Mathematische Annalen (2021) 379:1685–1712
https://doi.org/10.1007/s00208-019-01952-y

Mathematische Annalen

Martingale optimal transport duality

Patrick Cheridito[1] · Matti Kiiski[1] · David J. Prömel[2] · H. Mete Soner[3]

Received: 10 April 2019 / Revised: 17 October 2019 / Published online: 24 January 2020

Abstract

We obtain a dual representation of the Kantorovich functional defined for functions on the Skorokhod space using quotient sets. Our representation takes the form of a Choquet capacity generated by martingale measures satisfying additional constraints to ensure compatibility with the quotient sets. These sets contain stochastic integrals defined pathwise and two such definitions starting with simple integrands are given. Another important ingredient of our analysis is a regularized version of Jakubowski's S-topology on the Skorokhod space.

Mathematics Subject Classification 60B05 · 60G44 · 91B24 · 91G20

1 Introduction

Kantorovich duality [42,43] is an important tool in the classical theory of optimal transport [3,13,57]. Abstractly it provides a dual representation for a convex, lower

Communicated by Y. Giga.

Research was partly supported by the Swiss National Foundation Grant SNF 200020-172815.

✉ H. Mete Soner
soner@princeton.edu

Patrick Cheridito
patrick.cheridito@math.ethz.ch

Matti Kiiski
matti.kiiski@math.ethz.ch

David J. Prömel
proemel@maths.ox.ac.uk

1 Department of Mathematics, ETH Zurich, Rämistrasse 101, 8092 Zurich, Switzerland

2 Mathematical Institute, University of Oxford, Andrew Wiles Building, Woodstock Road, Oxford OX2 6GG, UK

3 Department of Operations Research and Financial Engineering, Princeton University, Princeton, NJ 08540, USA

semicontinuous functional Φ defined on a *locally convex Riesz space* $\mathcal{X}$, i.e., a locally convex lattice-ordered topological vector space. In the Kantorovich setting, $\mathcal{X}$ is a set of real-valued functions defined on a topological space Ω. Typical examples are the set of all bounded continuous functions $\mathcal{C}_b(\Omega)$ or the set of all bounded Borel measurable functions $\mathcal{B}_b(\Omega)$ with the supremum norm.

For a *quotient set* given by a convex cone $\mathcal{I}$, we consider the extended real-valued functional given by

$$\Phi(\xi;\mathcal{I}) := \inf\{c \in \mathbb{R} : c + \ell \geq \xi \text{ for some } \ell \in \mathcal{I}\}, \quad \xi \in \mathcal{X}.$$

There are several immediate properties of Φ. For instance, it follows directly from the definition that Φ is monotone and convex. Also, it is clear that for any constant c, one has $\Phi(c;\mathcal{I}) \leq c$ and $\Phi(\lambda\xi;\mathcal{I}) = \lambda\Phi(\xi;\mathcal{I})$ for every $\lambda \geq 0$. If additionally, one can establish that Φ is lower semicontinuous and proper (i.e., not identically equal to infinity and never equal to minus infinity), then one may apply the Fenchel-Moreau theorem [60, Theorem 2.4.14] to obtain the representation

$$\Phi(\xi;\mathcal{I}) = \sigma_{\partial\Phi}(\xi) := \sup_{\varphi\in\partial\Phi} \varphi(\xi), \quad \xi \in \mathcal{X}, \tag{1.1}$$

where the set of sub-gradients $\partial\Phi$ is the convex subset of the topological dual $\mathcal{X}^*$ of $\mathcal{X}$ given by $\partial\Phi = \partial\Phi(0;\mathcal{I}) := \{\varphi \in \mathcal{X}^* : \varphi(\xi) \leq \Phi(\xi;\mathcal{I}) \text{ for all } \xi \in \mathcal{X}\}$.

This formulation is similar to the one given in [13]. In addition to many other applications, it provides a natural framework for risk management [51,52]. Recently, it has also been used to reduce model dependency in pricing problems [8,33]. In these applications, Φ is the *super-replication functional* and $\mathcal{I}$ the *hedging set*. The main goal of this paper is to establish the dual representation (1.1) in the case where Ω is a suitable subset of the Skorokhod space taking also the trajectory of transportation into account.

In classical optimal transport, one has $\Omega = \mathbb{R}^d \times \mathbb{R}^d$ and the quotient set $\mathcal{I}_{ot}$ is defined through two given probability measures μ, ν on $\mathbb{R}^d$ by

$$\mathcal{I}_{ot} := \left\{ f \oplus h : f, h \in \mathcal{C}_b(\mathbb{R}^d) \text{ and } \mu(f) = \nu(h) = 0 \right\},$$

where $\mu(f) = \int f\, d\mu$, $\nu(h) = \int h\, d\nu$ and $(f \oplus h)(x, y) := f(x) + h(y)$. Let Φ_{ot} be the corresponding convex functional on $\mathcal{X} = \mathcal{C}_b(\Omega)$ with the supremum norm. Then, it is immediate that Φ_{ot} is proper and Lipschitz continuous. Moreover,

$$\partial\Phi_{ot} = \left\{ \varphi \in \mathcal{C}_b(\Omega)^* : \varphi \geq 0 \text{ and } \varphi(f \oplus h) = \mu(f) + \nu(h) \text{ for all } f, h \in \mathcal{C}_b(\mathbb{R}^d) \right\}.$$

Hence, any $\varphi \in \partial\Phi_{ot}$ is non-negative and has marginals μ and ν. It follows that φ is tight and therefore a Radon probability measure on Ω.

Alternatively, one could deduce the countable additivity of the dual elements by using the β_0-topology on $\mathcal{C}_b(\Omega)$ recalled in Appendix B below. If the topology on Ω is completely regular Hausdorff $\left(\mathrm{T}_{3\frac{1}{2}}\right)$, then the topological dual of $\mathcal{C}_b(\Omega)$ with the

β_0-topology is equal to the set of all signed Radon measures of finite total variation and the tightness argument is not needed. On the other hand, one then has to prove the continuity of Φ with respect to this topology. We use this observation in our study, which considers the problem on a more complex topological space Ω.

Kellerer [45] used Choquet's capability theorem [20] to show that the optimal transport duality also holds for measurable (and Suslin) functions if measurable functions are used in the definition of the quotient set. Similarly, for martingale or constrained optimal transport, one needs to enlarge the set $\mathcal{I}$ to achieve duality for more general functions with the same set of sub-gradients [9,29]. Alternatively, one could fix the quotient set $\mathcal{I}$ and obtain duality by extending the set of sub-gradients as it is done in [29]. We do not pursue this approach here.

In this paper, we study general martingale optimal transport on a subset Ω of the Skorokhod space $\mathcal{D}\left([0, T]; \mathbb{R}_+^d\right)$ of all $\mathbb{R}_+^d$-valued càdlàg functions, i.e., functions $\omega : [0, T] \mapsto \mathbb{R}_+^d$ that are continuous from the right and have finite left limits. We assume that Ω is a closed subset of $\mathcal{D}\left([0, T]; \mathbb{R}_+^d\right)$ with respect to Jakubowski's S-topology [39,40] and endow it with a regularized version of S. Our main goal is to prove duality with the same Choquet capacity defined by countably additive (martingale) measures, for different choices of $\mathcal{X}$ by appropriately extending the quotient set.

Martingale optimal transport was first introduced in a discrete time model in [8] and in continuous time in [33]. Since then it has been investigated extensively. The initial duality results [25–27,38] are proved by real-analytic techniques and only for uniformly continuous functions. Alternatively, [4–6,18,19] use functional analytical tools. In particular, [18] provides a general representation result. [19] proves duality in discrete time and [4–6] for a σ-compact set Ω. Our approach is similar to that of [5] but without the assumption of σ-compactness. Instead, we use the S-topology introduced by Jakubowski [39] which provides an efficient characterization of compact sets via up-crossings. This characterization allows us in Theorem 6.4 to construct an increasing sequence of compact sets K_n such that $\Phi(\mathbb{1}_{\Omega\setminus K_n})$ decreases to zero. This localization result is central to our approach. In a similar context, Jakubowski's S-topology was first used in [34–36] to prove several important properties of martingale optimal transport. Their set-up is related to [33] and differs from ours.

In martingale optimal transport, the quotient set $\mathcal{I}$ contains the "stochastic integrals". Since there is no a priori given probabilistic structure, the definition of the integral must be pathwise and is a delicate aspect of the problem. Starting from simple integrands, we first extend the integrands using the theory developed by Vovk [58,59], later by [49] and used in [7] to prove duality. This construction provides duality for upper semicontinuous functions. We then further enlarge the quotient set $\mathcal{I}$ by taking its Fatou-closure as defined in Sect. 2.3 and prove the duality for measurable functions by using Choquet's capacitability theorem as done earlier in [4–6,9]. These results are stated in Theorem 3.1. Section 8 provides examples showing the necessity of enlarging the set of integrands.

There are also deep connections between duality and the fundamental theorem of asset pricing (FTAP), which provides equivalent conditions for the dual set of measures to be non-empty. In the classical probabilistic setting, [37] proves it for the Black-Scholes model, [21] for discrete time and [22,23] in full generality. The robust discrete time model has first been studied in [1] and later in [15–17]. [10,14] on the other hand study probabilistic models with none or finitely many static options. We also obtain a general robust FTAP, Corollary 9.7, as an immediate consequence of our main duality result Theorem 3.1.

The paper is organized as follows. After providing the necessary structure and definitions in Sect. 2, we state the main result in Sect. 3. Important properties of the dual elements are proven in Sect. 4, and several approximation results are derived in Sect. 5. Section 6 analyses Φ on $\mathcal{C}_b(\Omega)$. The proof of the main result, Theorem 3.1, is given in Sect. 7. Several examples are constructed in Sect. 8. Section 9 discusses applications to model-free finance. The topological structures used in the paper and a sufficient condition for a probability measure to be a martingale measure are given in the Appendix.

2 Set-up

Let Ω be a non-empty subset of the Skorokhod space $\mathcal{D}\left([0,T];\mathbb{R}^d_+\right)$ of all càdlàg functions $\omega\colon [0,T] \to \mathbb{R}^d_+$ that is closed with respect to Jakubowski's S-topology [39,40]. We denote the relative topology of S on Ω again by S and, similarly to [46], endow Ω with the coarsest topology S^* making all S-continuous functions $\xi\colon \Omega \to \mathbb{R}$ continuous. More details on S and S^* are given in Appendix A, where it is shown that (Ω, S^*) is a perfectly normal Hausdorff space (T_6), and every Borel probability measure on (Ω, S^*) is automatically a Radon measure. Moreover, we know from [39,46] that for all $s < t$ and every $i = 1, \dots, d$, $\int_s^t \omega^i(u)\, du$ is continuous with respect to S, and

$$\|\omega\|_\infty := \sup_{0 \le t \le T} |\omega(t)|$$

is S-lower semicontinuous, where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^d$.

For $t \in [0,T]$, we denote by $X_t(\omega) = \omega(t)$ the coordinate map on Ω and let $\mathbb{F}^X = (\mathcal{F}^X_t)_{t\in[0,T]}$ be the natural filtration of X given by $\mathcal{F}^X_t = \sigma(X_s : s \le t)$. By $\mathbb{F} = (\mathcal{F}_t)$, we denote the right-continuous filtration given by $\mathcal{F}_t = \mathcal{F}^X_{t+} = \bigcap_{s>t} F^X_s$, $t < T$, and $\mathcal{F}_T = \mathcal{F}^X_T$. Adapted and predictable processes, as well as stopping times, are defined with respect to the filtration $\mathbb{F}$. In particular, for any open subset A of $\mathbb{R}^d_+$, the hitting time $\tau_A(\omega) = \inf\{t \ge 0 : X_t(\omega) \in A\}$ is a stopping time; see e.g. [24,50] for these facts. Moreover, arguments from [39,46] show that $\mathcal{F}_T = \mathcal{F}^X_T$ is equal to the collection of all Borel subsets of (Ω, S^*).

2.1 Riesz spaces

Let $\mathcal{B}(\Omega)$ be the set of all Borel measurable functions $\xi\colon \Omega \to [-\infty,\infty]$ and $\mathcal{B}_b(\Omega)$ the subset of bounded functions in $\mathcal{B}(\Omega)$. For $p \in [1,\infty)$, we define

$$\mathcal{B}_p(\Omega) := \left\{\xi \in \mathcal{B}(\Omega) : \omega \mapsto \xi(\omega)/(1+\|\omega\|^p_\infty) \text{ is bounded}\right\},$$

and

$$\mathcal{B}_0(\Omega) := \{\xi \in \mathcal{B}_b(\Omega) : \text{for all } \varepsilon > 0, \{\omega \in \Omega : |\xi(\omega)| > \varepsilon\} \text{ is relatively compact}\}.$$

By $\mathcal{U}_b(\Omega)$ and $\mathcal{U}_p(\Omega)$ we denote the sets of all upper semicontinuous functions in $\mathcal{B}_b(\Omega)$ and $\mathcal{B}_p(\Omega)$, respectively. $\mathcal{C}(\Omega)$ is the set of all real-valued continuous functions on Ω. $\mathcal{C}_b(\Omega)$ and $\mathcal{C}_p(\Omega)$ are defined analogously to $\mathcal{U}_b(\Omega)$ and $\mathcal{U}_p(\Omega)$. In addition, we need the set

$$\mathcal{C}_{q,p}(\Omega) := \left\{\xi \in \mathcal{C}(\Omega) : \xi^+ \in \mathcal{C}_q(\Omega),\ \xi^- \in \mathcal{C}_p(\Omega)\right\},$$

where $\xi^+ = \max(\xi, 0)$ and $\xi^- = \max(-\xi, 0)$.

By $\mathcal{M}(\Omega)$ we denote the set of all signed Radon measures of bounded total variation on Ω and by $\mathcal{P}(\Omega) \subset \mathcal{M}(\Omega)$ the subset of probability measures. For $Q \in \mathcal{P}(\Omega)$ and $\xi \in \mathcal{B}(\Omega)$, we define the expectation $\mathbb{E}_Q[\xi] \in [-\infty, \infty]$ by $\mathbb{E}_Q[\xi] := \mathbb{E}_Q[\xi^+] - \mathbb{E}_Q[\xi^-]$ with the convention $\infty - \infty = -\infty$. For $p \geq 1$, $\mathcal{L}^p(\Omega, Q)$ is the collection of all functions $\xi \in \mathcal{B}(\Omega)$ satisfying $\mathbb{E}_Q[|\xi|^p] < \infty$.

The β_0-topology on $C_b(\Omega)$ is generated by the semi-norms $\|.\,\eta\|_\infty$, $\eta \in \mathcal{B}_0^+(\Omega)$, where we use the superscript $^+$ to indicate the subset of non-negative elements. More details on the β_0-topology are given in Appendix B. Since (Ω, S^*) is a perfectly normal Hausdorff space, it is also completely regular, and it follows that the dual of $\mathcal{C}_b(\Omega)$ with the β_0-topology is $\mathcal{M}(\Omega)$; see e.g. [41,54].

