Martingale Optimal Transport

H. Mete Soner
ORFE, Princeton

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Mostly Joint with Yan Dolinsky, Hebrew University

Collaborators

- Matteo Burzoni, ETH Zürich, now Oxford
- Bruno Bouchard, Paris Dauphine,
- Patrick Cheridito, ETH Zürich,
- Ibrahim Ekren, Florida State,
- Matti Kiiski, ETH Zürich, now Mannheim
- Marcel Nutz, Columbia,
- Frank Riedel, University of Bielefeld,
- Dylan Possamai, Columbia,
- David Prömel, Oxford, now Mannheim
- Nizar Touzi, Ecole Polytechnique,
- Jianfeng Zhang, University of Southern California.

Related Works

I have benefited from the papers of many people including,

- Bartl, Kupper, Konstanz, Perkowski, Berlin, Siefried Trier,
- Biagini, Meyer-Brandis, Svindland, Munich, Huesmann, Bonn,
- Beiglböck, Källblad, Schachermayer, Vienna,
- De March, Denis, Martini, Matoussi, Ren, Tan, Paris,
- Cox, Bath, Bayraktar, Hermann, Michigan, Hobson, Warwick,
- Aksamit, Cohen, Guo, Hou, Obloj, Oxford, Biagini, Pisa,
- Stebegg, Columbia, Tangpi, Princeton, Acciaio, London,
- Cassese, Fritelli, Maggis, Milan, Henry-Labordère (SG)
- Hu, Rennes, Peng, Shandong, Zhou Singapore.

Outline

Knightian Uncertainty

Uncertainty in Finance

Risk Neutral pricing

An Example - VIX

Martingale Optimal Transport

Duality

Concluding

Knightian Uncertainty

Uncertainty, Risk and Discussions

- According to Frank Knight (1921) risk may be represented by numerical probabilities while uncertainty cannot.
- So risk is measurable uncertainty while uncertainty is unmeasurable uncertainty.
- Arrow (1951) finds this separation unneccessary and claims that nothing is gained.
- Schackle (1955) argues that probabilistic structure can not be used in decisions.
- Savage (1954) takes a systematic approach and shows that if the decisions are sufficiently orderly (i.e., if they satisfy certain axioms), then one can infer subjective probabilities.

After Savage, people taught that maybe all of Knight's "unmeasurable uncertainties" could have succumbed to measurement and "risk" would prevail instead of "uncertainty". Of course, this hinges upon the appropriateness of the axioms.

Ellsberg Experiment 1961

Consider the experiment with two urns containing red and black balls. We know that

- Urn 1 has exactly 50 red and 50 black balls - (risky urn) ;
- Urn 2 has red and black balls of unknown number - (uncertain urn).

Four random events are described :

- R_i : Pays $1 if a Red ball is from from Urn i ;
- B_i : Pays $1 if a Black ball is from from Urn i ;
- Most people strictly prefer R_1 over R_2 and B_2 and B_1 over R_2 and B_2 ;
- They are indifferent between R_1 and B_1 and R_2 and B_2 ;
- No probability measure can explain this.

This experiment refutes the claim of Arrow by showing that people react differently to risk and uncertainty. Hence, there is uncertainty aversion.

Uncertainty in Finance

Alan Greenspan

Alan Greenspan served as the Chairman of the Federal Reserve of the United States from 1987 to 2006. In his May 2004 speech,

INNOVATIONS AND ISSUES IN MONETARY POLICY: THE LAST FIFTEEN YEARS

Risk and Uncertainty in Monetary Policy

By Alan Greenspan*

I plan to sketch the key developments of the past decade and a half of monetary policy in the United States from the perspective of someone who has been in the policy trenches. I will offer some conclusions about what I believe has been learned thus far, though I suspect, as is so often

has been operating in an environment particularly conducive to the pursuit of price stability. The principal features of this environment included (i) increased political support for stable prices, which was the consequence of, and reaction to, the unprecedented peacetime inflation

Alan Greenspan

The Federal Reserve's experiences over the past two decades make it clear that uncertainty is not just a pervasive feature of the monetary policy landscape ; it is the defining characteristic of that landscape.

