Optimal Dividends

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- Consider a firm that generates random revenue from its fixed assets and is financed through its equity and debt.
- We model firm's cash flow by an exogenous diffusion process.
- Two controls are the size of the dividends and the size and the time of the issuance.
- It faces bankruptcy when it cannot meet its debt obligations.
- Following the literature we first study the problem in which dividends could be paid at any time.
- Then, restrict the dividends to an a-priori determined schedule and investigate the difference between the problems.
- This problem has its counter-parts in the insurance literature as the De Finnetti-Kramer problem.
- Also related to the real option context of Dixit & Pindyck.

Outline

Continuously paid Dividends

There is no literature on the periodic case expect our paper : *Discrete dividend payments in continuous time* J. Keppo, M. Reppen, H.M.S.

All the papers cited below are without this restriction.

- *Optimal dividends with random profitability* (Math. Finance 2019), M. Reppen, J.-C. Rochet, H.M.S.
- Corporate Liquidity and Capital Structure (RFS, 2012), by Anderson & Carverhill. This is very closely related to the above paper.
- *Free Cash-Flow, Issuance Costs and Stock Price Volatility* (JF, 2011), by Décamps, Mariotti, Rochet & Villeneuve.
- Capital supply uncertainty, cash holdings, and investment (RFS, 2015), by Hugonnier, Malamud & Morellec.

For insurance see the papers by Albrecher and collaborators.

Goal : Assign a value to a cash flow under a liquidity constraint.

Goal : Understand the capital allocation of a limited liability firm.

Means : Model maximization of expected value of future discounted dividends.

One Dimensional Models

There is no liquidity constraint and can borrow instantaneously. Then, the dividend rate is equal to the cash flow rate. Also it is not optimal to hold cash.

This is equivalent to cash flow evaluation with an option to exit (bankruptcy) and the firm value is then given by,

$$\boxed{V(x,\mu) = x + \sup_{\tau \geq 0} \mathbb{E}\left[\int_0^{\tau} e^{-rt}\, \mu_t dt\right].}$$

Here $\tau \geq 0$ is the time of exit, or equivalently, strategic bankruptcy. It is chosen optimally by the firm.

- Firm has cash reserves

$$dX_t^L = \mu dt + \sigma dW_t - dL_t, \quad X_0^L = x,$$

where $\mu, \sigma > 0$ and L_t (adapted, increasing, RCLL, $\Delta L_t \leq X_{t-}$) denotes *cumulative dividends paid up to time* t.

- Time of ruin : $\theta(L) := \inf\{t > 0 : X_t^L < 0\}$.
- The payoff of a dividend policy L is :

$$J(x; L) = \mathbb{E}\left[\int_0^{\theta(L)} e^{-rt} dL_t\right].$$

- The value of the cash flow/firm is the maximum value of :

$$V(x) = \sup_L J(x; L).$$

- Compared to Dixit & Pindyck, here the profitability μ is constant, dividends are always positive and bankruptcy is not chosen. Without this constraint, one can take $dL_t = \mu dt + \sigma dW_t$.

The value function solves

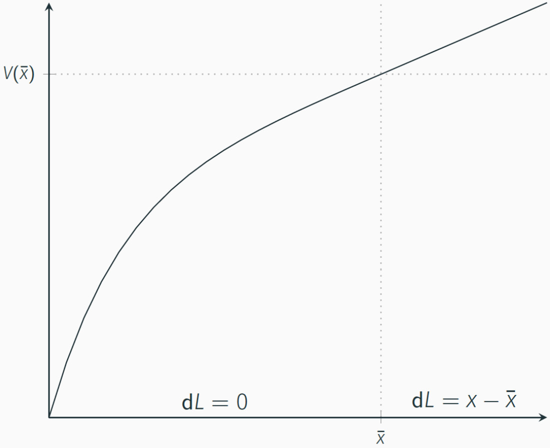
$$\boxed{\min\left\{rV - \mu V' - \frac{\sigma^2}{2}V'', V' - 1\right\} = 0, \quad V(0) = 0.}$$

The optimal solution is of barrier type, so the state space is divided into two regions :

No-dividend region : For some $\bar{x}$, it is not optimal to pay any dividends in $[0, \bar{x})$. In this region $rV - \mu V' - \frac{\sigma^2}{2}V'' = 0$.

