,\mathrm{d}s. \tag{9.1}
$$

We now let h tend to zero to arrive at the following inequality:

$$
-\partial_t \psi(t_0,\mu_0) \le \mathbb{E}[k^{\alpha^*}(t_0, X_{t_0}, \mu_{t_0})] = \mu_0(k^{\alpha^*}(t_0,\cdot,\mu_0)) \le H(\mu_0, \partial_\mu \psi(t_0,\mu_0)) + \epsilon.
$$

9.2. Supersolution. Suppose that, for $(t_0,\mu_0)\in[0,T)\times\mathcal{P}(\mathbb{T}^d)$ and a test function $\psi\in\mathcal{C}_s(\overline{\mathcal{O}})$,

$$0=(v-\psi)(t_0,\mu_0)=\min_{\overline{\mathcal{O}}}(v-\psi).$$

We may assume that the minimum is strict. Toward a counterposition, suppose that

$$-\partial_t\psi(t_0,\mu_0)<H(\mu_0,\partial_\mu\psi(t_0,\mu_0))=\inf_{\alpha\in\mathcal{C}_a}\{\mu_0(k^\alpha,\cdot,\mu_0))\},$$

where k^α is as in the previous subsection. By Definition 3.5 of test functions $\mathcal{C}_s(\mathcal{O})$, the map $(t,\mu)\in\overline{\mathcal{O}}\mapsto H(\mu,\partial_\mu\psi(t,\mu))$ is continuous. Therefore, there exist $\delta>0$ and a neighborhood $\mathcal{B}\subseteq\overline{\mathcal{O}}$ of (t_0,μ_0) such that

$$-\partial_t\psi(t,\mu)+\delta\le H(\mu,\partial_\mu\psi(t,\mu))=\inf_{\alpha\in\mathcal{C}_a}\{\mu(k^\alpha(t,\cdot,\mu))\}\qquad\forall\,(t,\mu)\in\mathcal{B}.$$

For $\boldsymbol{\alpha}\in\mathcal{A}$, set $X_s^{\boldsymbol{\alpha}}:=X_s^{t_0,\mu_0,\boldsymbol{\alpha}}$, $\mu_s^{\boldsymbol{\alpha}}:=\mathcal{L}_s^{t_0,\mu_0,\boldsymbol{\alpha}}$, and consider the (deterministic) time

$$\tau^{\boldsymbol{\alpha}}:=\inf\{s\in[t_0,T]:(s,\mu_s^{\boldsymbol{\alpha}})\notin\mathcal{B}\}$$

so that, for every $s\in[t_0,\tau^{\boldsymbol{\alpha}})$, $(s,\mu_s^{\boldsymbol{\alpha}})\in\mathcal{B}$, and consequently,

$$\mu_s^{\boldsymbol{\alpha}}(k^{\alpha_s}(s,\cdot,\mu_s^{\boldsymbol{\alpha}}))\ge H(\mu_s^{\boldsymbol{\alpha}},\partial_\mu\psi(s,\mu_s^{\boldsymbol{\alpha}}))\ge-\partial_t\psi(s,\mu_s^{\boldsymbol{\alpha}})+\delta.$$

Because $\mathbb{E}[k^{\alpha_s}(s,X_s^{\boldsymbol{\alpha}},\mu_s^{\boldsymbol{\alpha}})]=\mu_s^{\boldsymbol{\alpha}}(k^{\alpha_s}(s,\cdot,\mu_s^{\boldsymbol{\alpha}}))$,

$$\int_{t_0}^{\tau^{\boldsymbol{\alpha}}}\left(\mathbb{E}[k^{\alpha_s}(X_s^{\boldsymbol{\alpha}},\mu_s^{\boldsymbol{\alpha}})]+\partial_t\psi(s,\mu_s^{\boldsymbol{\alpha}})\right)\mathrm{d}s\ge\delta(\tau^{\boldsymbol{\alpha}}-t_0).$$