When confronted with uncertainty, especially Knightian uncertainty, human beings invariably attempt to disengage from medium - to long-term commitments in favor of safety and liquidity

In fact, uncertainty characterized virtually every meeting, and as the transcripts show, our ability to anticipate was limited.

A REGULATORY RESPONSE TO THE FINANCIAL CRISIS

"More fundamentally, however, it is important to realize that the assumption that past distribution patters carry robust inferences for the probability of future patterns is methodologically insecure. It involves applying to the world of social and economics relationships to a technique drawn from the world of physics, . . . it is unclear whether this analogy is valid when applied to economic and social relationships, or whether instead, we need to recognize that we are not dealing with mathematically modellable risk, but with inherent 'Knightian uncertainty'. "

Definitions

- Risk is when we use one dominating probability measure.
- Robust approach uses a set of non-dominated probability measures, $\mathcal{P}$. In this case, all inequalities or equalities are to be understood $\mathcal{P}$ quasi-surely, i.e., $\mathbb{P}$ almost surely, for every $\mathbb{P} \in \mathcal{P}$.
- Model independent approach takes the extreme point of view and allows all trajectories as possible stock price processes.

Risk Neutral pricing

Option Prices

- $(S_t)_{t\geq 0}$ future price process. It is random and not known.
- A contingent claim (or an option) maturing at a future date T is a deterministic function of the stock price path $(S_t)_{t\in[0,T]}$.
- Call option with strike k pays $(S_T - k)^+$. Put option with strike k pays $(k - S_T)^+$.
- Asian option with strike k pays $(\int_0^T S_t dt - k)^+$.
- Finance theory dictates that liquidly traded options are priced by a risk neutral measure $\mathbb{Q}$ and the discounted stock price is a $\mathbb{Q}$-martingale. Hence, $S_0 = \mathbb{E}_{\mathbb{Q}}[e^{-rT} S_t]$ for any t.
- Also,

$$C(k) = \mathbb{E}_{\mathbb{Q}}[e^{-rT}(S_T - k)^+], \quad P(k) = \mathbb{E}_{\mathbb{Q}}[e^{-rT}(k - S_T)^+].$$

Breeden & Litzenberger, 1978

- Prices of liquidly traded options can be use to estimate the risk neutral measure non-parametrically.
- Indeed, the set of functions

$$f_k : \mathbb{R}_+ \mapsto f_k(x) := (x - k)^+,$$

  with k ranging over positive real line, is a separating class.
- Let μ be the distribution of S_T under $\mathbb{Q}$ and assume $r = 0$. Then, $C(k) = \mu(f_k)$.
- Also, $(k - x)^+ = (x - k) - f_k(x)$ and $P(k) = S_0 - k + C(k)$.

Theorem (Breeden & Litzenberger, 1978)

The prices $\{C(k)\}_{k \geq 0}$ uniquely determines μ. In particular, call and put prices provide a good estimate of the risk neutral measure at a future date :

$$\mu([a, \infty)) = \mathbb{Q}(S_T \geq a) = -C'(a), \quad \forall a \geq 0.$$

Breeden & Litzenberger calculates $\mathbb{E}_{\mathbb{Q}}\left[\ln(\mathbb{S}_T)\right]$ in terms of the Call and Put prices. We start with the identity which holds for any f and $a > 0$,

$$f(x) = f'(a)(x-a) + \int_0^a f''(k)(k-x)^+ \, dk + \int_a^\infty f''(k)(x-k)^+ \, dk.$$

For the logarithm,

$$\ln(x/a) = \frac{x-a}{a} - \int_0^a \frac{(k-x)^+}{k^2} \, dk - \int_a^\infty \frac{(x-k)^+}{k^2} \, dk.$$

We take expected value with $x = S_T$ and $a = S_0$:

$$\boxed{\mathbb{E}_{\mathbb{Q}}\left[\ln(S_T/S_0)\right] = -\frac{2}{T}\left[\int_0^{S_0} \frac{P(k)}{k^2} \, dk + \int_{S_0}^\infty \frac{C(k)}{k^2} \, dk\right].}$$

Aït-Sahalia & Lo,1997 : estimating Q

- Suppose that the prices of all liquidly traded options are given by a risk neutral measure $\mathbb{Q}$.
- Let $\{\xi_a\}_{a\in A}$ be the future random pay-offs of liquidly traded claims and $\{p_a\}_{a\in A}$ be their prices.
- Then, we have the following linear constraints on the measure $\mathbb{Q}$:

$$\mathbb{E}_{\mathbb{Q}}[\xi_a] = p_a, \quad \forall a \in A.$$

- One can use these constraints to estimate the complete $\mathbb{Q}$..
- The seminal paper Aït-Sahalia & Lo(JF, 1997) provides a non-parametric estimation of $\mathbb{Q}$.
- Our approach is to use these constraints to give upper and lower bounds.