Dividend region : In the region $[\bar{x}, \infty)$, $V' = 1$, suggesting $V(x) = V(\bar{x}) + (x - \bar{x})$. Hence, any cash in excess of $\bar{x}$ is paid as dividends.

Dividends are paid according to a barrier strategy: lump sums above $\bar{x}$ and according to local time at $\bar{x}$.



Random profitability

- Cash flow rate :

$$d\mu_t = \kappa(\mu_t)dt + \tilde{\sigma}(\mu_t)dW_t, \quad \mu_0 = \mu.$$

- Examples of profitability are Ornstein–Uhlenbeck or CIR processes.
- Reserves at time t with cumulative dividends L (adapted, increasing, RCLL, $\Delta L_t \le X_{t-}$) :

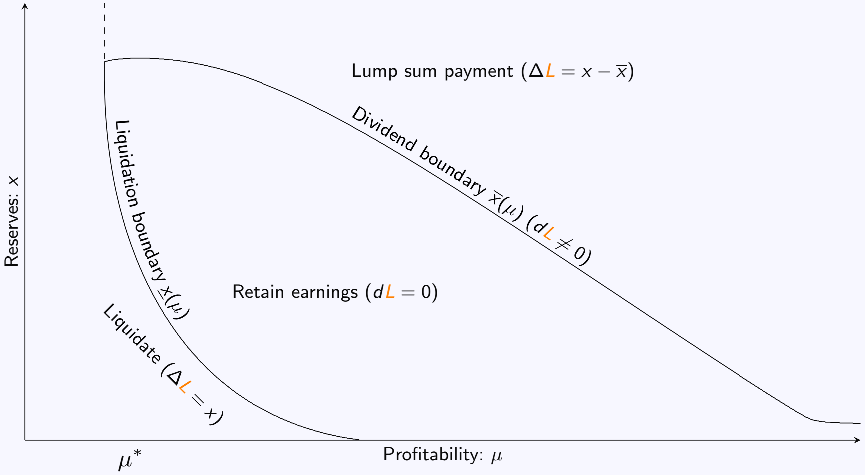
$$X_t^L = x + \int_0^t \mu_s ds + \sigma W_t - L_t.$$

- In particular, at certain times μ_t could be negative.
- For negative μ values, there is a balance between :

  liquidate immediately and collect x ;

  wait until the profitability is positive and pay the cost of discounting ;

  also waiting has the probability of bankruptcy due to depletion of the cash reserves.


Lump sum payment ($\Delta L = x - \overline{x}$)
Dividend boundary $\overline{x}(\mu)$ ($dL \neq 0$)
Liquidation boundary $\underline{x}(\mu)$
Retain earnings ($dL = 0$)
Liquidate ($\Delta L = x$)
Reserves: x
Profitability: μ
μ^*

As before, we maximize present value of dividends until ruin :

$$V(x,\mu) = \sup_{L} \mathbb{E}\left[\int_0^{\theta(L)} e^{-rt} dL_t\right].$$

Then,

$$\boxed{\min\{rV - \mathcal{A}V, V_x - 1\} = 0, \quad V(0,\cdot) \equiv 0,}$$

$$\mathcal{A} = \mu\partial_x + \kappa(\mu)\partial_\mu + \frac{\sigma^2}{2}\partial_{xx} + \frac{\rho\sigma\tilde{\sigma}}{2}\partial_{x\mu} + \frac{\tilde{\sigma}^2}{2}\partial_{\mu\mu}.$$

Theorem (Comparison)

Let u and v be upper and lower semicontinuous, polynomially growing viscosity sub- and super-solutions of the DPE. Then $u \leq v$ for $x = 0$ implies that $u \leq v$ everywhere.