2.2 The standing assumption

We fix *universal constants* $1 \leq p < q$. All our definitions and results depend on them, but we do not show this dependence in our notation.

Definition 2.1 For a convex cone $\mathcal{G} \subset \mathcal{B}(\Omega)$, we denote by $\mathcal{Q}(\mathcal{G})$ the (possibly empty) set of all probability measures $Q \in \mathcal{P}(\Omega)$ such that $\mathbb{E}_Q[\gamma] \leq 0$ for all $\gamma \in \mathcal{G}$ and the canonical map X is an $(\mathbb{F}, Q)$-martingale, i.e., for every $t \in [0, T]$, $X_t \in \mathcal{L}^1(\Omega, Q)$ and $\mathbb{E}_Q[Y \cdot (X_t - X_T)] = 0$ for all $\mathcal{F}_t$-measurable $Y \in \mathcal{B}_b(\Omega)^d$.

The following assumption is used throughout the paper. Although all results assume it, we do not always state this assumption explicitly.

Assumption 2.2 $\mathcal{G} \subset \mathcal{C}_{q,p}(\Omega)$ is a convex cone, and there exist $c_q \in \mathbb{R}_+$ and $\xi_q \in \mathcal{G}$ such that $|X_T|^q \leq c_q + \xi_q$.

Then, for every $Q \in \mathcal{Q}(\mathcal{G})$, $\mathbb{E}_Q\left[|X_T|^q\right] \leq c_q + \mathbb{E}_Q[\xi_q] \leq c_q$. We combine this with Doob's martingale inequality to conclude that

$$c_q^* := \sup_{Q \in \mathcal{Q}(\mathcal{G})} \mathbb{E}_Q\left[X_*^q\right] < \infty, \quad \text{where} \quad X_* := \sup_{t \in [0,T]} |X_t|. \tag{2.1}$$

In particular, $\mathbb{E}_Q[Y \cdot (X_t - X_T)] = 0$ for all $Q \in \mathcal{Q}(\mathcal{G})$, every $t \in [0, T]$ and any $\mathcal{F}_t$-measurable $Y \in \mathcal{B}_{q-1}(\Omega)^d$.

If, for a given $\mu \in \mathcal{P}(\mathbb{R}^d_+)$, $\mathcal{G}$ contains all functions $g(\omega(T)) - \mu(g)$ with $g \in \mathcal{C}_b(\mathbb{R}^d_+)$, then any element $Q \in \mathcal{Q}(\mathcal{G})$ has the marginal μ at the final time T. Hence, the above construction includes the classical example of given marginals. For this example, the celebrated result of Strassen [55] provides necessary and sufficient conditions for $\mathcal{Q}(\mathcal{G})$ to be non-empty; see also Corollary 9.7, below.

2.3 Integrals and quotient sets

A *simple integrand* H consists of a sequence of pairs $(\tau_n, h_n)_{n\in\mathbb{N}}$ such that $\tau_0 \le \tau_1 \le \tau_2 \le \cdots$ are $\mathbb{F}$-stopping times, and each $h_n \in \mathcal{B}_{q-1}(\Omega)^d$ is $\mathcal{F}_{\tau_n}$-measurable. We assume that for every $\omega \in \Omega$ there is an index $n(\omega)$ such that $\tau_{n(\omega)} \ge T$. The corresponding integral is defined directly as

$$(H \cdot X)_t(\omega) := \sum_{n=0}^{\infty} h_n(\omega) \cdot (X_{\tau_{n+1}\wedge t}(\omega) - X_{\tau_n \wedge t}(\omega)), \quad (t, \omega) \in [0, T] \times \Omega.$$

A simple integrand H is called *admissible* if for some $\lambda \in \mathcal{B}^+_q(\Omega)$

$$(H \cdot X)_{\tau_m \wedge t}(\omega) \ge -\lambda(\omega) \quad \text{for all } (t, \omega, m) \in [0, T] \times \Omega \times \mathbb{N}.$$

$\mathcal{H}_s$ denotes the set of all admissible simple integrands. An *admissible integrand* is a collection of simple integrands $H := (H^k)_{k\in\mathbb{N}} \subset \mathcal{H}_s$ satisfying $(H^k \cdot X)_t \ge -\Lambda$, for every $t \in [0, T]$, $k \in \mathbb{N}$, for some $\Lambda \in \mathcal{B}^+_q(\Omega)$. $\mathcal{H}$ denotes the set of all admissible integrands. The corresponding integral is defined pathwise by,

$$(H \cdot X)_t(\omega) := \liminf_{k\to\infty} (H^k \cdot X)_t(\omega) \quad \text{for all } (t, \omega) \in [0, T] \times \Omega.$$

We use the following quotient sets:

$$\mathcal{I}_s(\mathcal{G}) := \{\gamma + (H \cdot X)_T : \gamma \in \mathcal{G},\ H \in \mathcal{H}_s\},$$
$$\mathcal{I}(0) := \{(H \cdot X)_T : H \in \mathcal{H}\}, \quad \mathcal{I}(\mathcal{G}) := \{\gamma + (H \cdot X)_T : \gamma \in \mathcal{G},\ H \in \mathcal{H}\}.$$

Moreover, let $\widehat{\mathcal{I}}(\mathcal{G}) \subset \mathcal{B}(\Omega)$ be the Fatou-closure of $\mathcal{I}(\mathcal{G})$, i.e., the smallest set of extended real-valued Borel measurable functions containing $\mathcal{I}(\mathcal{G})$ with the property that for every sequence $\{\ell_n\}_{n\in\mathbb{N}} \subset \widehat{\mathcal{I}}(\mathcal{G})$ satisfying a uniform lower bound $\ell_n \ge -\lambda$ for some $\lambda \in \mathcal{B}^+_q$, $\liminf_n \ell_n \in \widehat{\mathcal{I}}(\mathcal{G})$. In the context of financial applications, similar integrals were first constructed in [58] and later used in [7,49,59]. Their properties have recently been studied in [48].

It is clear that $\mathcal{I}_s(\mathcal{G}) \subset \mathcal{I}(\mathcal{G}) \subset \widehat{\mathcal{I}}(\mathcal{G})$ and $\mathcal{I}(0)$ are all convex cones.

3 Main result

Theorem 3.1 *Under Assumption 2.2,*

$$\Phi(\xi;\mathcal{I}(\mathcal{G}))=\sigma_{\mathcal{Q}(\mathcal{G})}(\xi) \quad \text{for all } \xi\in\mathcal{U}_p(\Omega) \text{ and} \tag{3.1}$$

$$\Phi(\xi;\widehat{\mathcal{I}}(\mathcal{G}))=\sigma_{\mathcal{Q}(\mathcal{G})}(\xi) \quad \text{for all } \xi\in\mathcal{B}_p(\Omega), \tag{3.2}$$

where $\sigma_{\mathcal{Q}(\mathcal{G})}(\cdot):=\sup_{Q\in\mathcal{Q}(\mathcal{G})}\mathbb{E}_Q[\cdot]$ *is the support functional of* $\mathcal{Q}(\mathcal{G})$*.*

The proof is given in Sect. 7. If $\mathcal{Q}(\mathcal{G})$ is empty, by convention $\sigma_{\mathcal{Q}(\mathcal{G})}$ is identically equal to minus infinity, in which case both sides of the above equalities are equal to minus infinity; see Corollary 5.3. Counter-examples of Sect. 8 show that in general $\mathcal{I}_s(\mathcal{G})$ could be smaller than $\mathcal{I}(\mathcal{G})$ and (3.2) does not hold in general with $\mathcal{I}(\mathcal{G})$.

4 Properties of $\mathcal{Q}(\mathcal{G})$

Recall X_* in (2.1). If $\mathcal{Q}(\mathcal{G})$ is empty, all results of this section hold trivially. For $\xi\in\mathcal{B}(\Omega)$, we define for every constant $c\geq 0$,

$$\xi^c(\omega):=(c\wedge\xi(\omega))\vee(-c),\quad \omega\in\Omega. \tag{4.1}$$

Lemma 4.1 $\lim_{c\to\infty}\sigma_{\mathcal{Q}(\mathcal{G})}(\xi^c)=\sigma_{\mathcal{Q}(\mathcal{G})}(\xi)$ *for all* $\xi\in\mathcal{B}_p(\Omega)$*.*

Proof Fix $Q\in\mathcal{Q}(\mathcal{G})$ and $\xi\in\mathcal{B}_p(\Omega)$. There exists a constant $c_0>0$ so that $|\xi(\omega)|\leq c_0X_*^p(\omega)$ whenever $|\xi(\omega)|\geq c_0$. Using (2.1), we estimate that for $c\geq c_0$,

$$\begin{aligned}\mathbb{E}_Q\left[\left|\xi-\xi^c\right|\right]&\leq\mathbb{E}_Q\left[|\xi|\mathbb{1}_{\{|\xi|\geq c\}}\right]\leq c_0\mathbb{E}_Q\left[X_*^p\mathbb{1}_{\{X_*^p\geq c/c_0\}}\right]\\&\leq c_0\frac{\mathbb{E}_Q\left[X_*^q\mathbb{1}_{\{X_*^p\geq c/c_0\}}\right]}{(c/c_0)^{q/p-1}}\leq\frac{c_0^{q/p}c_q^*}{c^{q/p-1}}.\end{aligned}$$

Hence, by sub-additivity,

$$\left|\sigma_{\mathcal{Q}(\mathcal{G})}(\xi)-\sigma_{\mathcal{Q}(\mathcal{G})}(\xi^c)\right|\leq\sigma_{\mathcal{Q}(\mathcal{G})}\left(\left|\xi-\xi^c\right|\right)\leq\sup_{Q\in\mathcal{Q}(\mathcal{G})}\mathbb{E}_Q\left[\left|\xi-\xi^c\right|\right]\leq\frac{c_0^{q/p}c_q^*}{c^{q/p-1}}.$$

□

Lemma 4.2 *For every* $H\in\mathcal{H}$*,* $t\in[0,T]$*, and* $Q\in\mathcal{Q}(\mathcal{G})$*,* $\mathbb{E}_Q[(H\cdot X)_t]\leq 0$*. Consequently,* $\mathbb{E}_Q[\ell]\leq 0$ *for every* $\ell\in\mathcal{I}(\mathcal{G})$ *and* $Q\in\mathcal{Q}(\mathcal{G})$*.*

Proof Fix $Q\in\mathcal{Q}(\mathcal{G})$ and $H=(\tau_n,h_n)_{n\in\mathbb{N}}\in\mathcal{H}_s$. For $m\geq 1$, set

$$\ell_t^m:=(H\cdot X)_{\tau_m\wedge t}=\sum_{n=0}^{m-1}h_n\cdot(X_{\tau_{n+1}\wedge t}-X_{\tau_n\wedge t}),\quad t\in[0,T].$$

Since by definition each $h_n \in \mathcal{B}_{q-1}(\Omega)^d$ and X is an $(\mathbb{F}, Q)$-martingale, we have $E_Q[\ell_t^m] = 0$. By the admissibility of H, there exists $\lambda \in \mathcal{B}_q^+(\Omega)$ such that $\ell_t^m \geq -\lambda$ for each m and $t \in [0, T]$. Therefore, by Fatou's Lemma and (2.1),

$$\mathbb{E}_Q\left[(H \cdot X)_t\right] \leq \liminf_{m\to\infty} \mathbb{E}_Q\left[\ell_t^m\right] = 0 \quad \text{for all } t \in [0, T].$$

Let $H = (H^k)_{k\in\mathbb{N}} \in \mathcal{H}$. Then, by definition each $H^k \in \mathcal{H}_s$ and by the above result $\mathbb{E}_Q\left[(H^k \cdot X)_t\right] \leq 0$. Again by admissibility, there exists $\Lambda \in \mathcal{B}_q^+(\Omega)$ so that $(H^k \cdot X)_t \geq -\Lambda$ for each $k \geq 1$, $t \in [0, T]$. By Fatou's Lemma, for $t \in [0, T]$,

$$\mathbb{E}_Q\left[(H \cdot X)_t\right] = \mathbb{E}_Q\left[\liminf_{k\to\infty}(H^k \cdot X)_t\right] \leq \liminf_{k\to\infty} \mathbb{E}_Q\left[(H^k \cdot X)_t\right] \leq 0.$$

The final statement follows directly from the definitions. □

Lemma 4.3 *For every $\ell \in \widehat{\mathcal{I}}(\mathcal{G})$ and $Q \in \mathcal{Q}(\mathcal{G})$, $\mathbb{E}_Q[\ell] \leq 0$. Therefore,*

$$\sigma_{\mathcal{Q}(\mathcal{G})}(\xi) \leq \Phi(\xi; \widehat{\mathcal{I}}(\mathcal{G})) \leq \Phi(\xi; \mathcal{I}(\mathcal{G})) \quad \textit{for all } \xi \in \mathcal{B}(\Omega). \tag{4.2}$$

Proof Set $\mathcal{K}(\mathcal{G}) := \left\{\xi \in \mathcal{B}(\Omega) : \sigma_{\mathcal{Q}(\mathcal{G})}(\xi) \leq 0\right\}$. By Lemma 4.2, $\mathcal{I}(\mathcal{G}) \subset \mathcal{K}(\mathcal{G})$. Consider a sequence $\{\xi_n\}_{n\in\mathbb{N}} \subset \mathcal{K}(\mathcal{G})$ satisfying a uniform lower bound $\xi_n \geq -\lambda$ for some $\lambda \in \mathcal{B}_q^+(\Omega)$. Then, by Fatou's Lemma and the uniform bound (2.1),

$$\mathbb{E}_Q\left[\liminf_{n\to\infty} \xi_n\right] \leq \liminf_{n\to\infty} \mathbb{E}_Q[\xi_n] \leq 0 \quad \text{for all } Q \in \mathcal{Q}(\mathcal{G}).$$

Hence, $\liminf_n \xi_n \in \mathcal{K}(\mathcal{G})$. Since $\widehat{\mathcal{I}}(\mathcal{G})$, by its definition, is the smallest set of measurable functions with this property containing $\mathcal{I}(\mathcal{G})$, we conclude that $\widehat{\mathcal{I}}(\mathcal{G}) \subset \mathcal{K}(\mathcal{G})$.

