Trading with Impact

H. Mete Soner
ORFE, Princeton

SIAM Activity Group
Virtual Talk
April 16, 2020

I have benefited the collaboration of many people including : Albert Altarovici, Peter Bank, Umut Çetin, Grégoire Loeper, Ludovic Moreau, Dylan Possamaï, Max Reppen, Alexandre Roch, Moritz Voss, Chao Zhou and

Bruno Bouchard

Johannes Muhle-Karbe

Nizar Touzi

Consider a financial market in which our trades impact the current value of the stock. We would like to

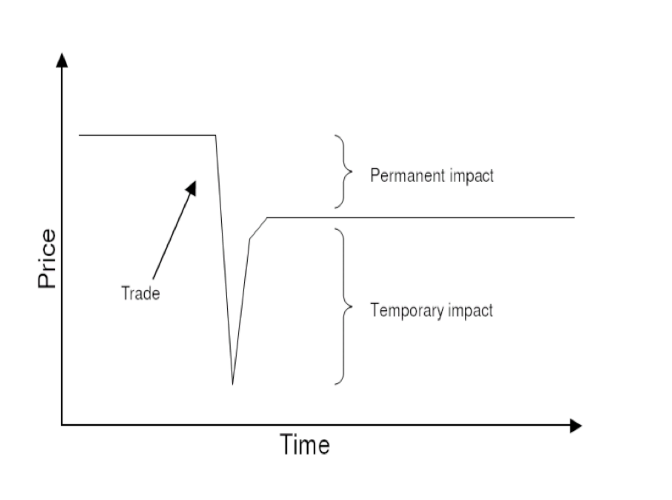
- model the frictions, in particular market impact,
- model the dynamics of this structure,
- study its impact on investment decisions.

The new computational techniques are making the computational approaches to high dimensional problems feasible. So, we should be able to incorporate more details in to our models and analyze them numerically.

Price Impact

Any model with friction is somehow related to this problem of price impact. Indeed, transaction costs can be taught as a particular price impact. Any positive amount of trade pushes the price to the ask-price and any sale to the bid.

In the context of hedging, Leland formally argued that transaction cost modifies the volatility. Later, Fukasawa and Rosenbaum & Tankov revisited this approach. To understand this modification, jointly with Barles we used asymptotics to obtain a modified Black & Scholes equation.


Price
Trade
Permanent impact
Temporary impact
Time

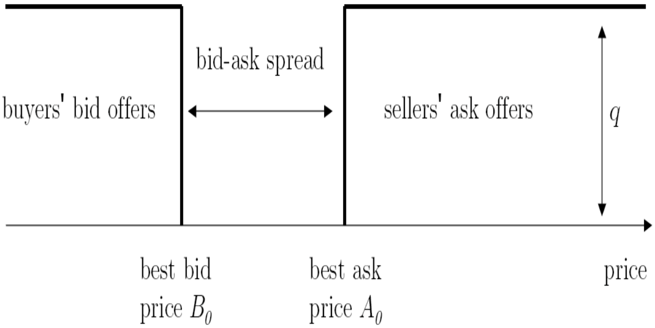


Fig. 1. The limit order book model before the large investor is active

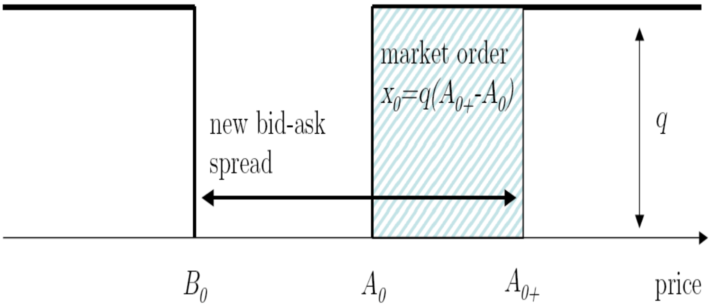


Fig. 2. Impact of a market buy order of x_0 shares

Outline

They postulate an exogenous supply curve

$$\boxed{\mathcal{S}(t, S_t, \nu), \qquad S_t = \mathcal{S}(t, S_t, 0)}$$

which gives the price per share for a transaction of size ν ($\nu > 0$ is a buy and $\nu < 0$ a sell). An example of the supply curve is the generalized Black-Scholes economy with liquidity parameter Λ :

$$\mathcal{S}(t, S_t, \nu) = S_t \exp(\Lambda \nu), \quad dS_t = S_t [\mu dt + \sigma dW_t].$$

We may simply take

$$\boxed{\mathcal{S}(t, S_t, \nu) = S_t + \Lambda \nu.}$$

This corresponds to a constant density LOB. And $1/\Lambda$ is the constant density. For a transaction of size ν we pay

$$\boxed{\nu S_t + \Lambda\, \nu^2 = \nu\left[\mathcal{S}(t, S_t, \nu) - S_t\right].}$$

Now imagine of splitting this order into two and execute them in tandem. Then we would pay

$$2\,\left[(\nu/2)S_t + \Lambda((\nu/2))^2\right] = \nu S_t + \frac{1}{2}\Lambda\, \nu^2.$$

Hence in this model, we would like to make small but many transactions. Hence the portfolio process should be continuous with no liquidity cost. But :

- In reality, each transaction takes some time to execute - finite speed of the portfolio. So, we carry the risk of stock price movements.
- Secondly, after the first transaction stock price moves and we will not be able to get the same price as before - i.e. there is resilience. But in the ideal model of Cetin, Protter and Jarrow, we do get the same price !