Then, by Lemma 9.1, we obtain the following inequality:

$$\begin{aligned}\psi(\tau^{\boldsymbol{\alpha}},\mu_{\tau^{\boldsymbol{\alpha}}}^{\boldsymbol{\alpha}})&=\psi(t_0,\mu_0)+\int_{t_0}^{\tau^{\boldsymbol{\alpha}}}(\partial_t\psi(s,\mu_s^{\boldsymbol{\alpha}})+\mathbb{E}[\mathcal{M}^{\alpha_s,\mu_s^{\boldsymbol{\alpha}}}[\partial_\mu\psi(s,\mu_s^{\boldsymbol{\alpha}})](X_s^{\boldsymbol{\alpha}})\,])\mathrm{d}s\\&=\psi(t_0,\mu_0)+\int_{t_0}^{\tau^{\boldsymbol{\alpha}}}(\partial_t\psi(s,\mu_s^{\boldsymbol{\alpha}})+\mathbb{E}[k^{\alpha_s}(s,X_s^{\boldsymbol{\alpha}},\mu_s^{\boldsymbol{\alpha}})]-\mathbb{E}[\ell^{\alpha_s}(X_s^{\boldsymbol{\alpha}},\mu_s^{\boldsymbol{\alpha}})])\,\mathrm{d}s\\&\ge\psi(t_0,\mu_0)-\int_{t_0}^{\tau^{\boldsymbol{\alpha}}}\mathbb{E}[\ell^{\alpha_s}(X_s^{\boldsymbol{\alpha}},\mu_s^{\boldsymbol{\alpha}})]\,\mathrm{d}s+\delta(\tau^{\boldsymbol{\alpha}}-t_0).\end{aligned}$$

Since $v\ge\psi$ and $\psi(t_0,\mu_0)=v(t_0,\mu_0)$, the above implies that

$$v(t_0,\mu_0)\le\int_{t_0}^{\tau^{\boldsymbol{\alpha}}}\mathbb{E}[\ell^{\alpha_s}(X_s^{\boldsymbol{\alpha}},\mu_s^{\boldsymbol{\alpha}})]\,\mathrm{d}s+v(\tau^{\boldsymbol{\alpha}},\mu_{\tau^{\boldsymbol{\alpha}}}^{\boldsymbol{\alpha}})-g(\boldsymbol{\alpha})\qquad\forall\,\boldsymbol{\alpha}\in\mathcal{A},$$

where $g(\boldsymbol{\alpha}):=\delta(\tau^{\boldsymbol{\alpha}}-t_0)+(v(\tau^{\boldsymbol{\alpha}},\mu_{\tau^{\boldsymbol{\alpha}}}^{\boldsymbol{\alpha}})-\psi(\tau^{\boldsymbol{\alpha}},\mu_{\tau^{\boldsymbol{\alpha}}}^{\boldsymbol{\alpha}}))$. We now claim that

$$\delta_0:=\inf_{\boldsymbol{\alpha}\in\mathcal{A}}g(\boldsymbol{\alpha})>0.$$

Indeed, since $v\ge\psi$, if $\tau^{\boldsymbol{\alpha}}=T$, then $g(\boldsymbol{\alpha})\ge\delta(T-t_0)$. On the other hand, if $\tau^{\boldsymbol{\alpha}}<T$, then $(\tau^{\boldsymbol{\alpha}},\mu_{\tau^{\boldsymbol{\alpha}}}^{\boldsymbol{\alpha}})\in\partial\mathcal{B}$. Because $\mathcal{B}$ is compact and $(t_0,\mu_0)\notin\partial\mathcal{B}$ is the strict minimizer of $v-\psi$, we have

$$(v-\psi)(\tau^{\boldsymbol{\alpha}},\mu_{\tau^{\boldsymbol{\alpha}}}^{\boldsymbol{\alpha}})\ge\inf_{(t,\mu)\in\partial\mathcal{B}}(v-\psi)(t,\mu)>0.$$

Hence, $\delta_0 > 0$, and the above inequalities imply that, for every $\boldsymbol{\alpha} \in \mathcal{A}$,

$$v(t_0,\mu_0) \le \int_{t_0}^{\tau^{\boldsymbol{\alpha}}} \mathbb{E}[\ell^{\alpha_s}(X_s^{\boldsymbol{\alpha}},\mu_s^{\boldsymbol{\alpha}})]\mathrm{d}s + v(\tau^{\boldsymbol{\alpha}},\mu_{\tau^{\boldsymbol{\alpha}}}^{\boldsymbol{\alpha}}) - \delta_0.$$

This contradiction to dynamic programming implies that $-\psi_t(t_0,\mu_0) \ge H(\mu_0,\partial_\mu\psi(t_0,\mu_0))$.

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