Fix $\xi \in \mathcal{B}(\Omega)$. Suppose that $\xi \leq c + \ell$ for some $c \in \mathbb{R}$ and $\ell \in \widehat{\mathcal{I}}(\mathcal{G})$. Since $\widehat{\mathcal{I}}(\mathcal{G}) \subset \mathcal{K}(\mathcal{G})$, $\mathbb{E}_Q[\xi] \leq \mathbb{E}_Q[c + \ell] \leq c$ for every $Q \in \mathcal{Q}(\mathcal{G})$. Hence, $\sigma_{\mathcal{Q}(\mathcal{G})}(\xi) \leq c$. Since $\Phi(\xi; \widehat{\mathcal{I}}(\mathcal{G}))$ is the infimum of all such constants, $\sigma_{\mathcal{Q}(\mathcal{G})}(\xi) \leq \Phi(\xi; \widehat{\mathcal{I}}(\mathcal{G}))$. The fact $\mathcal{I}(\mathcal{G}) \subset \widehat{\mathcal{I}}(\mathcal{G})$ implies that $\Phi(\xi; \widehat{\mathcal{I}}(\mathcal{G})) \leq \Phi(\xi; \mathcal{I}(\mathcal{G}))$. □

5 Approximation results

Lemma 5.1 $\hat{c}_q^* := \Phi(X_*^q; \mathcal{I}(\mathcal{G})) < \infty$.

Proof For $N \in \mathbb{N}$, $\{y_k\}_{k=0}^N \subset \mathbb{R}_+$ and $n \leq N$, let $y_n^* := \max_{0\leq k\leq n} y_k$.

Step 1. It is shown in [2, Proposition 2.1] that

$$\left(y_N^*\right)^q + d_q y_0^q \leq \sum_{n=0}^{N-1} h\left(y_n^*\right)(y_{n+1} - y_n) + \left(d_q y_N\right)^q,$$

where $d_q := q/(q-1)$ and $h(y) := -q d_q y^{q-1}$ for $y \in \mathbb{R}_+$.

Step 2. Set $\tau_0 := 0$ and for each $\omega \in \Omega$ and $n \in \mathbb{N}$ define recursively

$$\tau_n(\omega) := \inf\left\{t > \tau_{n-1}(\omega) : |X_t(\omega)|^q > |X_{\tau_{n-1}}(\omega)|^q + 1\right\} \wedge T.$$

Then, the τ_n's are stopping times. For $\omega \in \Omega$, $i = 1, \ldots, d$, $n = 0, 1, 2, \ldots$, set

$$h_n^{*,i}(\omega) := h\left(\max_{0 \le k \le n} X_{\tau_k}^i(\omega)\right), \quad h_n^*(\omega) := \left(h_n^{*,1}(\omega), \ldots, h_n^{*,d}(\omega)\right).$$

It is clear that $h_n^* \in \mathcal{B}_{q-1}(\Omega)^d$ and therefore $H^* := (\tau_n, h_n^*)_{n \in \mathbb{N}}$ is a simple integrand.

Step 3. We claim that H^* is admissible. Indeed, fix $t \in [0, T]$, $\omega \in \Omega$, $i = 1, \ldots, d$, and set $y_n := X^i_{\tau_n \wedge t}(\omega)$. For $k \in \mathbb{N}$, set

$$\tilde{n} = \tilde{n}(\omega, t, k) := \sup\{m : \tau_m(\omega) \le t\} \wedge (k-1).$$

Then, for $n \le \tilde{n}$, $y_n = X^i_{\tau_n}$ and therefore, $h_n^{*,i}(\omega) = h(y_n^*)$. For $\tilde{n} < n < k$, $X^i_{\tau_{n+1} \wedge t} = X^i_{\tau_n \wedge t} = X^i_t$ and $y_{\tilde{n}+1} = X^i_{\tau_k \wedge t}$. By Step 1,

$$\begin{aligned}
\sum_{n=0}^{k-1} h_n^{*,i}\left(X^i_{\tau_{n+1}\wedge t} - X^i_{\tau_n \wedge t}\right) &= \sum_{n=0}^{\tilde{n}} h\left(y_n^*\right)(y_{n+1} - y_n) \ge \left(y^*_{\tilde{n}+1}\right)^q - (d_q y_{\tilde{n}+1})^q \\
&= \sup_{n \le k}\left(X^i_{\tau_n \wedge t}\right)^q - \left(d_q X^i_{\tau_k \wedge t}\right)^q.
\end{aligned}$$

Hence, for every $t \in [0, T]$ and integer k,

$$\begin{aligned}
(H^* \cdot X)_{\tau_k \wedge t} &\ge \sum_{i \le d} \sup_{n \le k}\left(X^i_{\tau_n \wedge t}\right)^q - \sum_{i \le d}\left(d_q X^i_{\tau_k \wedge t}\right)^q \\
&\ge \sum_{i \le d} \sup_{n \le k}\left(X^i_{\tau_n \wedge t}\right)^q - c^* \left|X_{\tau_k \wedge t}\right|^q \ge -c^* X_*^q, \qquad (5.1)
\end{aligned}$$

for some constant c^* depending only on d and q. Hence, H^* is admissible.

Step 4. We let $t = T$ in (5.1) and send k to infinity to obtain

$$\sum_{i \le d} \sup_n \left(X^i_{\tau_n \wedge T}\right)^q \le (H^* \cdot X)_T + c^* \, |X_T|^q.$$

Choose a constant $\hat{c}^*$ so that for all $y = (y^1, \ldots, y^d) \in \mathbb{R}^d_+$, $|y|^q \le \hat{c}^* \sum_{i \le d} |y^i|^q$. Let c_q, ξ_q be as in Assumption 2.2. Then,

$$\begin{aligned}
0 \le \sup_n \left|X_{\tau_n \wedge T}\right|^q &\le \hat{c}^* \sum_{i \le d} \sup_n \left(X^i_{\tau_n \wedge T}\right)^q \le \hat{c}^* \left[(H^* \cdot X)_T + c^* \, |X_T|^q\right] \\
&\le (\hat{c}^* H^* \cdot X)_T + \hat{c}^* c^* (c_q + \xi_q) =: \ell^* + \hat{c}^* c^* c_q.
\end{aligned}$$

Since $H^* \in \mathcal{H}_s$, $\xi_q \in \mathcal{G}$ and $\mathcal{G}$ is a cone, $\ell^* := (\hat{c}^* H^* \cdot X)_T + \hat{c}^* c^* \xi_q \in \mathcal{I}(\mathcal{G})$.

Step 5. By definition of the τ_n's,

$$X_*^q \leq \sup_n \left|X_{\tau_n \wedge T}\right|^q + 1 \leq \ell^* + \hat{c}^* c^* c_q + 1,$$

from which one obtains $\Phi(X_*^q; \mathcal{I}(\mathcal{G})) \leq \hat{c}^* c^* c_q + 1 < \infty$. □

Corollary 5.2 *For any convex cone* $\mathcal{I} \supset \mathcal{I}(\mathcal{G})$ *and* $\xi \in \mathcal{B}_p(\Omega)$*, one has* $\lim_{c\to\infty} \Phi(\xi^c; \mathcal{I}) = \Phi(\xi; \mathcal{I})$.

Proof Fix $\xi \in \mathcal{B}_p(\Omega)$. There exists $c_0 > 0$ so that $|\xi| \leq c_0 X_*^p$ whenever $|\xi| \geq c_0$.

Step 1. For $c \geq c_0$,

$$(|\xi| - c)\mathbb{1}_{\{|\xi|\geq c\}} \leq c_0 X_*^p\, \mathbb{1}_{\{X_*^p \geq c/c_0\}} \leq \frac{c_0 X_*^q}{(c/c_0)^{q/p-1}} \mathbb{1}_{\{X_*^p \geq c/c_0\}} \leq \frac{c_0^{q/p}}{c^{q/p-1}} X_*^q.$$

Since $\mathcal{I}$ includes $\mathcal{I}(\mathcal{G})$, $\Phi((|\xi| - c)\mathbb{1}_{\{|\xi|\geq c\}}; \mathcal{I}) \leq c_0^{q/p} c^{1-q/p} \Phi(X_*^q; \mathcal{I}(\mathcal{G}))$, which in view of Lemma 5.1, gives $\limsup_{c\to\infty} \Phi((|\xi| - c)\mathbb{1}_{\{|\xi|\geq c\}}; \mathcal{I}) \leq 0$.

Step 2. Since $|\xi - \xi^c| \leq (|\xi| - c)\mathbb{1}_{\{|\xi|\geq c\}}$, one obtains from sub-additivity,

$$\Phi(\xi^c; \mathcal{I}) \leq \Phi(\xi^c - \xi; \mathcal{I}) + \Phi(\xi; \mathcal{I}) \leq \Phi((|\xi| - c)\mathbb{1}_{\{|\xi|\geq c\}}; \mathcal{I}) + \Phi(\xi; \mathcal{I}),$$

which by the previous step, gives $\limsup_{c\to\infty} \Phi(\xi^c; \mathcal{I}) \leq \Phi(\xi; \mathcal{I})$.

Step 3. Similarly,

$$\Phi(\xi; \mathcal{I}) \leq \Phi(\xi - \xi^c; \mathcal{I}) + \Phi(\xi^c; \mathcal{I}) \leq \Phi((|\xi| - c)\mathbb{1}_{\{|\xi|\geq c\}}; \mathcal{I}) + \Phi(\xi^c; \mathcal{I}),$$

and therefore, $\Phi(\xi; \mathcal{I}) \leq \liminf_{c\to\infty} \Phi(\xi^c; \mathcal{I})$. □

It is a direct consequence of the definition that $\Phi(\xi; \mathcal{I}(\mathcal{G})) \leq \|\xi\|_\infty$ for any $\xi \in \mathcal{B}_b(\Omega)$. In particular, $\Phi(0; \mathcal{I}(\mathcal{G})) \leq 0$.

Corollary 5.3 *We have the following alternatives:*

(i) *If* $\Phi(0; \mathcal{I}(\mathcal{G})) = 0$*, then* $|\Phi(\xi; \mathcal{I}(\mathcal{G}))| \leq \|\xi\|_\infty$ *for all* $\xi \in \mathcal{B}_b(\Omega)$.
(ii) *If* $\Phi(0; \mathcal{I}(\mathcal{G})) < 0$*, then* $\mathcal{Q}(\mathcal{G})$ *is empty, and* $\Phi(\cdot; \mathcal{I}(\mathcal{G})) \equiv \Phi(\cdot; \widehat{\mathcal{I}}(\mathcal{G})) \equiv -\infty$ *on* $\mathcal{B}_p(\Omega)$*. In particular,* (3.1) *and* (3.2) *hold trivially.*

Proof First, suppose that $\Phi(0; \mathcal{I}(\mathcal{G})) = 0$ and let $\xi \in \mathcal{B}_b(\Omega)$. Since $\xi + \|\xi\|_\infty \geq 0$, one has

$$\Phi(\xi; \mathcal{I}(\mathcal{G})) = -\|\xi\|_\infty + \Phi(\xi + \|\xi\|_\infty; \mathcal{I}(\mathcal{G})) \geq -\|\xi\|_\infty + \Phi(0; \mathcal{I}(\mathcal{G})) = -\|\xi\|_\infty.$$

Now assume that $\Phi(0; \mathcal{I}(\mathcal{G})) < 0$. Then, there exist $c < 0$, $\ell \in \mathcal{I}(\mathcal{G})$ such that $c + \ell \geq 0$. Also, for any constant $\lambda > 0$, $\lambda(c + \ell) \geq 0$. Since $\mathcal{I}(\mathcal{G})$ is a cone,

$\lambda\ell \in \mathcal{I}(\mathcal{G})$ and consequently, $\Phi(0; \mathcal{I}(\mathcal{G})) \leq c\lambda$. As $\lambda > 0$ above was arbitrary, we have $\Phi(0; \mathcal{I}(\mathcal{G})) = -\infty$ and

$$\Phi(\xi; \mathcal{I}(\mathcal{G})) \leq \|\xi\|_\infty + \Phi(\xi - \|\xi\|_\infty; \mathcal{I}(\mathcal{G})) \leq \|\xi\|_\infty + \Phi(0; \mathcal{I}(\mathcal{G})) = -\infty.$$

This shows that $-\infty \leq \sigma_{\mathcal{Q}(\mathcal{G})}(\cdot) \leq \Phi(\cdot; \widehat{\mathcal{I}}(\mathcal{G})) \leq \Phi(\cdot; \mathcal{I}(\mathcal{G})) \equiv -\infty$ on $\mathcal{B}_b(\Omega)$, and by Corollary 5.2, also on $\mathcal{B}_p(\Omega)$.

Moreover, (4.2) implies that if $\mathcal{Q}(\mathcal{G})$ is non-empty, $\Phi(0; \widehat{\mathcal{I}}(\mathcal{G})) = 0$. Hence if $\Phi(0; \widehat{\mathcal{I}}(\mathcal{G})) < 0$, $\mathcal{Q}(\mathcal{G})$ must be empty. □

For an $\mathbb{R}^d$-valued càdlàg process Y, set

$$\ell_Y(\omega) := \int_0^T Y_u(\omega) \cdot (X_u(\omega) - X_T(\omega))\, du.$$

Lemma 5.4 *Let Y be an $\mathbb{R}^d$-valued, adapted, càdlàg process. Suppose that there exists $\lambda \in \mathcal{B}_{q-1}(\Omega)$ satisfying $|Y_u| \leq \lambda$ for every $u \in [0, T]$. Then, $\ell_Y \in \mathcal{I}(0)$ and for any quotient set $\mathcal{I}$ containing $\mathcal{I}(0)$, $\Phi(\ell_Y; \mathcal{I}) \leq 0$*

Proof For $k \in \mathbb{N}$ and $n = 0, \ldots, k$ set $\tau_n^k := nT/k$, $Y_n^k := Y_{\tau_n^k}$, $X_n^k := X_{\tau_n^k}$, $h_0^k := -(T/k)Y_0$ and $h_n^k := h_{n-1}^k - (T/k)Y_n^k$ for $n \geq 1$. Since $\lambda \in \mathcal{B}_{q-1}(\Omega)$, the simple integrand $H^k := (\tau_n^k, h_n^k)_{n=0}^k$ is admissible. Moreover,

$$(H^k \cdot X)_T = \sum_{n=0}^{k-1} h_n^k \cdot \left(X_{n+1}^k - X_n^k\right) = \frac{T}{k} \sum_{n=0}^{k-1} Y_n^k \cdot \left(X_n^k - X_T\right).$$

Let $H := (H^k)_{k\in\mathbb{N}}$. Since both Y and X are càdlàg,

$$(H \cdot X)_T = \lim_{k\to\infty} (H^k \cdot X)_T = \ell_Y.$$

One can directly verify that $H \in \mathcal{H}$. Hence, $\ell_Y \in \mathcal{I}(0)$. □

6 Continuity on $\mathcal{C}_b(\Omega)$

We use the compact notation $\Phi(\cdot) = \Phi(\cdot; \mathcal{I}(\mathcal{G}))$.

Lemma 6.1 $\limsup_{c\to\infty} \Phi(\mathbb{1}_{\{X_*>c\}}) \leq 0$.