Simply, we restrict the portfolio and its gamma to be a semi-martingales with some bounds on the characteristics. Let $\mathcal{A}_{t,s}$ be this set of all admissible portfolios.

For a given a European contingent claim with payoff g, the super-replication cost is defined by

$$\boxed{V(t,s) = \inf\left\{y : Y_T^{t,y,Z} \geq g(S_T^{t,s}) \ a.s. \text{ for some } Z \in \mathcal{A}_{t,s}\right\}}$$

where

$$dS_t = S_t\left[\mu dt + \sigma dW_t\right]$$

$$Y_t = y + \int_0^t Z_u dS_u - \sum_{n=0}^{N-1} z_n\left[\mathcal{S}\left(\tau_n, z^n\right) - \mathcal{S}\left(\tau_n, 0\right)\right] 1_{\{t<\tau_{n+1}\}} - \int_0^t \frac{\partial \mathcal{S}}{\partial \nu}(u, S_u, 0)\ \Gamma_u^2 \sigma^2 S_u^2 du.$$

$$Z_t = \sum_{n=0}^{N-1} z_n 1_{\{t<\tau_{n+1}\}} + \int_0^t \alpha_u du + \int_0^t \Gamma_u dS_u.$$

Together with Çetin and Touzi, we showed that the super-replicating cost satisfies

$$0 = -V_t + \sup_{\beta \geq 0} \left(-\frac{1}{2} s^2 \sigma^2 (V_{ss} + \beta) - \Lambda s^2 \sigma^2 (V_{ss} + \beta)^2 \right),$$

together with terminal cost $V(T, s) = g(s)$. For a convex pay-off g, the solution also remains convex and the equation simplifies to

$$V_t = -\frac{1}{2} s^2 \sigma^2 V_{ss} - \Lambda s^2 \sigma^2 (V_{ss})^2 = -\frac{1}{2} s^2 \hat{\sigma}^2(t,s) V_{ss},$$

where

$$\boxed{\hat{\sigma}^2(t,s) = \sigma^2 \left[1 + 2\Lambda V_{ss}(t,s)\right].}$$

Hence the effect of liquidity is to increase the *effective volatility* as in Leland and Fukasawa. Same is true for non-convex pay-off's as well.

- By an application of maximum principle we have

$$V(t,s) \geq V_{BS}(t,s)$$

  and they are equal only when g is an affine function.
- This implies that there exists a strict liquidity premium, a difference between the super-replicating cost and the Black-Scholes value of the claim.
- The reason why there are contradicting results between CJP and the above is the trading strategy constraints.
- However, without introducing the resilience explicitly, the liquidity premium is weak and is does not impact the utility maximization problems.

Penalizing the speed

This is a phenomenological model by Almgren & Chriss (also important contributions by Rogers & Sign, Garleanu & Pedersen), considers an impact functional of the form

$$\mathcal{S}(t, S_t, Z'_t) = S_t + \Lambda Z'_t, \quad \text{where} \quad Z'_t := \frac{d}{dt} Z_t.$$

Then, the dynamics are given by

$$Y_t = \int_0^t Z_u dS_u - L_t$$

$$L_t = \Lambda \int_0^t \left(Z'_u\right)^2 du.$$

In these models, it is not possible to avoid the liquidity premium.

Consider a utility maximization problem

$$\boxed{\sup_{Z} \mathbb{E}\left[\ U\left(\mathcal{R}_T^Z\right)\ \right],}$$

where $\mathcal{R}_T$ is the risk adjusted liquidation cost of Schöneborn and is given by,

$$\mathcal{R}_T^Z := Y_T^Z - C\Lambda^2(Z_T - \theta_T^*)^2,$$

where C is a constant derived from the model and θ^* is optimal portfolio for the frictionless (i.e., $\Lambda = 0$) market.

There are two difficulties :

- Due to the price impact, we could only use portfolios that are differentiable in time. If the target portfolio θ^* is rough, the optimisation problem gives us a way to approximate this target portfolio.
- In addition to continuous targeting error, we have both initial and final liquidation costs.
- Initially, we might far from the optimal location and need to move there efficiently.
- Also, closer to maturity one must consider the final portfolio position.