Why comparison useful

Facts

- *The value function is continuous and satisfies the dynamic programming principle.*
- *The value function is a viscosity solution.*
- *There is comparison for the dynamic programming equation in an appropriate class.*
- *The numerical scheme converges to the unique viscosity solution ; hence, to the value function.*

- We now add the possibility of equity issuance or capital injection.
- It can be done at any time but with proportional $\lambda_f \geq 0$ and fixed costs $\lambda_f \geq 0$.
- Then,

$$dX_t^{L,I} = dC_t - dL_t + dI_t.$$

- *Pay-off functional :*

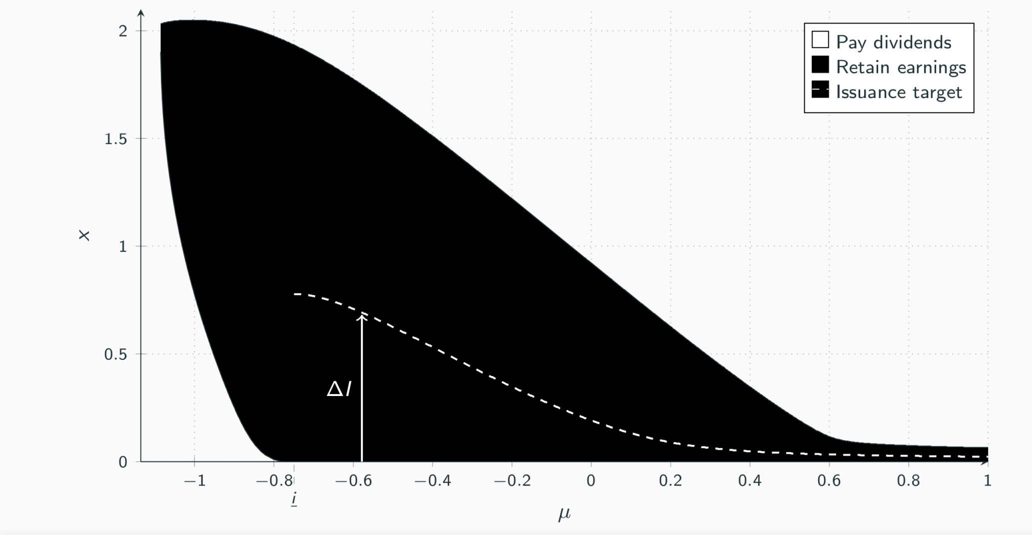
$$\boxed{J(x; L, I) = \mathbb{E}\left[\int_0^{\theta(L,I)} e^{-rt} dL_t - \sum_{t\geq 0} e^{-rt}(\lambda_f + (1+\lambda_p)\Delta I_t)1_{\{\Delta I_t > 0\}}\right],}$$

where as before $\theta(L, I) := \inf\{t > 0 : X_t^{L,I} < 0\}$. *and* $V(x) = \sup_{L,I} J(x; L, I)$.

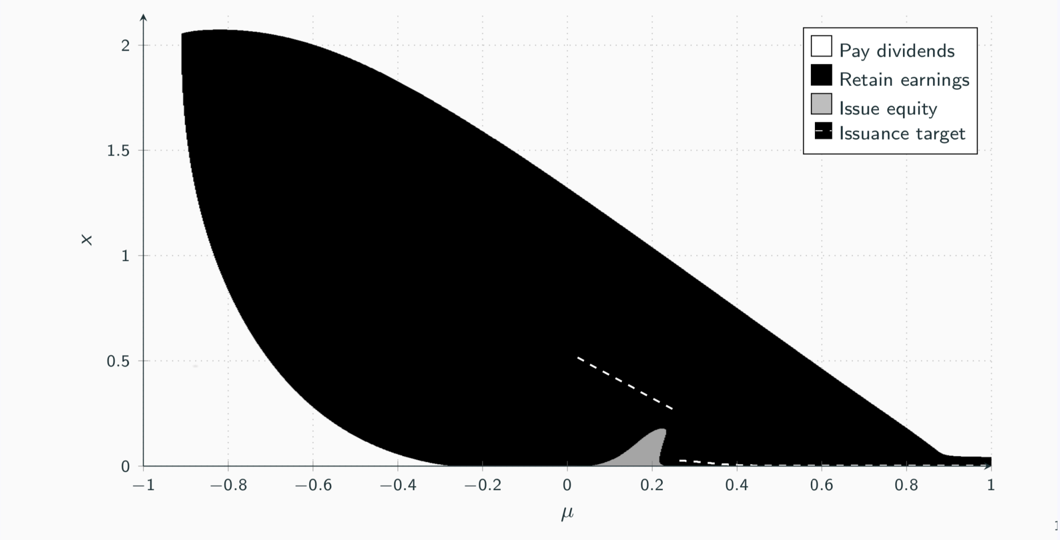
- *The control I could be used to avoid bankruptcy.*