An Example - VIX

The powerful and flexible trading and risk management tool from the Chicago Board Options Exchange

THE CBOE VOLATILITY INDEX® - VIX®

Goal is to price the future, random, realized volatility over the a period of $[0, T]$. We restrict stock processes to diffusions :

$$dS_t = S_t[rdt + \sigma_t dW_t],$$

where W is Brownian motion, σ_t is unknown. Then, $VIX := 100\sqrt{\mathbb{E}_{\mathbb{Q}}[\frac{1}{T}\int_0^T \sigma_t^2 dt]}$ with unknown $\mathbb{Q}$. By Ito's formula, $d\ln(S_t) = -\frac{1}{2}\sigma_t^2 dt - \frac{dS_t}{S_t}$. Hence,

$$\frac{1}{T}\int_0^T \sigma_t^2 dt = -\frac{2}{T}\left[\ln(S_T/S_0) + \int_0^T \frac{1}{S_t} dS_t\right].$$

So the price of $100VIX^2$ is given by

$$(\frac{VIX}{100})^2 = -\frac{2}{T}\mathbb{E}_{\mathbb{Q}}[\ln(S_T/S_0)] = \frac{2}{T}\left[\int_0^{S_0} \frac{P(k)}{k^2}\, dk + \int_{S_0}^{\infty} \frac{C(k)}{k^2}\, dk\right].$$

This is exactly the definition used by Chicago Board of Exchange.

THE VIX CALCULATION STEP-BY-STEP

Stock indexes, such as the S&P 500, are calculated using the prices of their component stocks. Each index employs rules that govern the selection of component securities and a formula to calculate index values.

VIX is a volatility index comprised of *options* rather than stocks, with the price of each option reflecting the market's expectation of future volatility. Like conventional indexes, VIX employs rules for selecting component options and a formula to calculate index values.

The generalized formula used in the VIX calculation[§] is:

$$\sigma^2 = \frac{2}{T}\sum_i \frac{\Delta K_i}{K_i^2} e^{RT} Q(K_i) - \frac{1}{T}\left[\frac{F}{K_0} - 1\right]^2 \qquad \textbf{(1)}$$

WHERE…

σ is	$VIX/100 \Rightarrow VIX = \sigma \times 100$
T	Time to expiration
F	Forward index level derived from index option prices
K_0	First strike below the forward index level, F
K_i	Strike price of i^{th} out-of-the-money option; a call if $K_i > K_0$ and a put if $K_i < K_0$; both put and call if $K_i = K_0$.
ΔK_i	Interval between strike prices – half the difference between the strike on either side of K_i:

$$\Delta K_i = \frac{K_{i+1} - K_{i-1}}{2}$$

Martingale Optimal Transport

Forward Start Option

- Here trading is at two future points $0 < t_1 < T$.
- Let μ be the t_1 marginal of $\mathbb{Q}$ and ν at time T.
- So if sufficiently many call or puts expiring at t_1 and at T are liquidly traded we can approximate μ and ν effectively.
- Problem is to find model independent bounds for the forward-start option $c(S_{t_1}, S_T) = (S_T - S_{t_1})^+$.
- Let $\mathcal{M}(\mu, \nu)$ be the set of all martingale measures such that the marginals at t_1 and T are μ and ν.
- Then,

$$\inf_{\mathbb{Q} \in \mathcal{M}(\mu,\nu)} \mathbb{E}_{\mathbb{Q}}[c(S_{t_1}, S_T)], \qquad \sup_{\mathbb{Q} \in \mathcal{M}(\mu,\nu)} \mathbb{E}_{\mathbb{Q}}[c(S_{t_1}, S_T)],$$

  are lower and upper bounds for the price of forward-start options.
- Without the martingality constraint this is classical optimal transport.