The dynamic equation is for the value function $v(t,s,y,z)$ where s is the initial stock price, y is the initial wealth and z is the initial shares of stocks. Then, v solves

$$\boxed{-v_t - s\mu[v_s + v_y] - \frac{s^2\sigma^2}{2}[v_{ss} + v_{yy} + 2v_{ys}] + \inf_{\alpha}\left\{-\Lambda\alpha^2 v_y - \alpha v_z\right\} = 0.}$$

More realistic models have been analyzed ; see for instance Moreau, Muhle-Karbe & Soner Math. Fin. (2017), Bouchard, Loeper, Soner & Zhou SICON (2019).

Until recently it would be impossible to numerically solve the equation. However, recent papers by Hure, Pham & Warin ; Pham, Warin ; Bachoud, Hure, Langrene & Pham ; Han & E ; Buehler, Gonon, Teichmann & Wood allow for high-dimensional optimal control problems.

The main advantage of these methods is the ease at which they can handle complexity of the problems. In particular, one can easily add factors, frictions and other constraints in to the code.

The actual problem is not quite tractable and together with Moreau & Muhle-Karbe we considered the asymptotics as Λ gets smaller.

We have asymptotic results for the value function and also for optimal portfolio.

The rigorous proof uses machinery from viscosity solutions, developed jointly with Possamaï and Touzi, which I do not report here. Only I only outline the asymptotic structure of the hedge.

Let $Z^* = Z^{*,\Lambda}$ be the optimal portfolio for the utility maximization problem with small but non-zero impact $\Lambda > 0$. Recall that θ^* is the frictionless optimizer.

Asymptotically,

$$\boxed{\frac{d}{dt} Z_t^* = c\, \Lambda^{-1/2} \left(Z_t^* - \theta_t^*\right), \quad \text{where} c = \frac{\sigma}{\sqrt{2R_t}},}$$

and R_t is the frictionless investor's indirect risk-tolerance process, i.e., the risk tolerance of the frictionless value.

As Λ gets smaller, Z^* moves very quickly towards the frictionless optimizer θ^*.

Outline

Joint with Bank & Voss, we consider the following tracking problem for a given portfolio process θ_t^*,

$$\text{minimize} \quad J(u) := J(u; x, \theta^*),$$

where

$$J(u; x, \theta^*) := \frac{1}{2} \int_0^T \left[(Z_t - \theta_t^*)^2 + \Lambda (Z_t')^2 \right] dt,$$

$$Z_t := z + \int_0^t Z_s' ds.$$

The above model is motivated by recent papers of Bank & Voss and also Kallsen & Muhle-Karbe. It was also considered in Rogers & Sign but solved only approximately.

The optimizer $Z^* = Z^{*\Lambda}$ has a very similar structure to the asymptotic formula already discussed in the impact model. Indeed, it solves

$$\boxed{\frac{d}{dt} Z_t^* = c\ \Lambda^{-1/2}\ (\theta_t^* - (\mathcal{L}\theta^*)_t),}$$

where $\mathcal{L}\theta^*$ is a linear map of θ^* depending on the parameter Λ. Roughly, it is the adapted projection of the forward convolution of θ^*.

So, instead of targeting directly the target portfolio θ_t^* at time t, we target an estimate of the possible future values of the target. This was also obtained by Garleanu & Pedersen.

Garleanu & Pedersen quote Wayne Gretzky, "*A great hockey player skates to where the puck is going to be, not where it is.*"

Equilibrium

There has been quite a lot of recent work. In particular,

- Lukas Gonon, Johannes Muhle-Karbe, Xiaofei Shi, 2020
- Martin Herdegen, Johannes Muhle-Karbe, Dylan Possamaï, 2019
- Xiao Chen, Jin Hyuk Choi, Kasper Larsen, Duane Seppi, 2019
- Peter Bank, Ibrahim Ekren, Johannes Muhle-Karbe, 2018
- Jin Hyuk Choi, Kasper Larsen, Duane Seppi, 2018
- Hao Xing, Gordan Zitkovic, 2017
- Kostas Kardaras, Hao Xing, Gordan Zitkovic, 2015

- There are a rich class of models for illiquid markets with price impact.
- Another use of this approach is to assume that target portfolio is given but not implementable. This would give us away to provide implementable approximations.
- Asymptotics makes things tractable.
- New computational techniques allow for more complex models.

- Second order stochastic target problems with generalized market impact, Bouchard, Loeper, Soner & Zhou, SICON, 2019,
- A primer on portfolio choice with small transaction costs, Muhle-Karbe, Reppen & Soner, Annual Review of Financial Economics, (2017).
- Hedging with temporary price impact, Bank, Soner &Voss, MAFE, (2017).
- Trading with small impact, Moreau, Muhle-Karbe & Soner, Mathematical Finance, (2017).
- Homogenization and asymptotics for small transaction costs, Soner & Touzi, SICON, (2013).
- Option hedging for small investors under liquidity costs, Çetin, Soner & Touzi, F&S, (2010).

THANK YOU FOR YOUR ATTENTION.