Numerical Results

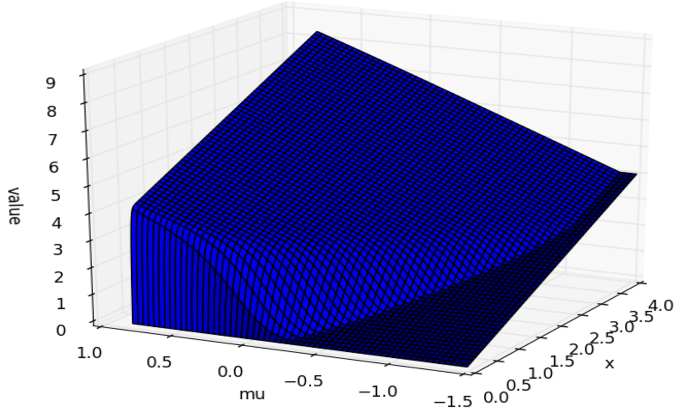
Random Profitability



Random Profitability - II



Value Function



- Let L^1 and L^2 be two strategies and define $\bar{L} = \frac{L^1+L^2}{2}$.
- Then, $\theta(\bar{L}) < \theta(L^1) \vee \theta(L^2)$ may happen, suggesting dividend payment after ruin for the midpoint strategy.
- Due to this, the value function is not necessarily concave.

Discrete dividends :
Abstract description

Abstract formulation

State process $X^{\alpha,\beta}$ controlled by

- a continuous control α at $t \in \mathbb{R}_{\geq 0}$ with a continuous cost rate of $F_t^{\alpha} = F(\alpha_t, X_t^{\alpha,\beta})$ and
- a discrete control β at $t = 0, T, 2T, \ldots$ moving state to $x + \beta$ and accumulating cost of G_t^{β} at discrete times.

The value function is

$$\boxed{V(x) := \sup_{\alpha,\beta} \mathbb{E}\left[\int_0^{\infty} e^{-rt} dF_t^{\alpha} + \sum_{k=0}^{\infty} e^{-rkT} G_{kT}^{\beta} \,\middle|\, X_{0-}^{\alpha,\beta} = x\right],}$$

Applying the dynamic programming principle,

$$\boxed{V(x) = \sup_{\alpha,\beta} \mathbb{E}\left[\int_0^T e^{-rt} dF_t^{\alpha} + G(\beta, x) + e^{-rT} V(X_{T-}^{\alpha,\beta}) \,\middle|\, X_{0-}^{\alpha,\beta} = x\right],}$$

Define continuous and discrete operators

$$\mathcal{L}\varphi(x) := \sup_{\alpha} \mathbb{E}\left[\int_0^T e^{-rt} dF_t^{\alpha} + e^{-rT} \varphi(X_T^{\alpha,0}) \,\middle|\, X_0^{\alpha,0} = x\right],$$

$$\mathcal{D}\varphi(x) := \sup_{\beta} \left(\varphi(x+\beta) + G(\beta, x)\right).$$

Let $\mathcal{T} := \mathcal{D} \circ \mathcal{L}$. Then,

$$\boxed{V = \mathcal{T} V.}$$

Recall that $V = \mathcal{T}V$ with

$$\mathcal{L}\varphi(x) := \sup_{\alpha} \mathbb{E}\left[\int_0^T e^{-rt} dF_t^{\alpha} + e^{-rT}\varphi(X_T^{\alpha,0}) \,\middle|\, X_0^{\alpha,0} = x \right],$$

$$\mathcal{D}\varphi(x) := \sup_{\beta} \left(\varphi(x+\beta) + G(\beta, x) \right).$$

Goal : Find $(\mathcal{X}, d)$ such that $\mathcal{T}$ is a strict contraction and $V \in \mathcal{X}$. Then, for any $\varphi \in \mathcal{X}$,

$$\boxed{V = \lim_{n\to\infty} \mathcal{T}^n \varphi.}$$

This is possible due to discounting.