Pierre Henry-Labordère introduced this problem :

- Given two probability measures μ, ν on $\mathbb{R}^d$.
- $\mathbb{Q} \in \mathcal{M}(\mu, \nu)$ is a probability measure on $\mathbb{R}^d \times \mathbb{R}^d$ with marginals μ, ν and

$$\mathbb{E}_{\mathbb{Q}}[\gamma(S_{t_1}) \cdot (S_T - S_{t_1})] = \int_{\mathbb{R}^d \times \mathbb{R}^d} \gamma(x) \cdot (y - x) \mathbb{Q}(dxdy) = 0,$$

  for every bounded, Borel $\gamma : \mathbb{R}^d \to \mathbb{R}^d$ (this means $\mathbb{E}_{\mathbb{Q}}[S_T \, | S_{t_1}] = S_{t_1}$.)
- Due to Strassen (1965) $\mathcal{M}(\mu, \nu)$ is non empty iff μ and ν are in convex increasing order.
- For any Borel function ξ on $\mathbb{R}^d \times \mathbb{R}^d$, the primal (or pricing) problem is

$$\mathcal{P}(\xi) := \sup_{\mathbb{Q} \in \mathcal{M}(\mu,\nu)} \mathbb{E}_{\mathbb{Q}}[\xi].$$

Multi Periods

- Beiglböck, Henry-Labordère, Penkner (MF, 2013) discrete time.
- Galichon, Henry-Labordère, Touzi (AAP, 2014) stochastic control in continuous time.
- Dolinsky, Soner (PTRF, 2014) in continuous time.

Duality

Theorem (Pricing-Hedging Duality)

For any continuous function ξ *on* $\mathbb{R}^d \times \mathbb{R}^d$,

$$\mathcal{P}(\xi) := \sup_{\mathbb{Q}\in\mathcal{M}(\mu,\nu)} \mathbb{E}_{\mathbb{Q}}[\xi]$$
$$= \Phi(\xi) := \inf\{\mu(f) + \nu(h) \ : \ \exists\gamma \ s.t. \ f(x) + h(y) + \gamma(x)\cdot(y-x) \geq \xi(x,y)\}.$$

Note that for the Monge-Kantorovich problem, the dual is given by

$$\Phi_{ot}(\xi) := \inf\{\mu(f) + \nu(h) \ : \ f(x) + h(y) \geq \xi(x,y)\}.$$

The difference is the term $\gamma(x)\cdot(y-x)$.

Proof

- Let $\mathcal{X} = \mathcal{C}_b(\mathbb{R}^d \times \mathbb{R}^d)$ be the Banach space of bounded, continuous functions.

$$\Phi : \xi \in \mathcal{X} \mapsto \Phi(\xi) := \inf\{\mu(f) + \nu(h) \ : \ \exists \gamma \ s.t. \ f(x) + h(y) + \gamma(x) \cdot (y - x) \geq \xi(x, y)\}.$$

- Φ is Lipschitz continuous and convex and homogenous : $\mathcal{D}(\lambda\xi) = \lambda\mathcal{D}(\xi)$ for every $\lambda > 0$.
- By Fenchel - Moreau theorem,

$$\Phi(\xi) = \sup_{\varphi \in \mathcal{X}^*} [\varphi(\xi) - \Phi^*(\varphi)].$$

- The convex dual of Φ is given by,

$$\Phi^*(\varphi) = \sup_{\xi \in \mathcal{X}} [\varphi(\xi) - \Phi(\xi)], \quad \varphi \in \mathcal{X}^*.$$

- It follows directly that Φ^* is either zero or infinity, and

$$\mathcal{M}(\mu, \nu) = \{\Phi^* = 0\} = \{\varphi \in \mathcal{X}^* \ : \ \varphi(\xi) \leq \Phi(\xi) \ \forall \xi \in \mathcal{X} \ \}.$$

Extensions

- This duality extends to finite discrete time directly.
- It extends to upper-semicontinuous functions by standard arguments.
- It does not hold for general measurable functions.
- This is in contrast to optimal transport ; Kellerer 1984 proves duality for measurable functions by an application of the Choquet theorem.
- It extends to continuous time but for uniformly continuous functions.