Proof Fix $c > 0$, $i \in \{1, \ldots, d\}$ and set $X_*^i := \sup_{t\in[0,T]} X_t^i$. Since $X_T^i \geq 0$,

$$c\mathbb{1}_{\{X_*^i>c\}}(\omega) \leq X_T^i(\omega) + (c - X_T^i(\omega))\mathbb{1}_{\{X_*^i>c\}}(\omega) \quad \text{for all } \omega \in \Omega.$$

Set $h_1^i = -1$, $h_1^j = 0$ for $j \neq i$, $\tau_0(\omega) := \inf\{t \geq 0 : X_t^i \in (c, \infty)\} \wedge T$, $\tau_1 := T$ and let H be the corresponding integrand. Then, $(H \cdot X)_T = (X_{\tau_1}^i - X_T^i)$. By right-continuity, we have $X_{\tau_1}^i \geq c$ on the set $\{X_*^i > c\}$ and $\tau_1 = T$ on its complement.

Consequently, $(H \cdot X)_T \geq (c - X^i_T)\mathbb{1}_{\{X^i_*>c\}}$ and $\Phi\left((c - X^i_T)\,\mathbb{1}_{\{X^i_*>c\}}\right) \leq 0$. Therefore, $\Phi(\mathbb{1}_{\{X^i_*>c\}}) \leq \ \Phi\left(X^i_T\right)/c$, which, by Lemma 5.1, converges to zero as c tends to infinity. Since $\{X_* > \sqrt{d}c\} \subset \cup_i \{X^i_* > c\}$, the claim of the lemma follows from the sub-additivity of Φ. □

Definition 6.2 For $\omega \in \mathcal{D}([0,T];\mathbb{R}_+)$, $t \in [0,T]$, and $a < b$, the number of up-crossings up to t, $U^{a,b}_t(\omega)$, is the largest integer n for which one can find $0 \leq t_1 < \cdots < t_{2n} \leq t$ such that $\omega(t_{2k-1}) < a$ and $\omega(t_{2k}) > b$ for $k = 1, \ldots, n$.

For $\omega \in \mathcal{D}([0,T];\mathbb{R}^d_+)$, we set $U^{a,b,i}_t(\omega) := U^{a,b}_t(\omega^i)$.

Lemma 6.3 *For $0 < a < b$ and $i = 1, \ldots, d$, there exists $H^{a,b,i} \in \mathcal{H}_s$ such that*

$$(H^{a,b,i} \cdot X)_t(\omega) \geq -a + (b-a)U^{a,b,i}_t(\omega) \quad \textit{for all } (t,\omega) \in [0,T] \times \Omega.$$

Proof For $k \geq 1$, set $I_k := [0,a)$ if k is an odd integer and $I_k := (b,\infty)$ if k is even, and $\tau_0 := 0$. Recursively define a sequence of random times by

$$\tau_k(\omega) := \inf\left\{t \geq \tau_{k-1}(\omega) \ : \ X^i_t(\omega) \in I_k\right\} \wedge T,$$

where the infimum over an empty set is infinity. Since X is càdlàg and I_k is open, τ_k's are $\mathbb{F}$-stopping times. Define $h_k = (h^1_k, \ldots, h^d_k)$ as follows: $h^i_k := 1$ when k is odd, $h^i_k := 0$ for k even and $h^j_k = 0$ for $j \neq i$. Let $H^{a,b,i}$ be the corresponding simple integrand. It is clear that for every $t \in [0,T]$, $\omega \in \Omega$, $(H^{a,b,i} \cdot X)_t(\omega) \geq -a + (b-a)U^{a,b,i}_t(\omega)$. Hence, $H^{a,b,i} \in \mathcal{H}_s$. □

6.1 Localization

Theorem 6.4 *There exists an increasing sequence of compact subsets $\{K_n\}_{n\in\mathbb{N}}$ of Ω satisfying,*

$$\lim_{n\to\infty} \Phi(\mathbb{1}_{\Omega\setminus K_n}) \leq 0.$$

Proof We complete the proof in several steps.

Step 1. Let D be a countable dense subset of $(0,\infty)$ and $\{(a_j,b_j) : j \in \mathbb{N}\}$ an enumeration of the countable set $\{(a,b) \in D \times D : 0 < a < b\}$. For all $n \in \mathbb{N}$, define

$$K^{i,j}_n := \left\{\omega \in \Omega : U^{a_j,b_j,i}_T(\omega) \leq c^j_n\right\}, \quad \widehat{K}_n := \bigcap_{i=1}^{d} \bigcap_{j\in\mathbb{N}} K^{i,j}_n, \quad K_n := B_n \cap \widehat{K}_n,$$

where $c^n_j := 2^{j+n}(a_j \vee 1)/(b_j - a_j)$ and $B_n := \{\omega \in \Omega : X_*(\omega) \leq n\}$. Since Ω is S-closed, one obtains from [39, Corollary 2.10] that $K^{i,j}_n$ and B_n are S-closed subsets of $\mathcal{D}([0,T];\mathbb{R}^d_+)$. Hence, all K_n are S-compact and therefore also S^*-compact subsets of Ω; see Appendix A or [46, Corollary 5.11]. Moreover,

$$(\Omega\backslash K_n) \subset O_n \cup (\Omega\backslash B_n), \quad \text{where} \quad O_n := \bigcup_{i,j} \left(\Omega\backslash K_n^{i,j}\right).$$

Step 2. Let $H^{a,b,i}$ be as in Lemma 6.3 and set $H_n^{i,j} := (c_j^n(b_j - a_j))^{-1} H^{a_j,b_j,i}$. Then, for every $t \in [0,T]$,

$$(H_n^{i,j} \cdot X)_t \geq -\frac{a^j}{c_j^n(b_j - a_j)} + \frac{U_t^{a_j,b_j,i}}{c_j^n} \geq -2^{-(j+n)} + \frac{U_t^{a_j,b_j,i}}{c_j^n}.$$

Hence, $H_n^{i,j} \in \mathcal{H}_s$ and also $(H_n^{i,j} \cdot X)_T \geq -2^{-(j+n)} + \mathbb{1}_{\Omega\backslash K_n^{i,j}}$.

For $k \geq 1$, set $H_n^k := \sum_{i=1}^d \sum_{j=1}^k H_n^{i,j}$. Then, for every $k \geq 1$ and $t \in [0,T]$, $(H_n^k \cdot X)_t \geq -d\, 2^{-n}$. Hence, for each n, $H_n := (H_n^k)_{k\in\mathbb{N}} \in \mathcal{H}$ and

$$(H_n \cdot X)_T = \liminf_{k\to\infty} \left(H_n^k \cdot X\right)_T \geq \sum_{i=1}^d \sum_{j=1}^\infty \left(\mathbb{1}_{\Omega\backslash K_n^{i,j}} - 2^{-(j+n)}\right) \geq \mathbb{1}_{O_n} - d\, 2^{-n}.$$

Therefore, $\Phi(\mathbb{1}_{O_n}) \leq d\, 2^{-n}$.

Step 3. By the previous steps and Lemma 6.1,

$$\limsup_{n\to\infty} \Phi(\mathbb{1}_{\Omega\backslash K_n}) \leq \limsup_{n\to\infty} \left(\Phi(\mathbb{1}_{\Omega\backslash B_n}) + \Phi(\mathbb{1}_{O_n})\right) \leq 0.$$

Finally, since for each pair (i,j), the sets $K_n^{i,j}$ are increasing in n, we conclude that K_n is also increasing in n. □

6.2 β_0-continuity

Proposition 6.5 *Suppose that $\Phi(0) = 0$. Then Φ is real-valued and β_0-continuous on $\mathcal{C}_b(\Omega)$.*

Proof By Corollary 5.3, Φ is real-valued and the compact sets constructed in Theorem 6.4 satisfy $\Phi(\mathbb{1}_{\Omega\backslash K_n}) \downarrow 0$ as n tends to infinity. Let K_0 be the empty set and by relabeling, we may assume that $\Phi(\mathbb{1}_{\Omega\backslash K_k}) \leq 2^{-2k}$, for every $k \geq 0$. Define

$$\eta^* := \sum_{k=1}^\infty 2^{-k} \mathbb{1}_{K_k\backslash K_{k-1}}.$$

Since on the complement of K_{k-1}, $|\eta^*| \le 2^{-k}$, $\eta^* \in \mathcal{B}_0(\Omega)$. Fix an integer n and $\xi \in \mathcal{C}_b(\Omega)$. Since on $K_k \backslash K_{k-1}$, $\eta^* = 2^{-k}$, on $K_k \backslash K_{k-1}$, $(\eta^*)^{-1} = 2^k$, so, on $K_n = \cup_{k=1}^n (K_k \backslash K_{k-1})$,

$$|\xi| \mathbb{1}_{K_n} = |\xi| \eta^* \, (\eta^*)^{-1} \mathbb{1}_{K_n} \le \|\xi \eta^*\|_\infty \, (\eta^*)^{-1} \mathbb{1}_{K_n} = \|\xi \eta^*\|_\infty \sum_{k=1}^n 2^k \, \mathbb{1}_{K_k \backslash K_{k-1}}.$$

In view of the hypothesis $\Phi(0) = 0$, $|\Phi(\xi \mathbb{1}_{K_n})| \le \Phi(|\xi| \mathbb{1}_{K_n})$ and consequently,

$$\begin{aligned}\left|\Phi\left(\xi \mathbb{1}_{K_n}\right)\right| &\le \|\xi \eta^*\|_\infty \sum_{k=1}^n 2^k \Phi\left(\mathbb{1}_{K_k \backslash K_{k-1}}\right) \le \|\xi \eta^*\|_\infty \sum_{k=1}^n 2^k \Phi\left(\mathbb{1}_{\Omega \backslash K_{k-1}}\right) \\ &\le \|\xi \eta^*\|_\infty \sum_{k=1}^n 2^k 2^{-2(k-1)} \le 4 \|\xi \eta^*\|_\infty = 4 \|\xi\|_{\eta^*}.\end{aligned}$$

Therefore, by Theorem 6.4,

$$|\Phi(\xi)| \le \limsup_{n \to \infty} \left(|\Phi(\xi \mathbb{1}_{K_n})| + \|\xi\|_\infty \Phi(\mathbb{1}_{\Omega \backslash K_n})\right) \le 4 \|\xi \eta^*\|_\infty.$$

For $\xi, \zeta \in \mathcal{C}_b(\Omega)$, by sub-additivity, $\Phi(\xi) = \Phi((\xi - \zeta) + \zeta) \le \Phi(\xi - \zeta) + \Phi(\zeta)$. Hence, $\Phi(\xi) - \Phi(\zeta) \le \Phi(\xi - \zeta) \le 4 \|(\xi - \zeta) \, \eta^*\|_\infty$. Switching the roles of ξ and ζ, we conclude that $|\Phi(\xi) - \Phi(\zeta)| \le 4 \|(\xi - \zeta) \, \eta^*\|_\infty$. Since the β_0-topology is generated by the semi-norms $\| \, . \, \eta\|_\infty$ for arbitrary $\eta \in \mathcal{B}_0^+(\Omega)$ and $\eta^* \in \mathcal{B}_0^+(\Omega)$, the above inequality yields that Φ is β_0-continuous on $\mathcal{C}_b(\Omega)$ (see Appendix B below).

□

6.3 Sub-differential

Proposition 6.6 $\mathcal{Q}(\mathcal{G}) = \partial \Phi := \{\varphi \in \mathcal{M}(\Omega) : \varphi(\xi) \le \Phi(\xi; \mathcal{I}(\mathcal{G})), \ \xi \in \mathcal{C}_b(\Omega)\}$.

Proof The lower bound (4.2) implies that $\mathcal{Q}(\mathcal{G}) \subset \partial \Phi$. To prove the opposite inclusion, fix $Q \in \partial \Phi \subset \mathcal{M}(\Omega)$. The monotonicity of Φ implies that $Q \ge 0$. Since $\Phi(c) \le c$ for every constant c, we conclude that $Q \in \mathcal{P}(\Omega)$.

Step 1. Let $\xi \in \mathcal{C}^+(\Omega)$, and define ξ^c for $c \ge 0$ as in (4.1). Then, $\xi^c \le \xi$ and by the defining property of Q, $\mathbb{E}_Q[\xi^c] \le \Phi(\xi^c) \le \Phi(\xi)$. So, by monotone convergence, $\mathbb{E}_Q[\xi] = \lim_{c \to \infty} \mathbb{E}_Q[\xi^c] \le \Phi(\xi)$.

Step 2. For $\varepsilon > 0$, set

$$X_t^\varepsilon(\omega) := \frac{1}{\varepsilon} \int_t^{t+\varepsilon} X_{u \wedge T} \, du, \quad t \in [0, T].$$

Since the map X_T and time integrals are S-continuous [39], X^ε is S^*-continuous. Hence, for every $t \in [0, T]$, $|X_t^\varepsilon| \in \mathcal{C}_1(\Omega)$ and $\left|X_t^\varepsilon\right| \le X_*$, where X_* is as in (2.1).

Also, $\lim_{\varepsilon\to 0} X_t^\varepsilon(\omega) = X_t(\omega)$ for every $\omega \in \Omega$. Fix $t \in [0,T]$ and choose $\xi = |X_t^\varepsilon|^q$ in Step 1 to obtain, $\mathbb{E}_Q[|X_t^\varepsilon|^q] \le \Phi(|X_t^\varepsilon|^q) \le \Phi(X_*^q)$. By Fatou's Lemma,

$$\mathbb{E}_Q[|X_t|^q] \le \liminf_{\varepsilon\to 0} \mathbb{E}_Q[|X_t^\varepsilon|^q] \le \Phi(X_*^q) = \hat{c}_q^* < \infty,$$

where $\hat{c}_q^*$ is as in Lemma 5.1. Hence, $X_t \in \mathcal{L}^q(\Omega, Q)$ for every $t \in [0,T]$.

Step 3. Fix $t \in [0,T)$ and an $\mathcal{F}_t$-measurable $Y \in \mathcal{C}_b(\Omega)^d$. For $\varepsilon \in (0, T-t]$, set

$$\ell_{Y,\varepsilon} := \frac{1}{\varepsilon}\int_t^{t+\varepsilon} \frac{Y}{|X_u|+1}\cdot (X_u - X_T)\, du, \quad \text{and} \quad \ell_Y := \frac{Y}{|X_t|+1}\cdot (X_t - X_T).$$

Observe that $\ell_{Y,\varepsilon} \in \mathcal{C}_1(\Omega)$, $\lim_{\varepsilon\to 0} \ell_{Y,\varepsilon}(\omega) = \ell_Y(\omega)$, for all $\omega \in \Omega$ and in view of Corollary 5.4, $\Phi(\ell_{Y,\varepsilon}) \le 0$. Moreover,

$$\ell_{Y,\varepsilon} \ge -\|Y\|_\infty\, [1 + |X_T|] \quad \Rightarrow \quad \ell_{Y,\varepsilon}^c \ge -\|Y\|_\infty\, [1+|X_T|] \in \mathcal{L}^q(\Omega, Q).$$

Then, by Fatou's Lemma, $\mathbb{E}_Q[\ell_Y] \le \liminf_{\varepsilon\to 0} \mathbb{E}_Q[\ell_{Y,\varepsilon}]$. We now use again Fatou's Lemma, the sub-differential inequality, and Corollary 5.2 to obtain $\mathbb{E}_Q[\ell_{Y,\varepsilon}] \le \liminf_{c\to\infty} \mathbb{E}_Q[\ell_{Y,\varepsilon}^c] \le \lim_{c\to\infty} \Phi(\ell_{Y,\varepsilon}^c) = \Phi(\ell_{Y,\varepsilon}) \le 0$. Since this argument also holds for $-Y$, we conclude that $\mathbb{E}_Q[\ell_Y] = 0$.