- This structure could exist in several problems where there is periodic control or monitoring.
- We will discuss the case of periodic dividends in the next section.
- Another application is Leveraged Exchange Traded Funds (LETF).
- in LETF, one monitors the daily returns of an underlying and tries to match a pre-determined multiple of this return. Hence, it is daily monitoring with continuous trading.
- LEFT is studied in a paper with Min Dai, Steve Kuo, M.S., Chen Yang.

Discrete dividends

- Firm with cash flow C and reserves

$$dX_t^{L,I} = dC_t - dL_t + dI_t.$$

- Time of ruin $\theta(L, I) = \inf\{t > 0 : X_t^{L,I} < 0\}$.
- *Continuous payoff functional :*

$$J(x; L, I) = \mathbb{E}\left[\int_0^{\theta(L,I)} e^{-rt} dL_t - \sum_{t\geq 0} e^{-rt}(\lambda_f + (1+\lambda_p)\Delta I_t)1_{\{\Delta I_t > 0\}}\right].$$

- *Discrete payoff functional :*

$$J(x; L, I) = \mathbb{E}\left[\sum_{t=0}^{\theta(L,I)} e^{-rt}\Delta L_t - \sum_{t\geq 0} e^{-rt}(\lambda_f + (1+\lambda_p)\Delta I_t)1_{\{\Delta I_t > 0\}}\right].$$

Like in the abstract formulation,

$$\mathcal{L}\varphi(x) = \sup_{I} \mathbb{E}\left[-\sum_{0\leq t<T} e^{-rt}(\lambda_f + (1+\lambda_p)\Delta I_t)1_{\{\Delta I_t>0\}} + e^{-rT}\varphi(X^{0,I}_{T-})1_{\{T<\theta(L,I)\}} \right]$$

$$\mathcal{D}\varphi(x) = \sup_{\ell\leq x}(\varphi(x-\ell)+\ell);$$

Numerically we want to find the fixed point of $\mathcal{T}$: $V = \lim_{n\to\infty}(\mathcal{D}\circ\mathcal{L})^n\varphi$.

To compute $\mathcal{L}$, observe that $\mathcal{L}\varphi = u(0,\cdot)$, where u solves $u(T,\cdot) = \varphi$ and

$$\boxed{\min\left\{-(\partial_t + \mathcal{A} - r)u(t,x),\ \ u(t,x) - \sup_{i\geq 0}\left(u(t,x+i) - (1+\lambda_p)i - \lambda_f\right)\right\} = 0,}$$

with $\mathcal{A}u = \mu u' + \frac{\sigma^2}{2}u''$ is the infinitesimal generator of C.

- Define the issuance operator as

$$\mathbf{M}u(t,x) = \sup_{i \geq 0} \left(u(t, x+i) - (1+\lambda_p)i - \lambda_f \right).$$

- The fixed point can be characterized by the PDE

$$\min\left\{-(\partial_t + \mathcal{A} - r)v(t,x),\ \ v(t,x) - \mathbf{M}v(t,x)\right\} = 0,$$

with the boundary condition

$$v(t,0) = \mathbf{M}v(t,x) \vee 0, \quad t \in [0,T]$$

and a periodic final condition,

$$\boxed{v(T,x) = e^{-rT} \sup_{\ell \leq x} (v(0, x-\ell) + \ell).}$$

Fix φ and consider the pure issuance problem up to time T.