This is from Cheredito, Kiiski, Prömel, Soner (2019)

- $\mathcal{D}([0,T];\mathbb{R}^d_+)$ is the Skorokhod space of all càdlàg functions, i.e., functions $\omega : [0,T] \to \mathbb{R}^d_+$ that are continuous from the right and have finite left limits.
- Uniform norm is not appropriate. We endow $\mathcal{D}([0,T];\mathbb{R}^d_+)$ with the S-topology of Jakubowski.
- Ω is a closed subset of $\mathcal{D}([0,T];\mathbb{R}^d_+)$ (In OT or MOT, $\Omega = \mathbb{R}^d \times \mathbb{R}^d$.)
- Ω represents all possible stock price paths.
- $\mathcal{X} = \mathcal{C}_b(\Omega)$ be the set of continuous and bounded functions on Ω (could be more general).
- $\mathcal{G} \subset \mathcal{X}$ is a cone of functions who are liquidly traded and have price at most zero. In MOT

$$\mathcal{G}_{mot} := \{\gamma(x,y) = f(x) - \mu(f) + h(y) - \nu(h) + \gamma(x) \cdot (y-x)\ \}.$$

Potential pricing functionals $\mathcal{Q}(\mathcal{G})$ must respect the market data $\mathcal{G}$.

Definition (Risk Neutral Measures)

A (countably additive) probability measure $\mathbb{Q}$ is in $\mathcal{Q}(\mathcal{G})$ provided that

- $\mathbb{E}_{\mathbb{Q}}[\gamma] \leq 0, \quad \forall \gamma \in \mathcal{G}$.
- *The canonical process $S_t(\omega) := \omega(t)$ is a $(\Omega, \mathcal{F}, \mathbb{Q})$ martingale :*

1. *for every $t \in [0, T]$, $X_t \in \mathcal{L}^1(\Omega, \mathbb{Q})$;*
2. $\mathbb{E}_{\mathbb{Q}}[Y \cdot (X_t - X_T)] = 0$ *for every $t \in [0, T]$, $\mathcal{F}_t$ measurable, bounded Y ;*
3. $\mathcal{F} = (\mathcal{F}_t)_{t \in [0,T]}$ *is the right-continuous version of the canonical filtration.*

Pricing - Hedging Duality

Theorem (Cheredito, Kiiski, Prömel, Soner)

For any continuous ξ with appropriate growth

$$\sup_{\mathbb{Q}\in\mathcal{Q}(\mathcal{G})} \mathbb{E}_{\mathbb{Q}}[\xi] = \inf\{c \; : \exists\, \gamma \in \mathcal{G}, \;\; H \in \mathcal{H}, \; s.t. \; c + \gamma + (H \cdot S)_T \geq \xi\},$$

where $(H \cdot S)_t$ is the stochastic integral and the class $\mathcal{H}$ is an appropriate class of integrands.

- The technical challenge is to prove that countably additive measures are sufficient.
- There is an extention to measurable claims. It uses the Choquet theorem but requires a larger class of integrals.
- All results are essentially optimal.

Concluding

- One can use the market data to restrict possible prices.
- Combined with model assumptions this approach reduces model dependency.
- In risk management it has many applications as recorded in the book of Rachev & Rüschendorf.
- Martingale Optimal Transport or more generally Pricing-Hedging duality is a central tool to understand Knightian Uncertainty - see Burzoni, Riedel, Soner for viability, arbitrage in this context.
- Less restrictive setting of robust finance requires calculus with multiple probability measures as developed by S., Touzi, Zhang.

Thank you for your Attention

- *Martingale Optimal Transport Duality.* Cheredito, Kiiski, Prömel, Soner, (2019) ;
- *Viability and Arbitrage under Knightian Uncertainty.* Burzoni, Riedel, Soner, (2019) ;
- *Constrained Optimal Transport.* Ekren, Soner (Archive for Rat. Mech. and Analysis, 2018) ;
- *Martingale Optimal Transport in the Skorokhod Space.* Dolinsky, Soner, (SPA, 2015) ;
- *Martingale Optimal Transport and Robust Hedging in Continuous Time.* Dolinsky, Soner, (PTRF, 2014).