Step 4. Let Y be as in the previous step. For $c > 0$, set $Y_c := Y[(|X_t|+1) \wedge c]$. Since $X_t, X_T \in \mathcal{L}^q(\Omega, Q)$, by dominated convergence,

$$\mathbb{E}_Q[Y \cdot (X_t - X_T)] = \lim_{c\to\infty} \mathbb{E}_Q\left[\frac{Y_c}{|X_t|+1}\cdot (X_t - X_T)\right] = 0.$$

The above equality, the integrability proved in Step 2 and Lemma C.1 imply that X is an $(\mathbb{F}, Q)$-martingale. As in (2.1), this also implies that $\mathbb{E}_Q\left[X_*^q\right] < \infty$.

Step 5. Let $\xi \in \mathcal{C}_p(\Omega)$. Then, $|\xi| \le c_\xi(1 + X_*^q)$ for some constant c_ξ. Since $X_* \in \mathcal{L}^q(\Omega, Q)$, dominated convergence yields that $\mathbb{E}_Q[\xi] = \lim_{c\to\infty} \mathbb{E}_Q[\xi^c]$. Also, by Corollary 5.2, $\lim_{c\to\infty} \Phi(\xi^c) = \Phi(\xi)$ and the sub-differential inequality at $\xi^c \in \mathcal{C}_b(\Omega)$ imply that $\mathbb{E}_Q[\xi^c] \le \Phi(\xi^c)$. Hence, $\mathbb{E}_Q[\xi] = \lim_{c\to\infty} \mathbb{E}_Q[\xi^c] \le \lim_{c\to\infty} \Phi(\xi^c) = \Phi(\xi)$ for every $\xi \in \mathcal{C}_p(\Omega)$.

Step 6. Fix $\gamma \in \mathcal{G}$. Then, by Assumption 2.2, $\gamma \in \mathcal{C}_{q,p}(\Omega)$ and hence, $\gamma^- \in \mathcal{C}_p(\Omega)$. For $a > 0$, set $\gamma_a := \gamma \wedge a$. Since $\gamma_a \le \gamma$, $\Phi(\gamma_a) \le \Phi(\gamma) \le 0$. Also, $\gamma_a \in \mathcal{C}_p(\Omega)$ and by the previous step, $\mathbb{E}_Q[\gamma_a] \le \Phi(\gamma_a) \le 0$. Moreover, $|\gamma_a| \le |\gamma| \le c_\gamma(1 + X_*^q)$ for some $c_\gamma > 0$. Since $X_* \in \mathcal{L}^q(\Omega, Q)$, dominated convergence yields $\mathbb{E}_Q[\gamma] = \lim_{a\to\infty} \mathbb{E}_Q[\gamma_a] \le 0$. Hence, $\mathbb{E}_Q[\gamma] \le 0$ for every $\gamma \in \mathcal{G}$. This and Step 4 imply that $\partial\Phi \subset \mathcal{Q}(\mathcal{G})$. □

The above results also prove the compactness of the set $\mathcal{Q}(\mathcal{G})$.

Corollary 6.7 *$\mathcal{Q}(\mathcal{G})$ is convex and both compact as well as sequentially compact with respect to the topology induced by the pairing $\langle \mathcal{C}_b(\Omega), \mathcal{M}(\Omega)\rangle$.*

Proof It is clear that $\hat{\omega} \in \Omega$ is convex. Let K_n be as in the proof of Theorem 6.4. Then, by (4.2), $\sigma_{\mathcal{Q}(\mathcal{G})}(\mathbb{1}_{\Omega\setminus K_n}) \leq \Phi(\mathbb{1}_{\Omega\setminus K_n}) =: \alpha_n$. By Theorem 6.4, α_n tends to zero. Hence, $Q(K_n) \geq 1-\alpha_n$ uniformly over $Q \in \mathcal{Q}(\mathcal{G})$. Since α_n converges to zero, $\mathcal{Q}(\mathcal{G})$ is uniformly tight.

By Proposition 6.6, $\mathcal{Q}(\mathcal{G}) = \bigcap_{\xi\in\mathcal{C}_b(\Omega)} \left\{Q \in \mathcal{P}(\Omega) : \mathbb{E}_Q[\xi] \leq \Phi(\xi)\right\}$. Hence, $\mathcal{Q}(\mathcal{G})$ is weak* closed. Then, by Prokhorov's theorem for completely regular Hausdorff spaces [12, Theorem 8.6.7], $\mathcal{Q}(\mathcal{G})$ is weak* compact. By the second assertion of [12, Theorem 8.6.7], since the compact sets K_n above are metrizable [46, Proposition 5.7], $\mathcal{Q}(\mathcal{G})$ is also sequentially weak* compact. □

7 Proof of Theorem 3.1

7.1 Duality on $\mathcal{C}_b(\Omega)$

Proposition 7.1 $\Phi(\xi; \mathcal{I}(\mathcal{G})) = \sigma_{\mathcal{Q}(\mathcal{G})}(\xi)$ *for all* $\xi \in \mathcal{C}_b(\Omega)$.

Proof In view of Corollary 5.3, we may assume that $\Phi(0; \mathcal{I}(\mathcal{G})) = 0$. Then, by the results of Sect. 6, Φ is convex, finite-valued and β_0-continuous. Hence, the hypotheses of the Fenchel-Moreau theorem on the topological space $\mathcal{C}_b(\Omega)$ with the locally convex β_0-topology are satisfied [60, Theorem 2.3.3]. Since Φ is positively homogenous, $\Phi(\xi; \mathcal{I}(\mathcal{G})) = \sigma_{\partial\Phi}(\xi)$ for every $\xi \in \mathcal{C}_b(\Omega)$. We then complete the proof of duality on $\mathcal{C}_b(\Omega)$ by Proposition 6.6. □

7.2 Duality on $\mathcal{U}_p(\Omega)$

We first extend the duality from $\mathcal{C}_b(\Omega)$ to $\mathcal{U}_b(\Omega)$ by a minimax argument.

Lemma 7.2 *The duality* $\Phi(\xi; \mathcal{I}(\mathcal{G})) = \sigma_{\mathcal{Q}(\mathcal{G})}(\xi)$ *holds for all* $\xi \in \mathcal{C}_b(\Omega)$ *if and only if it holds for all* $\xi \in \mathcal{U}_b(\Omega)$.

Proof Assume that the duality holds on $\mathcal{C}_b(\Omega)$ and let $\eta \in \mathcal{U}_b(\Omega)$. In view of (4.2), we need to show that $\sigma_{\mathcal{Q}(\mathcal{G})}(\eta) \geq \Phi(\eta; \mathcal{I}(\mathcal{G}))$. Since S^* is perfectly normal by Lemma A.6 below, for every $Q \in \mathcal{P}(\Omega)$, $\mathbb{E}_Q[\eta] = \inf_{\eta\leq\xi\in\mathcal{C}_b(\Omega)} \mathbb{E}_Q[\xi]$. Clearly, $\{\xi \in \mathcal{C}_b(\Omega) : \eta \leq \xi\}$ is a convex subset of $\mathcal{C}_b(\Omega)$ and the mapping that takes (ξ, Q) to $\mathbb{E}_Q[\xi]$ is continuous and bilinear on $\mathcal{C}_b(\Omega) \times \mathcal{Q}(\mathcal{G})$. Moreover, by Corollary 6.7, $\mathcal{Q}(\mathcal{G})$ is a convex, weak* compact subset of $\mathcal{P}(\Omega)$. Hence, the assumptions of a standard minimax argument are satisfied, see e.g. [60, Theorem 2.10.2]. Since Φ is monotone,

$$\begin{aligned}\sigma_{\mathcal{Q}(\mathcal{G})}(\eta) &= \sup_{Q\in\mathcal{Q}(\mathcal{G})} \inf_{\eta\leq\xi\in\mathcal{C}_b(\Omega)} \mathbb{E}_Q[\xi] = \inf_{\eta\leq\xi\in\mathcal{C}_b(\Omega)} \sup_{Q\in\mathcal{Q}(\mathcal{G})} \mathbb{E}_Q[\xi]\\ &= \inf_{\eta\leq\xi\in\mathcal{C}_b(\Omega)} \sigma_{\partial\Phi}(\xi) = \inf_{\eta\leq\xi\in\mathcal{C}_b(\Omega)} \Phi(\xi; \mathcal{I}(\mathcal{G})) \geq \Phi(\eta; \mathcal{I}(\mathcal{G})).\end{aligned}$$

Therefore, the duality holds on $\mathcal{U}_b(\Omega)$. □

Proposition 7.3 $\Phi(\xi; \mathcal{I}(\mathcal{G})) = \sigma_{\mathcal{Q}(\mathcal{G})}(\xi)$ *for all* $\xi \in \mathcal{U}_p(\Omega)$.

Proof Fix $\xi \in \mathcal{U}_p(\Omega)$ and ξ^c be as in (4.1). Then, $\xi^c \in \mathcal{U}_b(\Omega)$ and duality holds at ξ^c. We now combine this with Lemma 4.1 and Corollary 5.2 to arrive at

$$\sigma_{\mathcal{Q}(\mathcal{G})}(\xi) = \lim_{c\to\infty} \sigma_{\mathcal{Q}(\mathcal{G})}(\xi^c) = \lim_{c\to\infty} \Phi(\xi^c; \mathcal{I}(\mathcal{G})) = \Phi(\xi; \mathcal{I}(\mathcal{G})).$$

□

7.3 Duality on $\mathcal{B}_p(\Omega)$

In this section, we follow the approach of [45] and extend the duality to measurable functions by the Choquet capacitability theorem [20].

Proposition 7.4 $\Phi(\xi; \widehat{\mathcal{I}}(\mathcal{G})) = \sigma_{\mathcal{Q}(\mathcal{G})}(\xi)$ *for all* $\xi \in \mathcal{B}_p(\Omega)$.

Proof We write $\widehat{\Phi}(\cdot)$ instead of $\Phi(\cdot; \widehat{\mathcal{I}}(\mathcal{G}))$ and $\Phi(\cdot)$ for $\Phi(\cdot; \mathcal{I}(\mathcal{G}))$ as before.

Step 1. Since $\widehat{\mathcal{I}}(\mathcal{G}) \supset \mathcal{I}(\mathcal{G})$, $\widehat{\Phi} \le \Phi$. By Proposition 7.3 and (4.2), for every $\eta \in \mathcal{U}_b(\Omega)$, $\sigma_{\mathcal{Q}(\mathcal{G})}(\eta) \le \widehat{\Phi}(\eta) \le \Phi(\eta) = \sigma_{\mathcal{Q}(\mathcal{G})}(\eta)$. Hence, $\Phi = \widehat{\Phi}$ on $\mathcal{U}_b(\Omega)$.

Step 2. Consider a sequence $\{Q_n\}_{n\in\mathbb{N}}$ in $\mathcal{M}(\Omega)$ converging to Q^* in the weak* topology. Then, $\mathbb{E}_{Q_n}[\xi]$ converges to $\mathbb{E}_{Q^*}[\xi]$ for every $\xi \in \mathcal{C}_b(\Omega)$. Fix $\eta \in \mathcal{U}_b(\Omega)$. Since S^* is perfectly normal by Lemma A.6 below, there is a decreasing sequence $\{\xi_k\}_{k\in\mathbb{N}} \subset \mathcal{C}_b(\Omega)$ converging to η and $\mathbb{E}_{Q^*}[\xi_k]$ converges to $\mathbb{E}_{Q^*}[\eta]$. We use this and the weak* convergence of Q_n to arrive at

$$\limsup_{n\to\infty} \mathbb{E}_{Q_n}[\eta] \le \inf_k \lim_{n\to\infty} \mathbb{E}_{Q_n}[\xi_k] = \inf_k \mathbb{E}_{Q^*}[\xi_k] = \mathbb{E}_{Q^*}[\eta].$$

Since by Corollary 6.7, $\mathcal{Q}(\mathcal{G})$ is weak* compact, the above property implies that for every $\eta \in \mathcal{U}_b(\Omega)$ there is $Q_\eta \in \mathcal{Q}(\mathcal{G})$ satisfying, $\mathbb{E}_{Q_\eta}[\eta] = \sigma_{\mathcal{Q}(\mathcal{G})}(\eta)$.

Step 3. Suppose that a sequence $\{\eta_n\}_{n\in\mathbb{N}} \subset \mathcal{U}_b(\Omega)$ decreases monotonically to a function $\eta^* \in \mathcal{U}_b(\Omega)$. Then, $Q_n := Q_{\eta_n}$ satisfies $\mathbb{E}_{Q_n}[\eta_n] = \sigma_{\mathcal{Q}(\mathcal{G})}(\eta_n)$. Since $\mathcal{Q}(\mathcal{G})$ is sequentially compact with respect to $\sigma(\mathcal{M}, \mathcal{C}_b)$, there is a subsequence (without loss of generality, again denoted by Q_n) and $Q^* \in \mathcal{Q}(\mathcal{G})$ such that Q_n converges to Q^* in the weak* topology. Then, by the previous step,

$$\limsup_{n\to\infty} \mathbb{E}_{Q_n}[\eta_n] \le \inf_k \limsup_{n\to\infty} \mathbb{E}_{Q_n}[\eta_k] \le \inf_k \mathbb{E}_{Q^*}[\eta_k] = \mathbb{E}_{Q^*}[\eta^*],$$

where we used monotone convergence in the final equality.