Theorem

Under some assumptions on φ. The value function u is a smooth, classical solution of

$$\boxed{-(\partial_t + \mathcal{A} - r)v(t,x) = 0, \quad t \in [0, T), x > 0,}$$

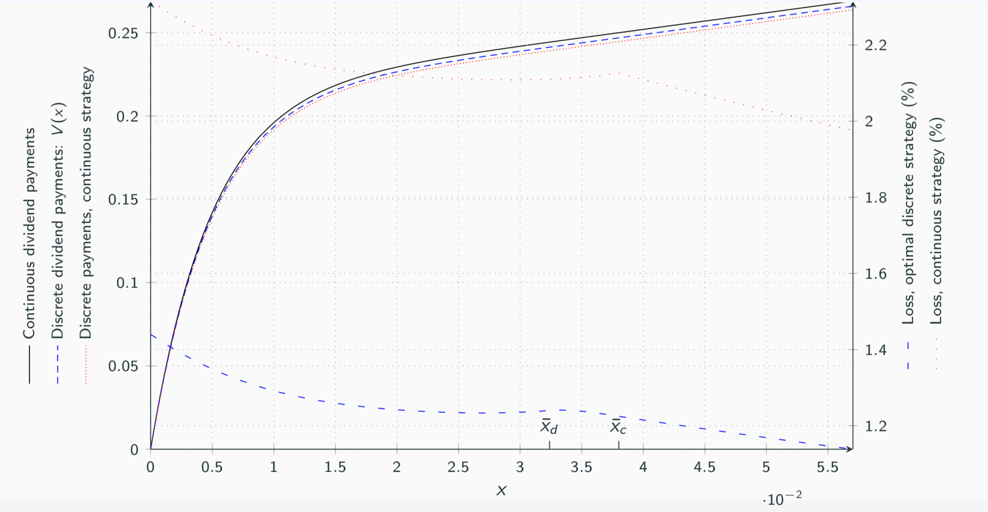
with the boundary condition

$$\boxed{v(t,0) = \mathbf{M}v(t,x) \vee 0, \quad t \in [0, T],}$$

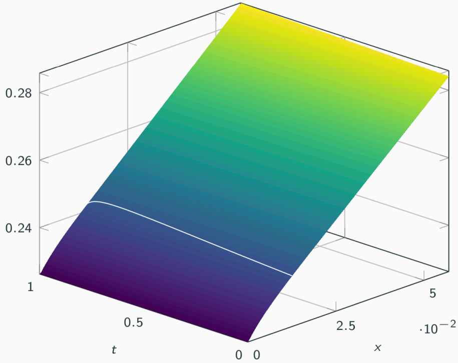
and $u(T,\cdot) = \varphi$. In particular, it is not optimal to make issuance when $x > 0$.

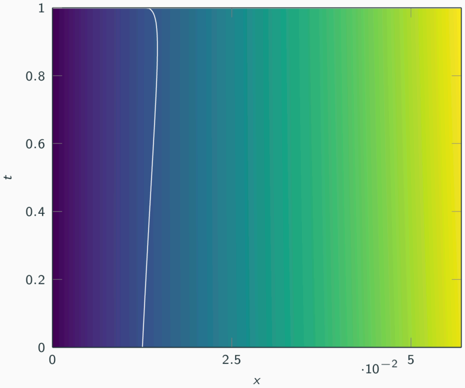
The assumptions on φ are natural and are satisfied in the dividend problem. It is proved by an iterative scheme.

VALUE FUNCTION WITH $C_t = \mu t + \sigma W_t$, $\mu, \sigma \in \mathbb{R}_{>0}$ AND NO ISSUANCE



VALUE FUNCTION WITH $C_t = \mu t + \sigma W_t, \;\; \mu, \sigma \in \mathbb{R}_{>0}$, AND ISSUANCE





- Let $dC_t = \mu_t dt + \sigma dW_t$ and $\mu$is a diffusion process.
- We follow the same steps :

$$\mathcal{D}\varphi(x,\mu) = \sup_{\ell \leq x}(\varphi(x-\ell,\mu)+\ell),$$

$$\mathcal{L}\varphi(x,\mu) = u(0,x,\mu),$$

where u solves

$$\boxed{\min\Big\{-(\partial_t + \mathcal{A} - r)u(t,x,\mu), u(t,x,\mu) - \mathbf{M}u(t,x,\mu))\Big\} = 0,}$$

$u(T,x,\mu) = \varphi(x,\mu)$, and $\mathcal{A}$ is the generator of (C,μ).

- We numerically compute $V = \lim_{n\to\infty}(\mathcal{D}\circ\mathcal{L})^n\varphi$.

Theorem

If there exists an $\alpha : \mathbb{R} \to [1, \infty)$ *so that*

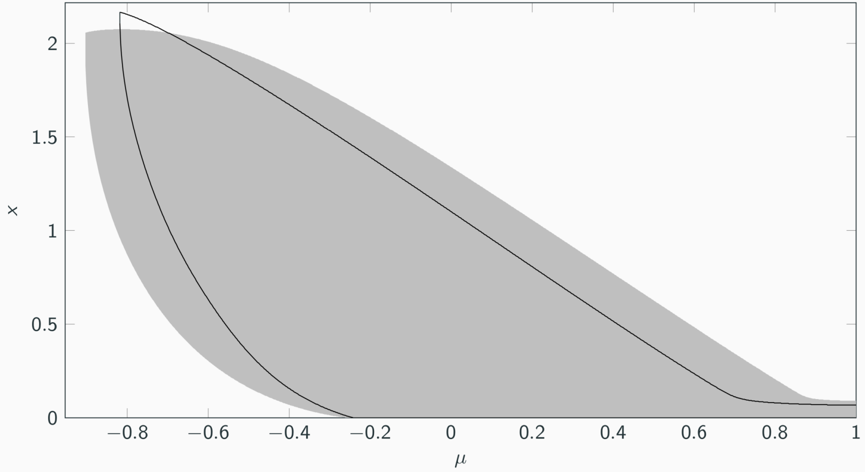
1. $\mathbb{E}[(x + C_T)^+ | \mu_0 = \mu] \leq x + C^* \alpha(\mu)$, $\forall \mu \in \mathbb{R}$, *for some* $C^* \geq 0$,
2. $\mathbb{E}[\alpha(\mu_T) | \mu_0 = \mu] \leq e^{rT/2} \alpha(\mu)$,

then there exists a metric space $(\mathcal{C}_\alpha, d_\alpha)$ *such that the operator* $\mathcal{T}$ *maps* $\mathcal{C}_\alpha$ *into itself and is a strict contraction.*

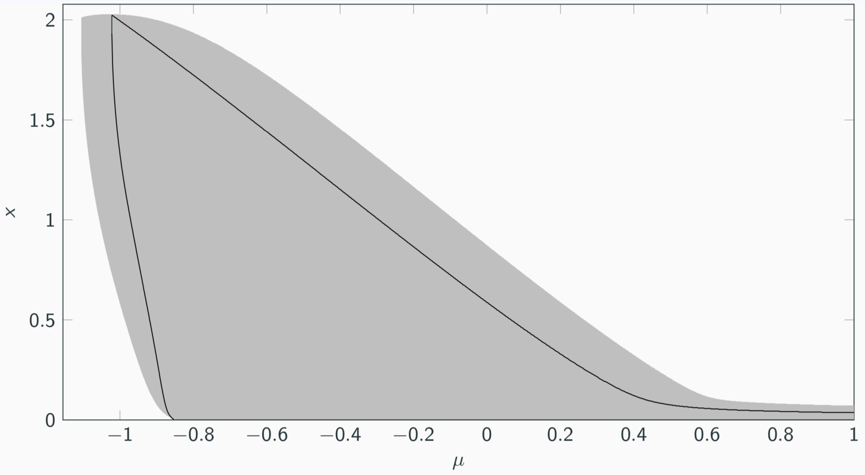
These assumptions are satisfied by Ornstein–Uhlenbeck profitability :