By the first step, $\Phi = \widehat{\Phi}$ on $\mathcal{U}_b(\Omega)$. Then, by Proposition 7.3 and (4.2),

$$\begin{aligned}\limsup_{n\to\infty} \widehat{\Phi}(\eta_n) &= \limsup_{n\to\infty} \sigma_{\mathcal{Q}(\mathcal{G})}(\eta_n) = \limsup_{n\to\infty} \mathbb{E}_{Q_n}[\eta_n] \\ &\le \mathbb{E}_{Q^*}[\eta^*] \le \sigma_{\mathcal{Q}(\mathcal{G})}(\eta^*) \le \widehat{\Phi}(\eta^*).\end{aligned}$$

Since η_n's are decreasing to η^*, the opposite inequality is immediate. Hence,

$$\lim_{n\to\infty} \widehat{\Phi}(\eta_n) = \widehat{\Phi}(\eta^*) \quad \text{whenever} \quad \mathcal{U}_b(\Omega) \ni \eta_n \downarrow \eta^* \in \mathcal{U}_b(\Omega) \quad \text{as } n \to \infty. \tag{7.1}$$

Step 4. Consider $\{\zeta_n\}_{n\in\mathbb{N}} \subset \mathcal{B}_b(\Omega)$ increasing monotonically to $\zeta^* \in \mathcal{B}_b(\Omega)$. Choose $\{\ell_n\}_{n\in\mathbb{N}} \subset \widehat{\mathcal{I}}(\mathcal{G})$ so that $\widehat{\Phi}(\zeta_n) + \frac{1}{n} + \ell_n(\omega) \geq \zeta_n(\omega)$, for every $\omega \in \Omega$. It is clear that $\widehat{\Phi}(\zeta_1) \leq \widehat{\Phi}(\zeta_n) \leq \widehat{\Phi}(\zeta^*)$. Since $\zeta_n \geq \zeta_1$, $\ell_n \geq (\zeta_1 - \widehat{\Phi}(\zeta^*) - 1) \wedge 0 =: -\lambda$. Then, by the definition of $\widehat{\mathcal{I}}(\mathcal{G})$, $\ell^* := \liminf_n \ell_n \in \widehat{\mathcal{I}}(\mathcal{G})$. Therefore,

$$\zeta^*(\omega) = \lim_{n\to\infty} \zeta_n(\omega) \leq \liminf_{n\to\infty} \left[\widehat{\Phi}(\zeta_n) + \frac{1}{n} + \ell_n(\omega)\right] = \lim_{n\to\infty} \widehat{\Phi}(\zeta_n) + \ell^*(\omega),$$

for every $\omega \in \Omega$. Hence, $\lim_{n\to\infty} \widehat{\Phi}(\zeta_n) \geq \widehat{\Phi}(\zeta^*)$. Again the opposite inequality is immediate. So we have shown that

$$\lim_{n\to\infty} \widehat{\Phi}(\zeta_n) = \widehat{\Phi}(\zeta^*) \quad \text{whenever} \quad \mathcal{B}_b(\Omega) \ni \zeta_n \uparrow \zeta^* \in \mathcal{B}_b(\Omega) \quad \text{as } n \to \infty. \tag{7.2}$$

Step 5. (7.1) and (7.2) imply that we can apply the Choquet capacitability theorem (see [45, Proposition 2.11] or [4, Proposition 2.1]) to the functional $\widehat{\Phi}$. Let $\mathcal{S}(\Omega)$ denote the family of all Suslin functions generated by $\mathcal{U}_b(\Omega)$ i.e. functions of the form $\sup_{\phi\in\mathbb{N}^{\mathbb{N}}} \inf_{k\geq 1} \xi_{\phi|k}$, where $\phi|k$ denotes the restriction of $\phi \in \mathbb{N}^{\mathbb{N}}$ to $\{1, \dots, k\}$ and each $\xi_{\phi|k}$ is an element of $\mathcal{U}_b(\Omega)$; we refer to [32, Section 42] for details. Since the S^*-topology on Ω is perfectly normal, by Lemma A.6 below, the family $\mathcal{S}(\Omega)$ contains $\mathcal{B}_b(\Omega)$. Moreover, $\widehat{\Phi} = \Phi$ on $\mathcal{U}_b(\Omega)$. Hence, $\widehat{\Phi}(\zeta) = \sup\{\Phi(\eta) : \eta \in \mathcal{U}_b(\Omega), \eta \leq \zeta\}$ for every $\zeta \in \mathcal{B}_b(\Omega)$. This approximation together with the duality proved in Lemma 7.2 yield,

$$\begin{aligned}\widehat{\Phi}(\zeta) &= \sup_{\{\eta\leq\zeta,\ \eta\in\mathcal{U}_b(\Omega)\}} \sup_{Q\in\mathcal{Q}(\mathcal{G})} \mathbb{E}_Q[\eta] = \sup_{Q\in\mathcal{Q}(\mathcal{G})} \sup_{\{\eta\leq\zeta,\ \eta\in\mathcal{U}_b(\Omega)\}} \mathbb{E}_Q[\eta] \\ &= \sup_{Q\in\mathcal{Q}(\mathcal{G})} E_Q[\zeta] \quad \text{for all } \zeta \in \mathcal{B}_b(\Omega).\end{aligned}$$

Hence, the duality holds on $\mathcal{B}_b(\Omega)$.

Step 5. We now follow the proof of Proposition 7.3 *mutatis mutandis* to extend the result to $\mathcal{B}_p(\Omega)$. □

8 Counter-examples

In this section, $d = 1$, $T = 1$, $p = 1$ and $q = 2$. For a given $\mu \in \mathcal{P}(\mathbb{R}_+)$, set

$$\mathcal{G}_\mu := \left\{g(X_1(\omega)) : g \in \mathcal{C}_{2,1}(\mathbb{R}_+),\ \mu(g) = 0\right\}.$$

Example 8.1 Suppose that μ is supported in $[1, 3]$ and let $\Omega = \mathcal{D}([0, 1]; [1, 3])$. Then, there exists a countable set $A \subset \Omega$ such that $0 = \sigma_{\mathcal{Q}(\mathcal{G}_\mu)}(\mathbb{1}_A) = \Phi(\mathbb{1}_A; \mathcal{I}(\mathcal{G}_\mu))$ and $\Phi(\mathbb{1}_A; \mathcal{I}_s(\mathcal{G}_\mu)) = 1$. In particular, $\mathcal{I}_s(\mathcal{G}_\mu) \neq \mathcal{I}(\mathcal{G}_\mu)$.

Proof For $\omega \in \Omega$, let $v_3(\omega) := \sup_\pi \sum_{k=1}^n |\omega(\tau_k) - \omega(\tau_{k-1})|^3$, where π ranges over all finite partitions $0 = \tau_0 < \tau_1 < \cdots < \tau_n = 1$ of $[0, 1]$. Set $t_0 = 1$ and for $k \geq 1$,

$t_k := 1/k$, $s_k := (t_{k+1}+t_k)/2$, $c_k := 2f(s_k)/(t_k - t_{k+1})$ with $f(x) := x^{1/3}$ for $x \geq 0$, and

$$\widehat{\omega}(t) := \sum_{k=1}^{\infty} c_k(t - t_{k+1})\mathbb{1}_{(t_{k+1},s_k]}(t) + [f(s_k) - c_k(t - s_k)]\mathbb{1}_{(s_k,t_k]}(t), \quad t \in [0,1].$$

It is clear that $\widehat{\omega} \in \Omega$ and $\widehat{\omega}(t_n) = 0$, $\widehat{\omega}(s_n) = f(s_n)$. Set $\widehat{\omega}_n(t) := \widehat{\omega}(t \wedge t_n)$ for $t \in [0,1]$. Then, $\widehat{\omega}_n(t) = 0$ for $t \in [t_n, 1]$ and

$$v_3(\widehat{\omega}_n) \geq \sum_{k=n}^{\infty}(f(s_k) - f(t_k))^3 = \sum_{k=n}^{\infty} s_k = \infty.$$

Let $\mathbb{Q}$ be the set of rational numbers in $[1,2]$. Set $A_q := \{q + \widehat{\omega}_n : n \in \mathbb{N}\}$ and $A := \cup_{q\in\mathbb{Q}} A_q$. Then, $A \subset \{\omega \in \Omega : v_3(\omega) = \infty\}$. Since for any martingale measure Q, $Q(\omega \in \Omega : v_3(\omega) = \infty) = 0$, we conclude that $\sigma_{\mathcal{Q}(\mathcal{G}_\mu)}(\mathbb{1}_A) = 0$. Suppose that for some $c \in \mathbb{R}$, $\gamma \in \mathcal{G}$ and $H = (\tau_m, h_m)_{m\in\mathbb{N}} \in \mathcal{H}_s$, we have $c + \gamma(\omega) + (H \cdot X)_1(\omega) \geq \mathbb{1}_A(\omega)$ for every $\omega \in \Omega$. Then, $\gamma(\omega) = g(X_1(\omega))$ with $\mu(g) = 0$ and $\gamma(\omega) = g(q)$ for every $\omega \in A_q$. Hence, $c + g(q) + (H \cdot X)_1(q + \widehat{\omega}_n) \geq 1$, for every $q \in \mathbb{Q}$, $n \geq 1$. By the adaptedness of H, $(H \cdot X)_1(q+\widehat{\omega}_n) = (H \cdot X)_{t_n}(q+\widehat{\omega})$, for each q. Therefore,

$$\lim_{n\to\infty} (H \cdot X)_1(q + \widehat{\omega}_n) = \lim_{n\to\infty} (H \cdot X)_{t_n}(q + \widehat{\omega}) = 0.$$

This implies that $c + g(q) \geq 1$. Moreover, g is continuous and $\mu(g) = 0$. Hence, $c \geq 1$. Since $\Phi(\mathbb{1}_A; \mathcal{I}_s(\mathcal{G}_\mu))$ is the smallest of all such constants, we conclude that $\Phi(\mathbb{1}_A; \mathcal{I}_s(\mathcal{G}_\mu)) \geq 1$. As $\mathbb{1}_A \leq 1$, $\Phi(\mathbb{1}_A; \mathcal{I}_s(\mathcal{G}_\mu)) = 1$.

We next proceed as in Lemma 6.3 to show that $\Phi(\mathbb{1}_A; \mathcal{I}(\mathcal{G}_\mu)) = 0$. Indeed, for $k \geq 2$, define $H^k = (\tau^k_m, h^k_m)_{m\in\mathbb{N}} \in \mathcal{H}_s$ as follows. Let $\tau^k_0 = 0$ and for $m \geq 1$, recursively define the stopping times by

$$\tau^k_{2m-1}(\omega) := \inf\{t > \tau^k_{2m-2}(\omega) : \omega(t) > \omega(0) + k^{-1/3}/2\} \wedge 1,$$
$$\tau^k_{2m}(\omega) := \inf\{t > \tau^k_{2m-1}(\omega) : \omega(t) < \omega(0) + k^{-1/3}/3\} \wedge 1.$$

For $m \geq 0$, set $h^k_{2m} = k^{-4/3}$, $h^k_{2m+1} = 0$. Let $U^k_t(\omega)$ be the crossings in the time interval $[0,t]$ between the lower boundary $\omega(0) + k^{-1/3}/3$ and the upper boundary $\omega(0) + k^{-1/3}/2$. Then, as in Lemma 6.3,

$$(H^k \cdot X)_t(\omega) \geq -\frac{\omega(0)}{k^{4/3}} + \frac{1}{6k^{5/3}} U^k_t(\omega) \quad \text{for all } t \in [0,1] \text{ and } \omega \in \Omega.$$

In particular, $H^k \in \mathcal{H}_s$. Observe that $U^k_t(q + \widehat{\omega}_n) \geq k - n$ for all $n \leq k$. Therefore, for any $q \in [1,2]$,

$$(H^k \cdot X)_t(q + \widehat{\omega}_n) \geq -\frac{2}{k^{4/3}} + \frac{(k-n)}{6k^{5/3}} \quad \text{for all } 1 \leq n \leq k.$$

For $\varepsilon > 0$, let $H^\varepsilon := \varepsilon(\widehat{H}^j)_{j\in\mathbb{N}}$, where $\widehat{H}^j := \sum_{k\le j} H^k$. Then, for each $j \ge 1$,

$$(H^\varepsilon \cdot X)_t(\omega) = \varepsilon \sum_{1\le k\le j} (H^k \cdot X)_t(\omega) \ge -\varepsilon \sum_{1\le k} \frac{2}{k^{4/3}} =: -\varepsilon C_* \quad \text{for all } t \in [0,1].$$

Hence, H^ε is admissible. Also, for $q \in [1,2]$,

$$\begin{aligned}(H^\varepsilon \cdot X)_t(q + \widehat{\omega}_n) &= \liminf_{j\to\infty} \sum_{1\le k\le j} \varepsilon(H^k \cdot X)_t(q + \widehat{\omega}_n) \\ &\ge \sum_{1\le k} -\frac{2\varepsilon}{k^{4/3}} + \frac{\varepsilon(k-n)^+}{6k^{5/3}} = \infty.\end{aligned}$$

Therefore, $\Phi(\mathbb{1}_A; \mathcal{I}(\mathcal{G}_\mu)) \le \varepsilon C_*$ for every $\varepsilon > 0$. □

The following example motivates the use of the Fatou-closure $\widehat{\mathcal{I}}(\mathcal{G})$ to establish the duality for measurable functions in Theorem 3.1.

Example 8.2 Let $\Omega = \mathcal{D}([0,1]; \mathbb{R}_+)$ and consider the quotient spaces given by $\mathcal{G} = \{g(X_1(\omega)) : g \in \mathcal{C}_{2,1}(\mathbb{R}_+), g(1) = 0\}$. Then, there exists an open set $B \subset \Omega$ such that $0 = \sigma_{\mathcal{Q}(\mathcal{G})}(\mathbb{1}_B) = \Phi(\mathbb{1}_B; \widehat{\mathcal{I}}(\mathcal{G})) < 1 = \Phi(\mathbb{1}_B; \mathcal{I}(\mathcal{G}))$. In particular, $\mathcal{I}(\mathcal{G}) \ne \widehat{\mathcal{I}}(\mathcal{G})$.

Proof Consider the open set $B := \{X_T \ne 1\}$, set $\omega^* \equiv 1$ and let Q^* be the Dirac measure at ω^*. Then, $\mathcal{Q}(\mathcal{G}) = \{Q^*\}$. Hence, $\sigma_{\mathcal{Q}(\mathcal{G})}(\mathbb{1}_B) = \mathbb{E}_{Q^*}[\mathbb{1}_B] = 0$.

Suppose that $\ell \in \mathcal{I}(\mathcal{G})$ and $c \in \mathbb{R}$ satisfy $c + \ell \ge \mathbb{1}_B$. By the definition of $\mathcal{I}(\mathcal{G})$, there are $H \in \mathcal{H}$ and $g(X_1(\cdot)) \in \mathcal{G}$ such that $\ell(\omega) = (H \cdot X)_1(\omega) + g(X_1(\omega))$. Consider a constant path $\omega \equiv x$. Then, for this path $(H \cdot X)_T(\omega) = 0$ and therefore, $\mathbb{1}_B(\omega) = 1 \le c + g(x)$ for every $x \ne 1$. Since $g(1) = 0$ and g is continuous, we conclude that $c \ge 1$. Hence, $\Phi(\mathbb{1}_B; \mathcal{I}(\mathcal{G})) = 1$. □

9 Financial applications

In this section we assume that X models the discounted prices of d assets. Alternatively, one could also model undiscounted prices and introduce an additional process representing a savings account. But this does not change the essential mathematical structure; see [19]. For related examples and discussions of the role of Ω as a prediction set, we refer to [5,6,38].