$$d\mu_t = k(\bar{\mu} - \mu_t)dt + \tilde{\sigma} d\tilde{W}.$$

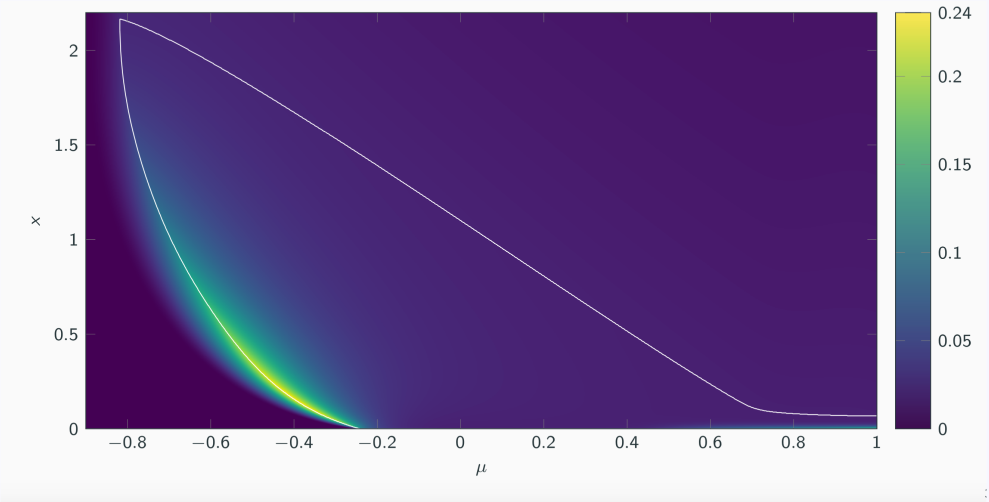
Value function ($I_t \equiv 0$)



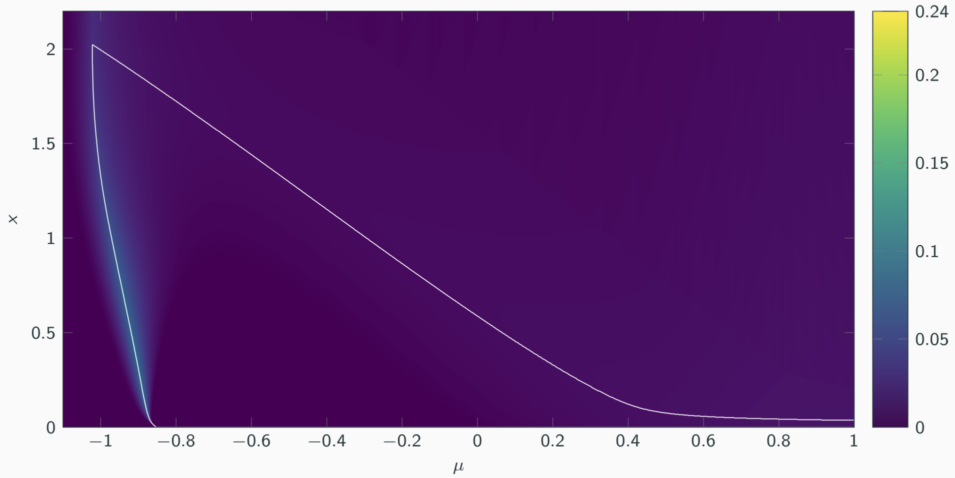
Value function with Issuance



Relative loss ($l_t \equiv 0$)



Relative loss with Issuance



Gambling for resurrection

Consider the possibility of entering a fair lottery at any time.

New control variable

$$G_t = \sum_{k=1}^{\infty} g_k 1_{\{t \geq \tau_k\}},$$

for predictable τ_k and $\mathcal{F}_{\tau_k}$-measurable random variable g_k satisfying $\mathbb{E}[g_k] = 0$ and $X_{\tau_k -} + g_k \geq 0$.

Define

$$\boxed{X_t = x + \int_0^t \mu_t dt + \sigma W_t - L_t + G_t,}$$

and the value function is as before, but including optimization over lotteries as wll.

The HJB equation is then

$$\boxed{\min\{rV - \mathcal{L}V, \quad V_x - 1, \quad -V_{xx}\} = 0, \quad V(0, \cdot) = 0.}$$

Concluding

Thank you for your attention

- Depending on the structure, there could be a substantial difference between the discrete and continuous problems.
- Periodic structure is conceptually natural.
- The regularity for the issuance problem in one step is also a new result.
- This is joint work with Jussi Keppo, Max Reppen, Jean-Charles Rochet :
- *Discrete dividend payments in continuous time*, J. Keppo, M. Reppen, M.S.
- *Optimal dividends with random profitability* (Math. Finance 2019), M. Reppen, J.-C. Rochet, M.S.