The set $\mathcal{G}$ represents the set of net outcomes of investments in liquid derivative instruments. Their initial prices are normalized to zero. Since we do not assume any probabilistic structure, this set plays an essential role in determining the pricing functionals. We give different examples of the set $\mathcal{G}$. They show that finite discrete-time models can be included in our framework by appropriately choosing the closed set Ω.

Example 9.1 (Final Marginal) In this example $\Omega = \mathcal{D}([0,T];\mathbb{R}^d_+)$. We fix a probability measure μ on $\mathbb{R}^d_+$ with finite q-th moments and set

$$\mathcal{G}_\mu := \left\{\gamma(\omega) = g(\omega(T)) - \mu(g) : g \in \mathcal{C}_{q,p}(\mathbb{R}^d_+)\right\}, \quad \text{where} \quad \mu(g) = \int_{\mathbb{R}^d_+} g\, d\mu.$$

Then, $\mathcal{Q}(\mathcal{G}_\mu)$ consists of all martingale measures Q whose final marginal is μ, i.e.,

$$\mathbb{E}_Q[h(X_T)] = \mu(h) \quad \text{for all } h \in \mathcal{B}_q\left(\mathbb{R}^d_+\right).$$

Remark 9.2 The duality in the setting of Example 9.3 with one fixed marginal does not immediately extend to the case of two marginals assuming that $\Omega = \mathcal{D}([0,T];\mathbb{R}^d_+)$. The difficulty arises from the fact that the coordinate mapping X_0 is not continuous. This issue can be removed by introducing a fictitious element X_{0-} on the Skorokhod space $\mathcal{D}([0,T];\mathbb{R}^d_+)$, i.e. one considers $\Omega_{x_0-} := \mathbb{R}^d_+ \times \mathcal{D}([0,T];\mathbb{R}^d_+)$.

Example 9.3 (Initial Value and Final Marginal) In addition to a final marginal, in this example we wish to fix the initial asset values $x_0 \in \mathbb{R}^d_+$. However, the canonical map, $X_t : \omega \in \mathcal{D}([0,T];\mathbb{R}^d_+) \mapsto \omega(t) \in \mathbb{R}^d_+$, is continuous only for $t = T$ and discontinuous at all other points. Therefore, $\Omega_{x_0} := \{\omega \in \Omega : \omega(0) = x_0\}$ is not an S-closed subset of $\mathcal{D}([0,T];\mathbb{R}^d_+)$. To overcome this difficulty, we fix a small time increment $h > 0$ and define

$$\Omega_{h,x_0} := \{\omega \in \Omega : \omega(t) = x_0 \text{ for all } t \in [0,h)\}.$$

One may directly verify that Ω_{h,x_0} is S-closed. We keep $\mathcal{G}_\mu$ as in the previous example. Then, the elements of $\mathcal{Q}(\mathcal{G}_\mu)$ restricted to Ω_{h,x_0} are martingale measures with the final marginal μ and satisfy

$$Q(X_t = x_0 \text{ for all } t \in [0,h)) = 1, \quad Q \in \mathcal{Q}(\mathcal{G}_\mu).$$

The set $\mathcal{Q}(\mathcal{G}_\mu)$ is non-empty provided that $\int x\, \mu(dx) = x_0$.

Example 9.4 (Multiple Marginals) In Example 9.1 we fixed the marginal of the dual measures at the final time. In a given application, marginals at other time points $\mathcal{T} = \{t_1, \ldots, t_N\}$ might be approximately known. So one may want to fix these marginals as well. Since X_{t_i} are all discontinuous, functions of the form $g(X_{t_i})$ are not necessarily S^*-continuous on $\mathcal{D}([0,T];\mathbb{R}^d_+)$. So, as in the previous example, we fix a small $h > 0$ and consider the set given by

$$\Omega_{\mathcal{T}} := \bigcap_{i=1}^{N} \left\{\omega \in \Omega \ : \ X_t(\omega) = X_{t_i}(\omega) \text{ for all } t \in [t_i, t_i + h)\right\}.$$

Then, $\Omega_{\mathcal{T}}$ is an S-closed subset of $\mathcal{D}([0, T]; \mathbb{R}^d_+)$. Moreover, for each i, X_{t_i} restricted to $\Omega_{\mathcal{T}}$ is S^*-continuous. Given probability measures $\{\mu_i\}_{i=1}^N$ on $\mathbb{R}^d_+$ with finite q-th moments, we consider the set

$$\mathcal{G}_{\mathcal{T}} := \left\{ \gamma(\omega) = \sum_{i=1}^{N} g_i(X_{t_i}(\omega)) - \mu_i(g_i) : g_i \in \mathcal{C}_{q,p}(\mathbb{R}^d_+) \text{ for all } i = 1, \dots, N \right\}.$$

Then, $\mathcal{G}_{\mathcal{T}} \subset C_{q,p}(\Omega_\tau)$. The measures $Q \in \mathcal{Q}(\mathcal{G}_{\mathcal{T}})$ are martingale measures and have marginal μ_i at times $t \in [t_i, t_i + h)$. Assume $0 \le t_1 < \ldots < t_N \le T$. In view of Strassen's result [55], $\mathcal{Q}(\mathcal{G}_{\mathcal{T}})$ is non-empty if and only if μ_i's are increasing in convex order, i.e, $\mu_1(\varphi) \le \ldots \le \mu_N(\varphi)$, for every convex function $\varphi : \mathbb{R}^d_+ \to \mathbb{R}$.

In the following examples, we collect some common option payoffs satisfying the assumptions of Theorem 3.1.

Example 9.5 The typical examples of S^*-continuous functions are the payoffs of Asian type options. Indeed, let $g : [0, T] \to \mathbb{R}$ be continuous. Then,

$$\xi(\omega) = \int_0^T g(t) X_t^i(\omega)\, dt,$$

for any $i \in \{1, \dots, d\}$, is S^*-continuous. However, the running maximum and minimum of X^i are only lower and upper semicontinuous, respectively; see [39]. We refer the reader to [39,46], for further examples.

In particular, the duality (3.2) holds for every derivative contract that is a measurable function of the underlying underlying assets.

Example 9.6 Since Ω is a measurable subset of $\mathcal{D}([0, T]; \mathbb{R}^d_+)$, we know from [44] that there exists an $\mathbb{F}$-progressively measurable $d \times d$-matrix-valued process $\langle X \rangle = (\langle X \rangle_t)_{t \in [0,T]}$ on Ω which equals the predictable quadratic variation of X Q-a.s., for every $\mathbb{F}$-martingale measure Q on Ω. We define the $d \times d$-matrix-valued volatility process $\sigma = (\sigma_t)_{t \in [0,T]}$ as the square-root of the non-negative, symmetric matrix-valued process

$$v_t(\omega) := \liminf_{\varepsilon \downarrow 0} \frac{\langle X \rangle_t(\omega) - \langle X \rangle_{(t-\varepsilon)\vee 0}(\omega)}{\varepsilon}, \quad (t, \omega) \in [0, T] \times \Omega.$$

In particular, σ is a measurable process on Ω. So Theorem 3.1 yields model-independent price bounds for derivative contracts written on σ. However, the construction of the quadratic variation process $\langle X \rangle$ relies on stopping times and therefore, on $\mathbb{F}$-progressively measurable partitions of the interval $[0, T]$, which in general are non-deterministic; we refer to [11] for details. In particular, derivative contracts depending on σ are in general not upper semicontinuous on Ω.

As a consequence of Theorem 3.1, we obtain a *fundamental theorem of asset pricing* relating the non-emptiness of $\mathcal{Q}(\mathcal{G})$ to an appropriate no-arbitrage condition. For classical versions of this result see e.g. [21] for discrete time, [22,23] for continuous time and the references therein. Robust versions have been derived in [1,19,22,26]. Our no-arbitrage conditions are the following.

Corollary 9.7 (Robust Fundamental Theorem of Asset Pricing) *Under Assumption* 2.2, the following are equivalent:

(i) $\mathcal{Q}(\mathcal{G})$ is non-empty.
(ii) $\Phi(\eta; \widehat{\mathcal{I}}(\mathcal{G}))$ is finite for all $\eta \in \mathcal{B}_p(\Omega)$.
(iii) $\Phi(0; \mathcal{I}(\mathcal{G})) = 0$.

A Appendix: *S* and *S**-topologies

The following definition is from Jakubowski [39,40].

Definition A.1 For $\{\nu^n\}_{n\in\mathbb{N}} \subset \mathcal{D}([0,T];\mathbb{R}^d_+)$ and $\nu^* \in \mathcal{D}([0,T];\mathbb{R}^d_+)$, we write $\nu^n \rightharpoonup_S \nu^*$ if for each $\varepsilon > 0$, there exist functions $\{\nu^{n,\varepsilon}\}_{n\in\mathbb{N}}$ and $\nu^{*,\varepsilon}$ in $\mathcal{D}([0,T];\mathbb{R}^d_+)$ which are of finite variation such that

$$\|\nu^* - \nu^{*,\varepsilon}\|_\infty \le \varepsilon, \quad \|\nu^n - \nu^{n,\varepsilon}\|_\infty \le \varepsilon \quad \text{for every } n \in \mathbb{N},$$

and

$$\lim_{n\to\infty} \int_{[0,T]} f(t)\, d\nu^{n,\varepsilon}_t = \int_{[0,T]} f(t)\, d\nu^{*,\varepsilon}_t, \tag{A.1}$$

for all $f \in \mathcal{C}_b([0,T];\mathbb{R}^d)$, where the integrals in (A.1) are Stieltjes integrals with $\nu^{n,\varepsilon}_{0-} = \nu^{*,\varepsilon}_{0-} = 0$. The topology generated by this sequential convergence is called the S-topology.

In particular, a subset $C \subset \mathcal{D}([0,T];\mathbb{R}^d_+)$ is S-closed if and only if it is sequentially closed for the above notion of convergence, i.e., if $\{\nu^n\}_{n\in\mathbb{N}} \subset C$ and $\nu^n \rightharpoonup_S \nu^*$, then $\nu^* \in C$. Open sets are the complements of the closed ones. One may directly verify that this collection of sets satisfies the definition of a topology.

Remark A.2 The (*a posteriori*) convergence in this topology could be different from the *a priori* convergence $\rightharpoonup_S$ defined above. This definition of a topology is known as the Kantorovich-Kisyński recipe; see [47] or [30, Sections 1.7.18, 1.7.19 on pages 63-64]. In particular, it is discussed in [40, Appendix] that $\{\nu^n\}_{n\in\mathbb{N}}$ converges to ν^* in the (*a posteriori*) S-topology, if every subsequence $\{\nu^{n_k}\}_{k\in\mathbb{N}}$ has a further subsequence $\{\nu^{n_{k_l}}\}_{l\in\mathbb{N}}$ such that $\nu^{n_{k_l}} \rightharpoonup_S \nu^*$.

As a different example, if one starts with almost-sure convergence as the *a priori* convergence (instead of the $\rightharpoonup_S$ convergence as above), then the resulting *a posteriori* convergence is the convergence-in-probability; see [40].

The following fact from [39,40] is an essential ingredient of our continuity proof. Recall the up-crossings $U^{a,b,i}_t$ of Definition 6.2.

Proposition A.3 (Jakubowski [39], Theorem 2.13; [40], Theorem 5.7). *A subset* $K \subset \mathcal{D}([0,T];\mathbb{R}^d_+)$ *is relatively S-compact if and only if*

$$\sup_{\omega\in K}\|\omega\|_\infty < \infty \quad \textit{and} \quad \sup_{\omega\in K} U_T^{a,b,i}(\omega) < \infty \textit{ for all } a < b \textit{ and } i = 1\ldots,d. \tag{A.2}$$

Let us denote the relative topology of S on Ω again by S. It is not known whether (Ω, S) is completely regular. As this property plays an important role in our analysis, we regularize S on Ω analogously to [46].

Definition A.4 The S^*-topology on Ω is the coarsest topology making all S-continuous functions $\xi : \Omega \to \mathbb{R}$ continuous.

It is clear from this definition that $S^* \subset S$, and a function $\xi\colon \Omega \to \mathbb{R}$ is S^*-continuous if and only if it is S-continuous. Moreover, (Ω, S) and (Ω, S^*) are both Hausdorff, and since compact sets stay compact if the topology is weakened, every S-compact subset of Ω is also S^*-compact.

The collection of finite intersections of sets of the form

$$O_{\xi,\varepsilon}(\omega_*) := \{\omega \in \Omega : |\xi(\omega) - \xi(\omega_*)| < 1\}$$

with arbitrary S-continuous functions $\xi\colon \Omega \to \mathbb{R}$, form a neighborhood basis at ω_*. In particular, for any S^*-open set O and $\omega_* \in O$, there is a neighborhood of ω_* of the form

$$\bigcap_{k=1}^{n} \{\omega \in \Omega : |\xi_k(\omega) - \xi_k(\omega_*)| < 1\},$$

contained in O, where each ξ_k is an S-continuous function from Ω to $\mathbb{R}$. For each $k \le n$, set $\eta_k(\omega) = |\xi_k(\omega) - \xi_k(\omega_*)| \wedge 1$ and $\eta(\omega) = \max_{k\le n}\eta_k(\omega)$. Then, η continuously maps Ω into $[0,1]$ and satisfies $\eta(\omega_*) = 0$ and $\eta(\omega) = 1$ for all $\omega \notin O$. This is the defining property of a completely regular space. Hence, (Ω, S^*) is a completely regular Hausdorff space, $(T_{3\frac{1}{2}})$. In fact, it turns out to be perfectly normal.

Lemma A.5 (Ω, S^*) *is perfectly normal Hausdorff* (T_6) *and a Lusin space. In particular, every Borel probability measure on* (Ω, S^*) *is a Radon measure.*

Proof It is well-known that the standard J_1-topology on the Skorokhod space is Polish. Moreover, by [39, Theorem 2.13 (vi)], $S \subset J_1$. So, since Ω is S-closed it is also J_1-closed. Therefore, if we denote the relative J_1-topology on Ω again by J_1, (Ω, J_1) is still Polish and $S^* \subset J_1$. As a consequence, the identity map from (Ω, J_1) to (Ω, S^*) is bijective and continuous, which shows that (Ω, S^*) is a Lusin space.

[31, Proposition I.6.1, page 19] proves that any completely regular Lusin space is perfectly normal. We note that [31] uses the terminology "*Espaces standards*" [31, Definition I.2.1, page 7] which is exactly a Lusin space and the term "*régulier*" as defined on page 18 in [31] corresponds to completely regular. The reader may also consult page 64 of [28] for a brief discussion of this implication.

Finally, on a Lusin space, every Borel probability measure is Radon; see e.g., [53, p. 122]. □

We also need the following facts about the S^*-topology.

Lemma A.6 *Every S^*-upper semicontinuous function from Ω to $\mathbb{R}$ is the pointwise limit of a decreasing sequence of S^*-continuous functions, and the family of Suslin functions generated by $\mathcal{U}_b(\Omega)$ includes $\mathcal{B}_b(\Omega)$.*

Proof The statement about approximation of upper semicontinuous functions by continuous ones is proved in [56, Theorem 3]. Also, see [24, Theorem 49 (c)] or [30, Page 61].

The statement about Suslin functions is proved in [4] (see the end of the proof of Theorem 2.2). Alternatively, by Proposition 421L in [32, page 143] on any topological space, every Baire set is a Suslin set. On perfectly normal Hausdorff spaces, Baire and Borel sets agree [12, Proposition 6.3.4]. Hence, bounded Borel functions with respect to S^* are Suslin. □

Remark A.7 [46] contains more results about the S^*-topology on $\mathcal{D}([0, T]; \mathbb{R}^d)$. In particular, the compact sets of S^* and S agree. Also, the S^*-topology is the strongest topology on the Skorokhod space for which the compactness criteria (A.2) holds and the Riesz representation theorem with the β_0-topology is true.

B Appendix: β_0-topology

Let E be a completely regular Hausdorff space and recall that $\mathcal{B}_0(E)$ is the set of real-valued, bounded, Borel measurable functions on E that vanish at infinity. Note that any perfectly normal topology, such as S^* on Ω, is completely regular.

For each $\eta \in \mathcal{B}_0^+(E)$ consider the semi-norm on $\mathcal{C}_b(E)$ given by,

$$\|\xi\|_\eta := \|\xi\eta\|_\infty := \sup_{x\in E} |\xi(x)\eta(x)|.$$

The β_0-topology on $\mathcal{C}_b(E)$ is generated by the semi-norms $\|.\|_\eta$ as η varies in $\mathcal{B}_0^+(E)$. Importantly, the topological dual of $\mathcal{C}_b(E)$ with the β_0-topology is the set of all signed Radon measures of bounded total variation on E; see e.g., [41, Theorem 3, page 141] or [54] for further details on the β_0-topology.

C Appendix: Martingale measures

Lemma C.1 *Let $Q \in \mathcal{P}(\Omega)$ such that $\sup_{t\in[0,T]} \mathbb{E}_Q[|X_t|^q] < \infty$ for some $q > 1$ and $\mathbb{E}_Q[Y \cdot (X_T - X_t)] = 0$ for every $t \in [0, T]$ and all $\mathcal{F}_t$-measurable $Y \in \mathcal{C}_b(\Omega)^d$. Then, the canonical map X is an $(\mathbb{F}, Q)$-martingale.*

Proof Fix $t < T$, and denote by $\mathcal{A}$ the family of all subsets of Ω that can be written as a finite intersection of sets of the form $X_{t_j}^{-1}(B_j)$ for $t_j \le t$ and a Borel subset B_j of $\mathbb{R}^d$. Let $i \in \{1, \ldots, d\}$. If we can show that

$$\mathbb{E}_Q[\mathbb{1}_A(X^i_T - X^i_t)] = 0 \quad \text{for all } A \in \mathcal{A}, \tag{C.1}$$

it follows from a monotone class argument that

$$\mathbb{E}_Q[\mathbb{1}_A(X^i_T - X^i_t)] = 0 \quad \text{for all } A \in \mathcal{F}^X_t.$$

By uniform integrability and right-continuity of X, this implies

$$\mathbb{E}_Q[\mathbb{1}_A(X^i_T - X^i_t)] = \lim_{\varepsilon \downarrow 0} \mathbb{E}_Q[\mathbb{1}_A(X^i_T - X^i_{t+\varepsilon})] = 0 \quad \text{for all } A \in \mathcal{F}_t,$$

which proves the lemma.

To show (C.1), note that for every set $A \in \mathcal{A}$ of the form

$$A = X^{-1}_{t_1}(B_1) \cap \cdots \cap X^{-1}_{t_k}(B_k)$$

for $t_1, \ldots, t_k \le t$ and Borel subsets $B_1, \ldots, B_k$ of $\mathbb{R}^d$, there exist bounded continuous functions $f^n_j : \mathbb{R}^d \to \mathbb{R}$ such that

$$\mathbb{E}_Q[\mathbb{1}_A(X^i_T - X^i_t)] = \lim_{n\to\infty} \mathbb{E}_Q[f^n_1(X_{t_1}) \cdots f^n_k(X_{t_k})(X^i_T - X^i_t)]. \tag{C.2}$$

On the other hand, for all n, one has

$$\mathbb{E}_Q[f^n_1(X_{t_1}) \cdots f^n_k(X_{t_k})(X^i_T - X^i_t)] = \lim_{\varepsilon\downarrow 0} \mathbb{E}_Q[f^n_1(X^\varepsilon_{t_1}) \cdots f^n_k(X^\varepsilon_{t_k})(X_T - X_{t+\varepsilon})], \tag{C.3}$$

for the S-continuous functions

$$X^\varepsilon_{t_j} = \frac{1}{\varepsilon} \int_{t_j}^{t_j+\varepsilon} X_{u\wedge T}\, du, \quad j = 1, \ldots, d.$$

Since $f^n_1(X^\varepsilon_{t_1}) \cdots f^n_k(X^\varepsilon_{t_k})$ is $\mathcal{F}_{t+\varepsilon}$-measurable and belongs to $\mathcal{C}_b(\Omega)$, it follows from the assumptions that (C.2)–(C.3) vanish, and the proof is complete. □

References

1. Acciaio, B., Beiglböck, M., Penkner, F., Schachermayer, W.: A model-free version of the fundamental theorem of asset pricing and the super-replication theorem. Math. Financ. **26**(2), 233–251 (2016)
2. Acciaio, B., Beiglböck, M., Penkner, F., Schachermayer, W., Temme, J.: A trajectorial interpretation of Doob's martingale inequalities. Ann. Appl. Prob. **23**(4), 1494–1505 (2013)
3. Ambrosio, L., Gigli, N.: A User's Guide to Optimal Transport. Springer, Berlin (2013)
4. Bartl, D., Cheridito, P., Kupper, M.: Robust expected utility maximization with medial limits. J. Math. Anal. Appl. **471**(1–2), 752–775 (2019)
5. Bartl, D., Kupper, M., Prömel, D. J., Tangpi, L.: Duality for pathwise superhedging in continuous time. arXiv:1705.02933 (2017)
6. Bartl, D., Kupper, M., Neufeld, A.: Pathwise superhedging on prediction sets. arXiv:1711.02764 (2017)

7. Beiglböck, M., Cox, A.M.G., Huesmann, M., Perkowski, N., Prömel, D.J.: Pathwise superreplication via Vovk's outer measure. Financ. Stoch. **21**(4), 1141–1166 (2017)
8. Beiglböck, M., Henry-Labordère, P., Penkner, F.: Model-independent bounds for option prices—a mass transport approach. Financ. Stoch. **17**(3), 477–501 (2013)
9. Beiglböck, M., Nutz, M., Touzi, N.: Complete duality for martingale optimal transport on the line. Ann. Prob. **45**(5), 3038–3074 (2017)
10. Biagini, S., Bouchard, B., Kardaras, C., Nutz, M.: Robust fundamental theorem for continuous processes. Math. Financ. **27**(4), 963–987 (2017)
11. Bichteler, K.: Stochastic integration and L^p-theory of semimartingales. Ann. Prob. **1**(2), 49–89 (1981)
12. Bogachev, V.I.: Measure Theory. Springer-Verlag, Berlin, Heidelberg (2007)
13. Bogachev, V.I., Kolesnikov, A.: The Monge–Kantorovich problem: achievements, connections, and perspectives. Russ. Math. Surv. **67**(5), 785–890 (2012)
14. Bouchard, B., Nutz, M.: Arbitrage and duality in non-dominated discrete-time models. Ann. Appl. Prob. **25**(2), 823–859 (2015)
15. Burzoni, M., Frittelli, M., Hou, Z., Maggis, M., Obłój, J.: Pointwise arbitrage pricing theory in discrete time. arXiv:1612.07618 (2018)
16. Burzoni, M., Frittelli, M., Maggis, M.: Model-free superhedging duality. Ann. Appl. Prob. **27**, 1452–1477 (2015)
17. Burzoni, M., Frittelli, M., Maggis, M.: Universal arbitrage aggregator in discrete time markets under uncertainty. Financ. Stoch. **20**, 1–50 (2016)
18. Cheridito, P., Kupper, M., Tangpi, L.: Representation of increasing convex functionals with countably additive measures. arXiv:1502.05763 (2015)
19. Cheridito, P., Kupper, M., Tangpi, L.: Duality formulas for robust pricing and hedging in discrete time. SIAM J. Financ. Math. **8**(1), 738–765 (2017)
20. Choquet, G.: Forme abstraite du théorème de capacitabilité. Ann. Inst. Fourier (Grenoble) **9**, 83–89 (1959)
21. Dalang, R.C., Morton, A., Willinger, W.: Equivalent martingale measures and no-arbitrage in stochastic securities market models. Stoch. Int. J. Prob.Stoch. Process. **29**(2), 185–201 (1990)
22. Delbaen, F., Schachermayer, W.: A general version of the fundamental theorem of asset pricing. Math. Ann. **300**(1), 463–520 (1994)
23. Delbaen, F., Schachermayer, W.: The fundamental theorem of asset pricing for unbounded stochastic processes. Math. Ann. **312**(2), 215–250 (1998)
24. Dellacherie, C., Meyer, P.A.: Probabilities and Potential, North-Holland Mathematics Studies, vol. 29. North-Holland Publishing Co., Amsterdam (1978)
25. Dolinsky, Y., Soner, H.M.: Martingale optimal transport and robust hedging in continuous time. Prob. Theory Relat. Fields **160**(1–2), 391–427 (2014)
26. Dolinsky, Y., Soner, H.M.: Robust hedging with proportional transaction costs. Financ. Stoch. **18**(2), 327–347 (2014)
27. Dolinsky, Y., Soner, H.M.: Martingale optimal transport in the Skorokhod space. Stoch. Process. Appl. **125**(10), 3893–3931 (2015)
28. Dudley, R.M.: A counterexample on measurable processes. In: *Proc. 6th Berkeley Symp. on Math. Stat. Proba*, vol. 2, pp. 57–66 (1972)
29. Ekren, I., Soner, H.M.: Constrained optimal transport. Arch. Ration. Mech. Anal. **227**(3), 929–965 (2018)
30. Engelking, R.: General Topology. Heldermann, Berlin (1989)
31. Fernique, X.: Processus linéaires, processus généralisés. Ann. Inst. Fourier (Grenoble) **17**(1), 1–92 (1967)
32. Fremlin, D.H.: Measure Theory, vol. 4. Torres Fremlin, Colchester (2000)
33. Galichon, A., Henry-Labordère, P., Touzi, N.: A stochastic control approach to no-arbitrage bounds given marginals, with an application to lookback options. Ann. Appl. Prob. **24**(1), 312–336 (2014)
34. Guo, G., Tan, X., Touzi, N.: On the monotonicity principle of optimal Skorokhod embedding problem. SIAM J. Control Optim. **54**(5), 2478–2489 (2016)
35. Guo, G., Tan, X., Touzi, N.: Optimal Skorokhod embedding under finitely many marginal constraints. SIAM J. Control Optim. **54**(4), 2174–2201 (2016)
36. Guo, G., Tan, X., Touzi, N.: Tightness and duality of martingale transport on the Skorokhod space. Stoch. Process. Appl. **127**(3), 927–956 (2017)

37. Harrison, J.M., Pliska, S.R.: Martingales and stochastic integrals in the theory of continuous trading. Stoch. Process. Appl. **11**(3), 215–260 (1981)
38. Hou, Z., Obłój, J.: Robust pricing-hedging dualities in continuous time. Financ. Stoch. **22**(3), 1–57 (2018)
39. Jakubowski, A.: A non-Skorohod topology on the Skorohod space. Electron. J. Prob. **2**, 1–21 (1997)
40. Jakubowski, A.: New characterizations of the S-topology on the Skorokhod space. arXiv:1609.00215 (2016)
41. Jarchow, H.: Locally Convex Spaces. Mathematische Leitfäden. B.G. Teubner, Leipzig (1981)
42. Kantorovich, L.V.: On the transfer of masses. Dokl. Akad. Nauk. SSSR (in Russian) **37**(2), 227–229 (1942)
43. Kantorovich, L.V.: On a problem of Monge. Uspekhi Mat. Nauk. **37**(3), 225–226 (1948)
44. Karandikar, R.L.: On pathwise stochastic integration. Stoch. Process. Appl. **1**, 11–18 (1995)
45. Kellerer, H.G.: Duality theorems for marginal problems. Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete **67**(4), 399–432 (1984)
46. Kiiski, M.: The Riesz representation theorem and weak* compactness of semimartingales. arXiv:1707.09382 (2018)
47. Kisyński, J.: Convergence du type L. Colloq. Math. **2**(7), 205–211 (1960)
48. Maggis, M., Meyer-Brandis, T., Svindland, G.: Fatou closedness under model uncertainty. Positivity **22**, 1325–1343 (2018)
49. Perkowski, N., Prömel, D.J.: Pathwise stochastic integrals for model free finance. Bernoulli **22**(4), 2486–2520 (2016)
50. Protter, P.E.: Stochastic Integration and Differential Equations. Stochastic Modelling and Applied Probability. Springer, Berlin, Heidelberg (2005)
51. Rachev, S.T., Rüschendorf, L.: Mass Transportation Problems: Volume I: Theory, vol. 1. Springer, Berlin (1998)
52. Rüschendorf, L.: Mathematical Risk Analysis. Springer Ser. Oper. Res. Financ. Eng. Springer, Heidelberg (2013)
53. Schwartz, L.: Radon measures on arbitrary topological spaces and cylindrical measures. Tata. Inst. Fund. Res., Oxford Univ., Oxford (1973)
54. Sentilles, F.D.: Bounded continuous functions on a completely regular space. Trans. Am. Math. Soc. **168**, 311–336 (1972)
55. Strassen, V.: The existence of probability measures with given marginals. Ann. Math. Stat. **36**(2), 423–439 (1965)
56. Tong, H.: Some characterizations of normal and perfectly normal spaces. Duke Math. J. **19**(2), 289–292 (1952)
57. Villani, C.: Optimal Transport. Old and New of Grundlehren der mathematischen Wissenschaften, vol. 338. Springer, Berlin Heidelberg (2009)
58. Vovk, V.: Rough paths in idealized financial markets. Lithuanian Math. J. **51**(2), 274 (2011)
59. Vovk, V.: Continuous-time trading and the emergence of probability. Financ. Stoch. **16**(4), 561–609 (2012)
60. Zalinescu, C.: Convex Analysis in General Vector Spaces. World Scientific Publishing, Singapore (2002)